

# Loops and Expansions

Advances in Quantum Field Theory – Dubna, October 11-14, 2021 – via Zoom

Matthias Steinhauser | 14. October 2021

ITPP KIT

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## Russian Loops





Kostja: 39 publications

Volodya: 43 publications



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1 paper where all three of us are co-authors

# Kostja



# Loops – conventions – automatization

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1. QCD

Правила Feynmana.

$$\mathcal{L} = -\frac{1}{4} F_{ij}^{\mu\nu} F_{ij\mu\nu} + \bar{\psi} (\not{D} - m) \psi - \frac{1}{2g} (\partial_\mu A_i^\mu)^2 + g \bar{\psi}_i (\not{\partial}^\mu \psi_i + g C_{ijk} A_j^\mu \psi_k).$$

$$F_i^{\mu\nu} = \partial^\mu A_i^\nu - \partial^\nu A_i^\mu + g G_{ijk} A_j^\mu A_k^\nu$$

$$D_{ab}^\mu = \delta_{ab} \partial^\mu - ig A_i^\mu (T_i)_{ab}$$

$a, b = 1, 2, \dots, n$  — число кварков / прядей локальной групппы цвета  $G$ .

$i, j, k = 1, \dots, N$  — размерность пр.  $G$ .

$$[T_i T_j] = i C_{ijk} T_k$$

$$Tr(T_i) = 0, \quad Tr(T_i T_i) = \alpha \delta_{ii}$$

$$i C_{ijk} = \frac{1}{\alpha} Tr(T_i T_j T_k - T_k T_j T_i).$$

$\alpha = \frac{1}{2}, \text{ если } T_i = (N)/2.$

2. Правила Feynmana.

$$\delta_{is} = \frac{1}{i} \left( \frac{g_{\mu\nu}}{k^2} + (\omega - 1) \frac{k_\mu k_\nu}{k^4} \right)$$

$$\delta_{ab} = \frac{1}{i} \frac{(\not{k} + m)}{m^2 - k^2}$$

$$\delta_{is} = \frac{1}{i} \frac{1}{-k^2} = \delta_{is} \frac{i}{k^2}$$

$i \not{k}_\mu T_{ab}$

$$ig (-C_{ijk}) \left\{ \begin{aligned} & (P_3 - P_2)_\mu g_{j\nu} \\ & (P_3 - P_1)_\nu g_{i\nu} \\ & * (P_2 - P_1)_\lambda g_{i\nu} \end{aligned} \right\}$$

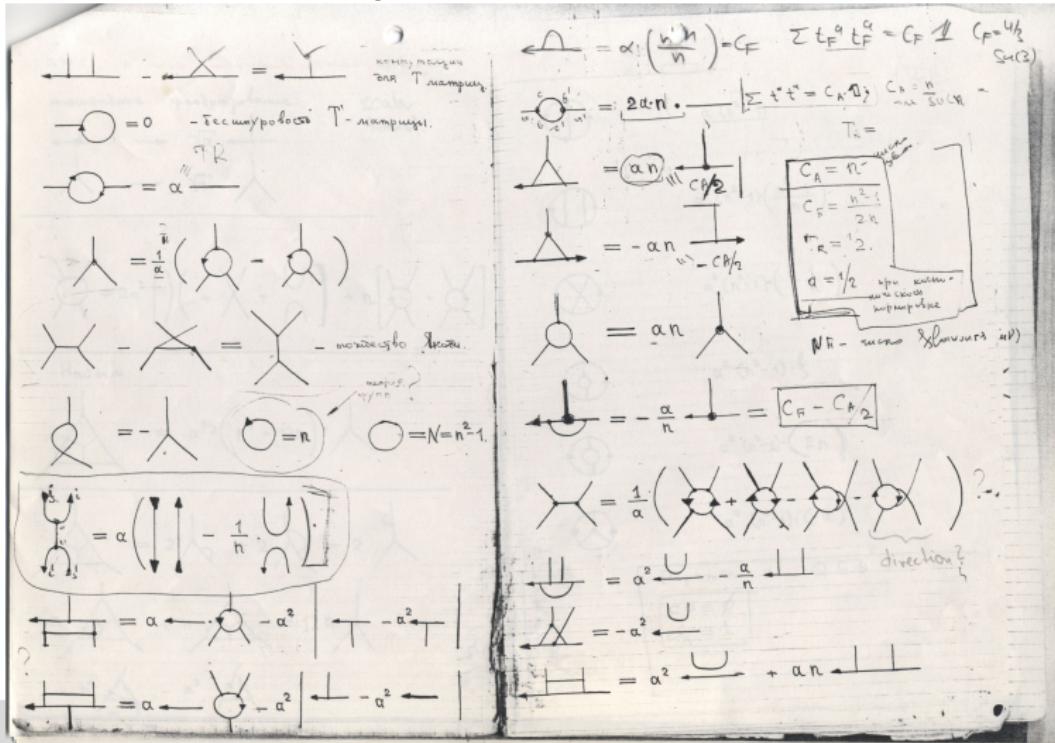
$$ig^2 \left\{ C_{i1j2m} C_{m1j2n} (g_{\mu_1\mu_2} g_{\mu_3\mu_4} - g_{\mu_1\mu_3} g_{\mu_2\mu_4}) \right.$$

$$C_{i1j3m} C_{m1j3n} (g_{\mu_1\mu_2} g_{\mu_3\mu_4} - g_{\mu_1\mu_3} g_{\mu_2\mu_4})$$

$$C_{i1j4m} C_{m1j4n} (g_{\mu_1\mu_2} g_{\mu_3\mu_4} - g_{\mu_1\mu_3} g_{\mu_2\mu_4})$$

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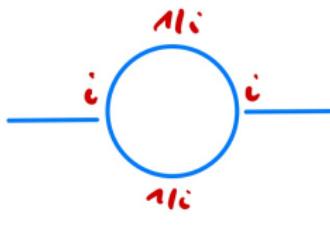
Hand-drawn Feynman diagrams and mathematical equations on lined paper. The top diagram shows a loop with two external gluons and two internal gluons. Below it, a vertical arrow labeled  $\uparrow$  points to a box containing the text "1) 3-matrix". To the right of the arrow is another box containing the equation  $\Delta(3t) = \langle \bar{q} q \rangle_0$ . In the center, there is a diagram of a circle with a diagonal line through it, labeled  $Z_{ij} x$  on the left and  $Z_{ij}$  on the right. Below this diagram is the expression  $Z_{ij} \delta(x-y)$ . On the far right, there is a small diagram of a loop with a central gluon and a box labeled  $\Delta C$ .

# Loops – conventions – automatization

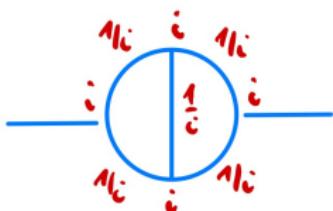
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$$i \frac{1}{i} \int d\mu \xrightarrow{d\mu^E} i \int d\mu^E$$



$$i \frac{1}{i} \frac{1}{i} \int d\mu_1 d\mu_2 \xrightarrow{d\mu_1^E d\mu_2^E} i \int d\mu_1^E d\mu_2^E$$

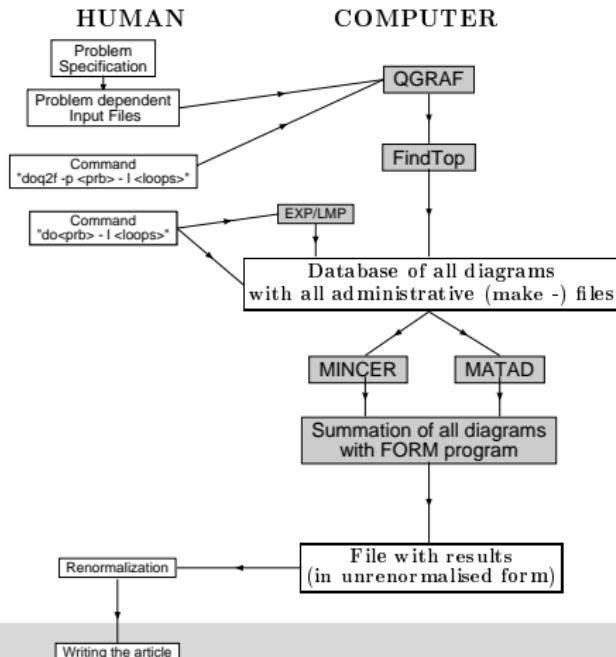
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## GEFICOM

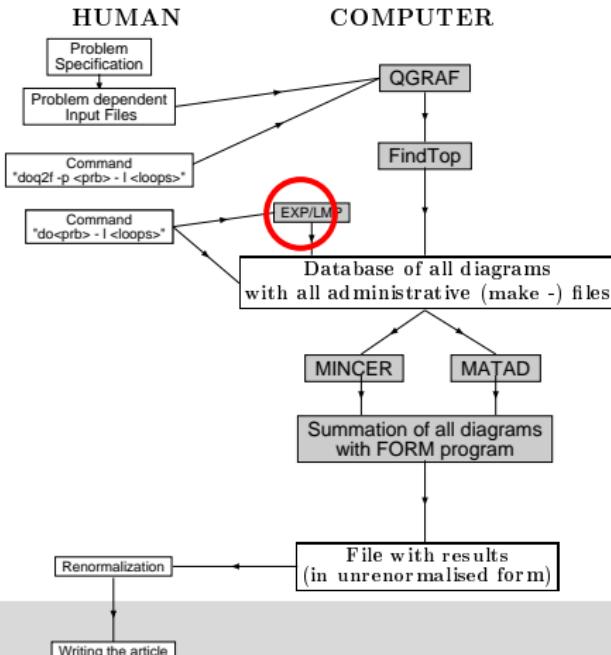
automatic GEneration, FInder topologies and COMputation of  
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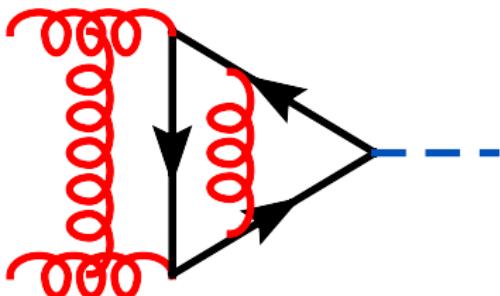


# First application: gluon-Higgs vertex

[Chetyrkin,Kniehl,Steinhauser'98]

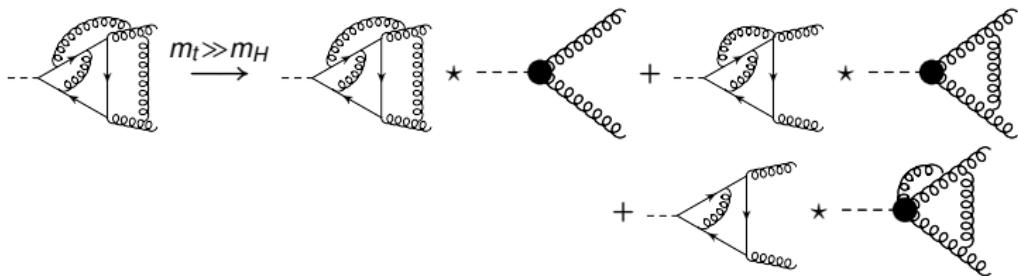
- $\mathcal{L}_{\text{eff}} = -\frac{H}{v} C_H (G_{\mu\nu})^2$
- 3-loop tadpoles
- “many” Feynman diagrams
- LET:  $C_H = -\frac{1}{\zeta_{\alpha_s}} \frac{\partial}{\partial \log m_t} \zeta_{\alpha_s}$

⇒ Gluon-Higgs coupling to 4 loops, N<sup>3</sup>LO in 1998



# Asymptotic expansion

[Davies,Herren,Steinhauser'19]...

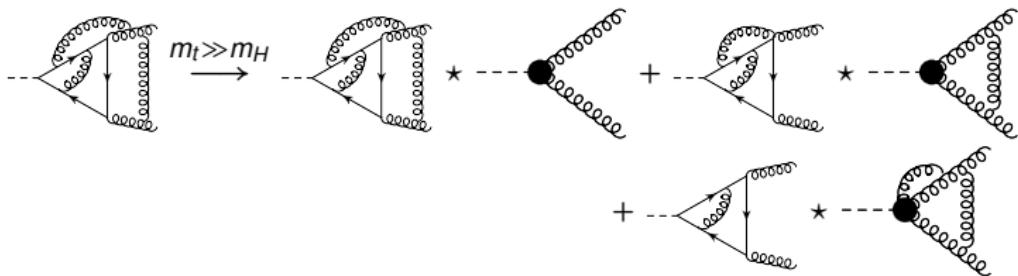


- automated nested Euclidian asymptotic expansion: `exp`

[Harlander,Seidensticker,Steinhauser'97;Seidensticker'97]

# Asymptotic expansion

[Davies,Herren,Steinhauser'19]...



- automated nested Euclidian asymptotic expansion: `exp`

[Harlander,Seidensticker,Steinhauser'97;Seidensticker'97]

Without Kostja — no `exp`

# Volodya

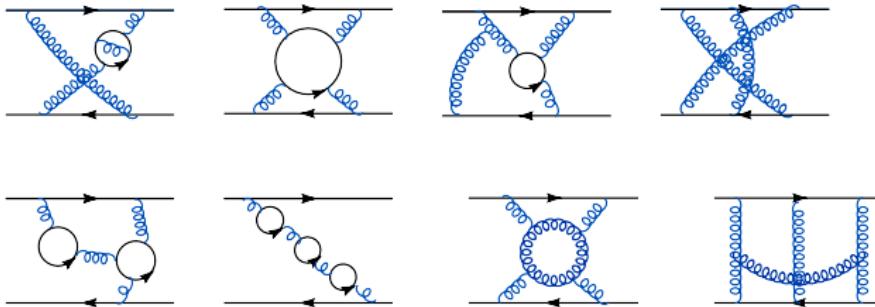
$$G_l = \int_k \frac{k^{\alpha\epsilon}}{k+q} - \Big|_m \frac{1}{k+m} = \int_k \frac{k^{\alpha\epsilon}}{k+q} - \frac{1}{m}$$
$$G_s = \int_0^\infty \frac{k^{-\epsilon} dk}{k+m} - \frac{\int_0^\infty \frac{k^{-\epsilon} dk}{k+m}}{\int_0^\infty |k^{-\epsilon-1}| dk}$$
$$\Rightarrow \int_0^\infty k^{l-n-1-\epsilon} dk - \frac{\int_0^\infty |k^{-\epsilon-1}| dk}{\int_0^\infty |k^{-\epsilon-1}| dk}$$
$$\rightarrow \int_0^\infty k^{l-n-1-\epsilon} dk$$

# Loops – expansions – more loops

- > 2000 Hamburg
- $N^3LO$  Hamiltonian for Potential NRQCD [Kniehl, Penin, Smirnov, Steinhauser'02]  
... and many applications
- Mercator professorship

# Loops – expansions – more loops

- > 2000 Hamburg
- N<sup>3</sup>LO Hamiltonian for Potential NRQCD [Kniehl, Penin, Smirnov, Steinhauser'02]  
... and many applications
- Mercator professorship
- > 2004 Karlsruhe
- annual visits
- Static potential: 2008, ..., 2009, ..., 2016 [Lee, Smirnov, Smirnov, Steinhauser]



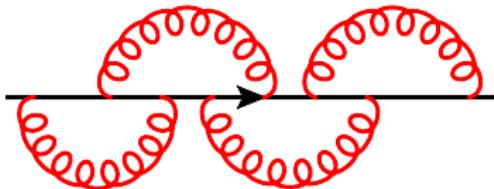
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- 2013: Forschungspreis der Alexander von Humboldt-Stiftung  
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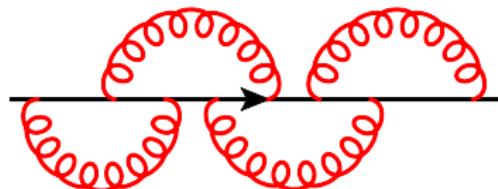
- 4-loop  $\overline{\text{MS}}$ -on-shell relation



[Marquard,Smirnov,Smirnov,Steinhauser,Wellmann'15'16]

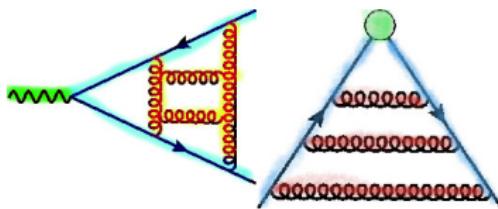
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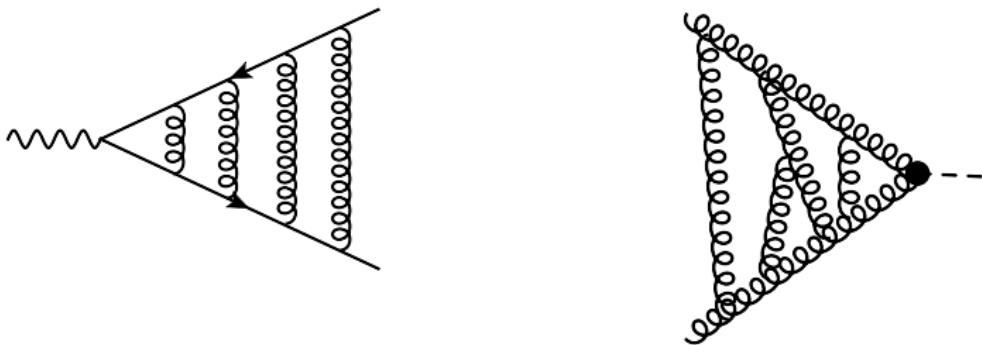
[Marquard,Smirnov,Smirnov,Steinhauser,Wellmann'15'16]

- 4-loop  $\overline{\text{MS}}$ -on-shell relation
- massive 3-loop form factor
- massless 4-loop form factor



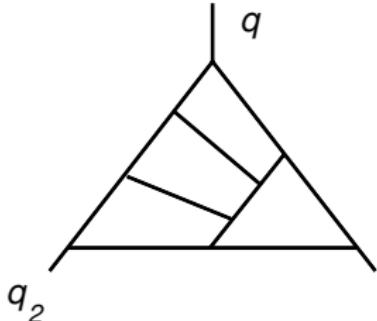
[Henn,Lee,Smirnov,Smirnov,Steinhauser,'16,..., + von Manteuffel,Schabinger'21]

# 4-loop form factor



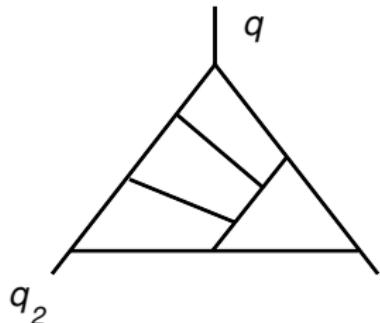
[...; Henn,Smirnov,Smirnov,Steinhauser'16; von Manteuffel,Schabinger'16'19; Henn,Lee,Smirnov,Smirnov,Steinhauser'16;  
Lee,Smirnov,Smirnov,Steinhauser'17'19; von Manteuffel,Panzer,Schabinger'20;  
Lee,von Manteuffel,Schabinger,Smirnov,Smirnov,Steinhauser'21; ...]

# 4-loop form factor



- reduction: FIRE [A. Smirnov],  
LiteRed [Lee]
- MIs:  $q^2 \neq 0$  and  
 $q_2^2 = (q_2 + q)^2 = 0$
- integrals simple if  $q_2^2 = q^2$
  
- idea: introduce arbitrary  $q_2^2 \Leftrightarrow$  differential equations [Henn,Smirnov,Smirnov'14]
- boundary conditions:  $q_2^2 = q^2$
- use canonical basis [Henn'13'14] [Lee'14] [Gituliar,Magerya'16; Meyer'16; Prausa'17]  
solution: iterated integrals  $\Leftrightarrow$  harmonic polylogarithms (HPLs)  
[Remiddi,Vermaseren'99][Maitre'05]
- $q_2^2 \rightarrow 0$ :  $(q_2^2/q^2)^{a\epsilon}$  extract term with  $a = 0$

# 4-loop form factor ... more details

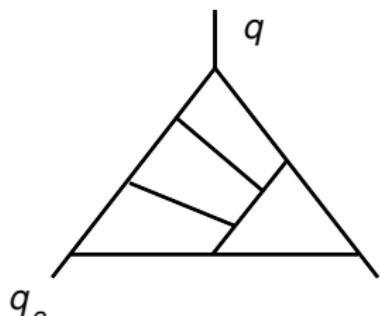


$\Leftrightarrow 76$  MIs

$q_2^2 \neq 0 \Leftrightarrow 332$  MIs

$$x = q_2^2/q^2$$

# 4-loop form factor ... more details



primary basis

⇒ 76 MIs

$$q_2^2 \neq 0 \Leftrightarrow 332 \text{ MIs} \quad x = q_2^2/q^2$$

canonical basis

$$f(x, \epsilon)$$

$$\xrightarrow{f=T \cdot g \text{ [Lee'14]}}$$

$$g'(x, \epsilon) = \epsilon A(x) \cdot g(x, \epsilon)$$

$$A(x) = \frac{a}{x} + \frac{b}{x-1}$$

# 4-loop form factor ... more details

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solve in terms of HPLs

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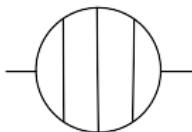
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solve in terms of HPLs  
boundary conditions for  $x = 1$ :

4-loop



2-point functions

[Baikov,Chetyrkin,Kühn'05'08; ...;  
Lee,Smirnov,Smirnov'11]

# 4-loop form factor ... more details

primary basis

$$f(x, \epsilon)$$

$$\xrightarrow{f = T \cdot g \text{ [Lee'14]}}$$

canonical basis

$$g'(x, \epsilon) = \epsilon A(x) \cdot g(x, \epsilon)$$

$$A(x) = \frac{a}{x} + \frac{b}{x-1}$$

solve in terms of HPLs  
boundary conditions for  $x = 1$ :

get  $x = 0$  result ("naive"):

1.  $g_{x \rightarrow 0} = x^{\epsilon a} h(\epsilon)$     a:  $332 \times 332$  matrix
- $\Leftrightarrow x^{\epsilon a}$  is  $332 \times 332$  matrix; each element is linear combination of  $x^{k\epsilon}$  terms with  $k \leq 0$

# 4-loop form factor ... more details

primary basis

$$f(x, \epsilon) \xrightarrow{f = T \cdot g \text{ [Lee'14]}}$$

canonical basis

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solve in terms of HPLs  
boundary conditions for  $x = 1$ :

get  $x = 0$  result (“naive”):

1.  $g_{x \rightarrow 0} = x^{\epsilon a} h(\epsilon)$
2. expand HPLs for  $x \rightarrow 0$

match 1. and 2.

$$\Leftrightarrow h(\epsilon)$$

$\Leftrightarrow$  get  $x^{0\epsilon}$  terms  $\cong$  “naive”

# 4-loop form factor ... more details

primary basis

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match 1. and 2.

$\Rightarrow h(\epsilon)$

$\Rightarrow$  get  $x^{0\epsilon}$  terms  $\cong$  "naive"

$$f(0, \epsilon)$$

$$\xleftarrow{f = T \cdot g}$$

332 MIs



76 MIs

$$\begin{aligned}
F_q^{(4)} \Big|_{\epsilon^0} = & n_f C_F^3 \left( \frac{1153615}{126} \zeta_7 + \frac{6316}{9} \zeta_5 \zeta_2 + \frac{229468}{135} \zeta_3 \zeta_2^2 + \frac{547270}{81} \zeta_3^2 + \frac{1341628}{2835} \zeta_2^3 - \frac{3467995}{162} \zeta_5 - \frac{192737}{81} \zeta_3 \zeta_2 - \frac{1420133}{1215} \zeta_2^2 \right. \\
& - \frac{38482147}{972} \zeta_3 + \frac{12734681}{648} \zeta_2 + \frac{7837713013}{419904} \Big) + n_f C_A C_F^2 \left( \frac{5669}{4} \zeta_7 - \frac{249194}{135} \zeta_5 \zeta_2 - \frac{417244}{405} \zeta_3 \zeta_2^2 - \frac{11438080}{729} \zeta_3^2 \right. \\
& - \frac{38510}{63} \zeta_2^3 + \frac{4959127}{243} \zeta_5 + \frac{408107}{81} \zeta_3 \zeta_2 + \frac{8388679}{2916} \zeta_2^2 + \frac{2642551543}{26244} \zeta_3 - \frac{111491363}{2187} \zeta_2 - \frac{326984889779}{3779136} \Big) \\
& + n_f C_A^2 C_F \left( \frac{6943}{24} \zeta_7 + \frac{2755}{9} \zeta_5 \zeta_2 + \frac{1912}{45} \zeta_3 \zeta_2^2 + \frac{202210}{27} \zeta_3^2 + \frac{128953}{1620} \zeta_2^3 - \frac{34844257}{3240} \zeta_5 - \frac{119489}{108} \zeta_3 \zeta_2 - \frac{1251893}{540} \zeta_2^2 \right. \\
& - \frac{111467677}{1944} \zeta_3 + \frac{123861583}{3888} \zeta_2 + \frac{21075909203}{279936} \Big) + n_f \frac{d_F^{abcd} d_F^{abcd}}{N_F} \left( -1240 \zeta_7 + \frac{992}{3} \zeta_5 \zeta_2 - \frac{3952}{15} \zeta_3 \zeta_2^2 + \frac{680}{9} \zeta_3^2 + \frac{41620}{189} \zeta_2^3 \right. \\
& + \frac{95098}{27} \zeta_5 + \frac{92}{3} \zeta_3 \zeta_2 + \frac{7552}{45} \zeta_2^2 - \frac{21566}{9} \zeta_2 - \frac{13414}{27} \zeta_3 + \frac{3190}{3} \Big) + n_f^2 C_F^2 \left( \frac{689582}{729} \zeta_3^2 + \frac{191252}{945} \zeta_2^3 + \frac{187364}{1215} \zeta_5 \right. \\
& + \frac{177800}{729} \zeta_3 \zeta_2 + \frac{4777}{135} \zeta_2^2 - \frac{90719803}{13122} \zeta_3 + \frac{44208841}{13122} \zeta_2 + \frac{5325319081}{944784} \Big) + n_f^2 C_A C_F \left( -\frac{1714}{3} \zeta_3^2 - \frac{2836}{315} \zeta_2^3 + \frac{150886}{135} \zeta_5 \right. \\
& - \frac{436}{3} \zeta_3 \zeta_2 - \frac{3722}{135} \zeta_2^2 + \frac{2897315}{486} \zeta_3 - \frac{5825827}{1296} \zeta_2 - \frac{3325501813}{279936} \Big) + n_f^3 C_F \left( -\frac{2194}{135} \zeta_5 - \frac{820}{81} \zeta_3 \zeta_2 + \frac{3322}{135} \zeta_2^2 - \frac{20828}{243} \zeta_3 \right. \\
& + \frac{145115}{729} \zeta_2 + \frac{10739263}{17496} \Big) + n_{q\gamma} \frac{d_F^{abc} d_F^{abc} C_F}{N_F} \left( 26624 \zeta_7 + 1792 \zeta_5 \zeta_2 + \frac{35584}{15} \zeta_3 \zeta_2^2 + \frac{30688}{9} \zeta_3^2 - \frac{179152}{189} \zeta_2^3 - \frac{170224}{27} \zeta_5 \right. \\
& - \frac{8656}{3} \zeta_3 \zeta_2 + \frac{60416}{45} \zeta_2^2 - \frac{100624}{9} \zeta_2 - \frac{71552}{27} \zeta_3 - \frac{89360}{9} \Big) + n_{q\gamma} \frac{d_F^{abc} d_F^{abc} C_A}{N_F} \left( -\frac{13972}{3} \zeta_7 - 1840 \zeta_5 \zeta_2 - \frac{784}{5} \zeta_3 \zeta_2^2 \right. \\
& - 13008 \zeta_3^2 - \frac{618328}{189} \zeta_2^3 + \frac{54436}{9} \zeta_5 + \frac{15112}{3} \zeta_3 \zeta_2 - \frac{95692}{45} \zeta_2^2 + \frac{45976}{9} \zeta_3 + \frac{107984}{9} \zeta_2 + \frac{65264}{9} \Big) + n_{q\gamma} n_f \frac{d_F^{abc} d_F^{abc}}{N_F} \left( 2304 \zeta_3^2 \right. \\
& + \frac{545792}{945} \zeta_2^3 - \frac{8512}{9} \zeta_5 - \frac{1888}{3} \zeta_3 \zeta_2 + \frac{12448}{45} \zeta_2^2 - \frac{3520}{9} \zeta_3 - \frac{16928}{9} \zeta_2 - \frac{11296}{9} \Big)
\end{aligned}$$

+ contributions without closed fermion loop.

[Lee,von Manteuffel,Schabinger,Smirnov,Smirnov,Steinhauser'21]



# Congratulations