SUPERCONFORMAL INDICES
AND
INTEGRABLE SYSTEMS

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$7 \times 70 \times$ HAPPY BIRTHDAYs!

Superconformal index

Flat $4d\ \mathcal{N}=1$ SUSY gauge field theory: $SU(2,2|1)\times G\times F$

Space-time symmetry $SU(2,2|1)$:
$J_i, \overline{J}_i$ = $SU(2)$ subgroups generators, or Lorentz rotations,
$P_\mu, Q_\alpha, \overline{Q}_{\dot\alpha}$ = supertranslations,
$K_\mu, S_\alpha, \overline{S}_{\dot\alpha}$ = special superconformal transformations,
H = dilations and $R = U(1)_R$-rotations.
Internal symmetries: gauge G and flavor F groups

For $Q\propto \overline{Q}_1$ and $Q^\dagger \propto \overline{S}_1$, one has $Q^2=(Q^\dagger)^2=0$ and

$$\{Q,Q^\dagger\}=2\mathcal{H},\qquad \mathcal{H}=H-2\overline{J}_3-3R/2$$

The superconformal index:
Romelsberger, 2005; Kinney, Minwalla, Maldacena, Raju, 2005

$$I(y;p,q)=\mathrm{Tr}_G\Big((-1)^{\mathcal{F}}p^{\mathcal{R}/2+J_3}q^{\mathcal{R}/2-J_3}\prod_k y_k^{F_k}\,e^{-\beta\mathcal{H}}\Big),$$

$\mathcal{F}$ = the fermion number, $\mathcal{R}=H-R/2$, $p,q,y_k,e^{-\beta}$ are group parameters (fugacities) for maximal Cartans commuting with Q (F_k - generators of F).

Flat $4d$ space-time $\Rightarrow S^3 \times S^1 \Rightarrow$ SCI is preserved + the only SUSY (Witten) index $\Rightarrow$ no divergencies

Counting of BPS states $\mathcal{H}|\psi\rangle = 0$ or cohomology space of Q, $Q^\dagger$ operators (hence, no β-dependence).

"Physical" (not rigorous) computation yields the matrix integral:

$$I(y;p,q) \ = \ \int_G d\mu(z)\, \exp\bigg(\sum_{n=1}^{\infty}\frac{1}{n}\mathrm{ind}\big(p^n,q^n,z^n,y^n\big)\bigg),$$

$d\mu(z)$ = the Haar measure, the single particle states index

$$\mathrm{ind}(p,q,z,y) = \frac{2pq-p-q}{(1-p)(1-q)}\chi_{adj_G}(z)$$
$$+\sum_j \frac{(pq)^{R_j/2}\chi_{r_F,j}(y)\chi_{r_G,j}(z)-(pq)^{1-R_j/2}\chi_{\bar{r}_F,j}(y)\chi_{\bar{r}_G,j}(z)}{(1-p)(1-q)},$$

$\chi_{R_F,j}(y)$ and $\chi_{R_G,j}(z)$ = characters of field representations, and R_j are the R-charges.

Romelsberger conjecture (2007):

SCIs of Seiberg-dual theories should coincide

Proof: Dolan, Obsborn (2008)

The simplest Seiberg electromagnetic duality (1994):

The "electric" theory: $G = SU(2)$, $F = SU(6)$, representations

1) vector superfield: $(adj, 1)$,

$$\chi_{SU(2),adj}(z) = z^2 + z^{-2} + 1,$$

2) chiral superfield: (f, f),

$$\chi_{SU(2),f}(z) = z + z^{-1}, \qquad R_f = 1/3,$$

$$\chi_{SU(6),f}(y) = \sum_{k=1}^{6} y_k, \quad \chi_{SU(6),\bar{f}}(y) = \sum_{k=1}^{6} y_k^{-1}, \quad \prod_{k=1}^{6} y_k = 1,$$

The "magnetic" theory: $G = 1$, $F = SU(6)$ with the single chiral superfield T_A : $\quad \Phi_{ij} = -\Phi_{ji}$,

$$\chi_{SU(6),T_A}(y) = \sum_{1 \le i < j \le 6} y_i y_j, \qquad R_{T_A} = 2/3.$$

A Wess-Zumino type theory for the confined colored particles.

For the unitary group $SU(N)$, $z=(z_1,\ldots,z_N)$, $\prod_{a=1}^N z_a=1$,

$$\int_{SU(N)} d\mu(z) \;=\; \frac{1}{N!}\int_{\mathbb{T}^{N-1}} \Delta(z)\Delta(z^{-1})\prod_{a=1}^{N-1}\frac{dz_a}{2\pi \mathrm{i} z_a},$$

$$\Delta(z) \;=\; \prod_{1\leq a<b\leq N}(z_a-z_b), \qquad \text{the Vandermonde determinant.}$$

Denote $t_k=(pq)^{1/6}y_k$, $k=1,\ldots,6 \quad \Rightarrow \quad \prod_{k=1}^6 t_k = pq$

The electric superconformal index (Dolan-Osborn, 2008):

$$I_E(t_k;p,q)=\frac{(p;p)(q;q)}{4\pi \mathrm{i}}\int_{\mathbb{T}}\frac{\prod_{j=1}^6\Gamma(t_jz^{\pm1};p,q)}{\Gamma(z^{\pm2};p,q)}\frac{dz}{z},$$

where $\mathbb{T}$ is the unit circle, $(q;q)=\prod_{k=1}^\infty(1-q^k)$,

$$\Gamma(z;p,q)=\prod_{j,k=0}^{\infty}\frac{1-z^{-1}p^{j+1}q^{k+1}}{1-zp^jq^k}, \qquad |p|,|q|<1,$$

the elliptic gamma function and

$$\Gamma(t_1,\ldots,t_k;p,q):=\Gamma(t_1;p,q)\cdots\Gamma(t_k;p,q),$$
$$\Gamma(tz^{\pm k};p,q):=\Gamma(tz^k;p,q)\Gamma(tz^{-k};p,q).$$

The magnetic superconformal index

$$I_M(t_k;p,q)=\prod_{1\leq j<k\leq 6}\Gamma(t_jt_k;p,q).$$

The elliptic beta integral

Theorem. For $|t_j| < 1$, $\quad I_E = I_M$. V.S., 2000

$=$ top generalization of Euler's beta integral

$$\int_0^1 x^{\alpha-1}(1-x)^{\beta-1}dx = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}.$$

$=$ proof of the confinement in the BPS-sector

$\Rightarrow$ A principally new class of transcendental special functions of mathematical physics = Elliptic Hypergeometric Integrals

My driving idea: search of the most general **exactly solvable** quantum mechanical system $\Leftrightarrow$ relation to **integrable systems**.

Elliptic analogue of the Euler-Gauss ${}_2F_1$ hypergeometric function:

$$V(t_1,\ldots,t_8;p,q) = \frac{(p;p)(q;q)}{4\pi \mathrm{i}} \int_{\mathbb{T}} \frac{\prod_{j=1}^8 \Gamma(t_j z^{\pm 1};p,q)}{\Gamma(z^{\pm 2};p,q)} \frac{dz}{z},$$

where $\prod_{j=1}^8 t_j = (pq)^2$, $|t_j| < 1$.

Symmetry transformation: V.S., 2003

$$V(\underline{t};p,q) = \prod_{1\leq j<k\leq 4} \Gamma(t_jt_k;p,q)\Gamma(t_{j+4}t_{k+4};p,q)\, V(\underline{s};p,q),$$

where $|t_j|, |s_j| < 1$ and

$$\begin{cases} s_j = \lambda t_j, & j=1,2,3,4 \\ s_j = \lambda^{-1} t_j, & j=5,6,7,8 \end{cases}; \quad \lambda = \sqrt{\frac{pq}{t_1t_2t_3t_4}} = \sqrt{\frac{t_5t_6t_7t_8}{pq}}$$

Seiberg duality for two $SU(2)\times SU(8)$ SUSY theories: $I_E = I_M$.

Elliptic hypergeometric equation: V.S., 2004

$$A(x)\,(f(qx)-f(x)) + A(x^{-1})\left(f(q^{-1}x)-f(x)\right) + \nu f(x) = 0,$$

$$A(x) = \frac{\prod_{k=1}^{8}\theta(\varepsilon_k x;p)}{\theta(x^2, qx^2;p)}, \quad \nu = \prod_{k=1}^{6}\theta\left(\frac{\varepsilon_k\varepsilon_8}{q};p\right), \quad \varepsilon_7 = \frac{\varepsilon_8}{q}$$

with two independent solutions $f(x) \propto V(\ldots)$.

Jacobi theta function:

$$\theta(x;p) = (x;p)(px^{-1};p), \qquad (x;p) = \prod_{j=0}^{\infty}(1-xp^j).$$

Relativistic quantum mechanical N-body problems.

Hamiltonian of the van Diejen (1994) (a generalized elliptic Ruijsenaars) integrable model

$$\mathcal{D}=\sum_{j=1}^{N}\Big(A_j(\underline{x})(T_j-1)+A_j(\underline{x}^{-1})(T_j^{-1}-1)\Big),$$

where $T_jf(\ldots,x_j,\ldots)=f(\ldots,qx_j,\ldots)$ and

$$A_j(\underline{x})=\frac{\prod_{m=1}^{8}\theta(t_mx_j;p)}{\theta(x_j^2,qx_j^2;p)}\prod_{\substack{k=1\\ \neq j}}^{n}\frac{\theta(tx_jx_k,tx_jx_k^{-1};p)}{\theta(x_jx_k,x_jx_k^{-1};p)}$$

with the constraint $t^{2N-2}\prod_{m=1}^{8}t_m=p^2q^2$.

The standard eigenvalue problem $\mathcal{D}\psi(\underline{x})=\lambda\psi(\underline{x})$. For $N=1$:

$$\frac{\prod_{j=1}^{8}\theta(t_jx;p)}{\theta(x^2,qx^2;p)}(f(qx)-f(x))$$
$$+\frac{\prod_{j=1}^{8}\theta(t_jx^{-1};p)}{\theta(x^{-2},qx^{-2};p)}(f(q^{-1}x)-f(x))=\lambda f(x).$$

Special parameters $\Rightarrow$ elliptic hypergeometric equation.

SCIs emerge as eigenfunctions of integrable systems $\Rightarrow$ new integrable models? (V.S., 2004)

Commuting difference operators via the residue calculus of SCIs (Gaiotto, Rastelli, Razamat, 2012; Razamat, 2020)

New rational integrable N-body problems

A very intricate degeneration limit of the elliptic hypergeometric equation

1) a hyperbolic degeneration:

$$p = e^{-2\pi v\omega_1},\ q = e^{-2\pi v\omega_2},\ t_a = e^{-2\pi v g_a},\ x = e^{-2\pi v z},\ v \to 0$$

2) a special rational degeneration: $\omega_1/\omega_2 = -1 + 2\mathrm{i}\delta$, $\delta \to 0$,

$z = \mathrm{i}\sqrt{\omega_1\omega_2}(n+y\delta),\ g_a = \mathrm{i}\sqrt{\omega_1\omega_2}(n_a+s_a\delta),\ n, n_a \in \mathbb{Z}, y, s_a \in \mathbb{C}.$

Then (G. Sarkissian, V.S., arXiv:2105.15031):

$$\frac{\prod_{j=1}^{8}(\beta_j + w)}{2w(2w+1)}(\psi(u+\mathrm{i}, m-1) - \psi(u,m))$$

$$+\frac{\prod_{j=1}^{8}(\beta_j - w)}{2w(2w-1)}(\psi(u-\mathrm{i}, m+1) - \psi(u,m)) + \prod_{k=1}^{6}(\beta_k+\beta_7)\,\psi(u,m) = 0,$$

where

$$w = \frac{\mathrm{i}u+m}{2}, \qquad \beta_k = \frac{\mathrm{i}s_k + n_k}{2}$$

with some additional constraints. This is a subcase of the Hamiltonian eigenvalue problem for a new rational finite-difference N-body integrable model (a special **new** degeneration of the van Diejen model). $\psi(u,m) \propto$ new complex hypergeometric function.

Elliptic Fourier transformation V.S., 2003

Definition

$$\beta(w,t) = M(t)_{wz}\alpha(z,t) = \frac{(p;p)_\infty (q;q)_\infty}{4\pi \mathrm{i}} \int_{\mathbb{T}} \frac{\Gamma(tw^{\pm 1}z^{\pm 1};p,q)}{\Gamma(t^2, z^{\pm 2};p,q)} \alpha(z,t) \frac{dz}{z}.$$

Inversion like in Fourier tr-n, $t \to t^{-1}$ (V.S., Warnaar, 2005)

$$M(t^{-1})_{wz} M(t)_{zx} f(x) = f(w).$$

Infinite chain of transformations:

$$\alpha'(w,st) = D(s;u,w)\alpha(w,t), \qquad D(s;u,w)D(s^{-1};u,w) = 1,$$

$$D(s;u,w) := \Gamma(\sqrt{pq}s^{-1}u^{\pm 1}w^{\pm 1};p,q),$$

$$\beta'(w,st) = D(t^{-1};u,w)M(s)_{wx}D(st;u,x)\beta(x,t).$$

Then $\beta'(w,st) = M(st)_{wz}\alpha'(z,st) \Rightarrow$

$$M(s)_{wx}D(st;u,x)M(t)_{xz} = D(t;u,w)M(st)_{wz}D(s;u,z).$$

Operator star-triangle relation = elliptic beta integral

Coxeter relations and YBE Derkachov, V.S., 2012

Let $\mathbf{t} = (t_1, t_2, t_3, t_4)$ and

$$s_1\mathbf{t} = (t_2, t_1, t_3, t_4), \quad s_2\mathbf{t} = (t_1, t_3, t_2, t_4), \quad s_3\mathbf{t} = (t_1, t_2, t_4, t_3).$$

Operators

$$\begin{aligned}
[\mathrm{S}_1(\mathbf{t})f](z_1, z_2) &:= M(t_1/t_2)_{z_1 z} f(z, z_2),\\
[\mathrm{S}_2(\mathbf{t})f](z_1, z_2) &:= D(t_2/t_3; z_1, z_2) f(z_1, z_2),\\
[\mathrm{S}_3(\mathbf{t})f](z_1, z_2) &:= M(t_3/t_4)_{z_2 z} f(z_1, z)
\end{aligned}$$

with the product rule $\mathrm{S}_j\mathrm{S}_k := \mathrm{S}_j(s_k\mathbf{t})\mathrm{S}_k(\mathbf{t})$ satisfy Coxeter rels

$$\mathrm{S}_j^2 = 1, \quad \mathrm{S}_i\mathrm{S}_j = \mathrm{S}_j\mathrm{S}_i \ \text{ for } \ |i-j| > 1, \quad \mathrm{S}_j\mathrm{S}_{j+1}\mathrm{S}_j = \mathrm{S}_{j+1}\mathrm{S}_j\mathrm{S}_{j+1}.$$

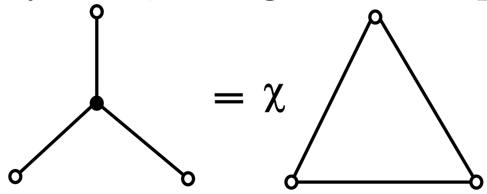
$M(t)$ = an intertwiner of the Sklyanin algebra
$\Rightarrow$ top solution of the Yang-Baxter equation, R-matrix $\propto$ elliptic hypergeometric integral operator:

$$\mathbb{R}_{12}(u) := \mathbb{P}_{12}\, \mathrm{S}_2\mathrm{S}_1\mathrm{S}_2\mathrm{S}_3, \quad t_1/t_2 = e^{2\pi i u},\ t_3/t_4 = e^{2\pi i v},$$

$$\mathbb{R}_{12}(u-v)\, \mathbb{R}_{13}(u)\, \mathbb{R}_{23}(v) = \mathbb{R}_{23}(v)\, \mathbb{R}_{13}(u)\, \mathbb{R}_{12}(u-v).$$

A $4d/2d$-correspondence

Elliptic beta integral = functional star-triangle relation for Ising type models on honeycomb, triangluar and square lattices:



Circles = spins $x, w, \ldots$, edges = Boltzmann weights W, black circle = integration (summation) over spins

Bazhanov, Sergeev (2010): for $\alpha + \beta + \gamma = \xi$,

$$\int_0^{2\pi} \rho(u) W_{\xi-\alpha}(x,u) W_{\xi-\beta}(y,u) W_{\xi-\gamma}(w,u) du$$
$$= \chi W_\alpha ; y, w) W_\beta(x,w) W_\gamma(x,y),$$

where

$$W_\alpha(x,y) = \Gamma(e^{-\alpha} e^{\mathrm{i}(\pm x \pm y)}; p, q)$$
$$\rho(u) = \frac{(p;p)_\infty (q;q)_\infty}{4\pi} \theta(e^{2\mathrm{i}u}; p) \theta(e^{-2\mathrm{i}u}; q),$$
$$\chi = \Gamma(e^{-\alpha}, e^{-\gamma}, e^{\alpha+\gamma-\xi}; p, q), \qquad e^{-\xi} = \sqrt{pq}.$$

$\Rightarrow$ $4d$ SCIs of quiver theories describe partition functions of integrable models of $2d$ spin systems: V.S., 2010

V-function symmetry = star-star relations $\Rightarrow$ IRF-type YBE

Seiberg duality = integrability condition