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Advances in Quantum Field Theory

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Exact β-functions in supersymmetric theories and the regularization by higher derivatives

Due to the well-known non-renormalization theorem

M. T. Grisaru, W. Siegel and M. Rocek, Nucl. Phys. **B 159** (1979) 429.

the superpotential in supersymmetric theories does not receive divergent quantum corrections. Consequently, it is possible (although not necessary) to renormalize the Yukawa couplings according to the prescription

$$\lambda^{ijk} = \lambda_0^{mnp} (Z_\phi^{1/2})_m{}^i (Z_\phi^{1/2})_n{}^j (Z_\phi^{1/2})_p{}^k,$$

where $(Z_\phi)_i{}^j$ is the renormalization constant for the matter superfields. In this case the Yukawa β-function

$$(\beta_\lambda)^{ijk}(\alpha,\lambda) \equiv \frac{d\lambda^{ijk}}{d\ln\mu}\Big|_{\alpha_0,\lambda_0=\text{const}}$$

is related to the anomalous dimension of the matter superfields by the exact equation

$$(\beta_\lambda)^{ijk} = \frac{1}{2}\Big((\gamma_\phi)_m{}^i \lambda^{mjk} + (\gamma_\phi)_m{}^j \lambda^{imk} + (\gamma_\phi)_m{}^k \lambda^{ijm}\Big) = \frac{3}{2}\,(\gamma_\phi)_m{}^{(i}\lambda^{jk)m}.$$

However, for other renormalization prescriptions this equation can be broken.

The exact expression for the gauge β-function of $\mathcal{N}=1$ supersymmetric theories was proposed by Novikov, Shifman, Vainshtein, Zakharov (NSVZ), and Jones.

V.Novikov, M.A.Shifman, A.Vainshtein, V.I.Zakharov, Nucl.Phys. **B 229** (1983) 381; Phys.Lett. **B 166** (1985) 329; M.A.Shifman, A.I.Vainshtein, Nucl.Phys. **B 277** (1986) 456; D.R.T.Jones, Phys.Lett. **B 123** (1983) 45.

It is usually called the exact NSVZ β-function. It also relates the β-function to the anomalous dimension of the matter superfields in $\mathcal{N}=1$ supersymmetric gauge theories,

$$\beta(\alpha,\lambda) = -\frac{\alpha^2\Big(3C_2 - T(R) + C(R)_i{}^j(\gamma_\phi)_j{}^i(\alpha,\lambda)/r\Big)}{2\pi(1-C_2\alpha/2\pi)}.$$

Here α and λ are the gauge and Yukawa coupling constants, respectively, and we use the notation

$$\mathrm{tr}\,(T^A T^B) \equiv T(R)\,\delta^{AB}; \qquad (T^A)_i{}^k (T^A)_k{}^j \equiv C(R)_i{}^j;$$
$$f^{ACD} f^{BCD} \equiv C_2 \delta^{AB}; \qquad r \equiv \delta_{AA} = \dim G.$$

The NSVZ equations can be written for phenomenologically interesting theories, including theories with multiple gauge couplings.

M. A. Shifman, Int. J. Mod. Phys. A **11** (1996), 5761.

For instance, the all-order exact MSSM β-functions are written as

$$\frac{\beta_3(\alpha,\lambda)}{\alpha_3^2} = -\frac{1}{2\pi(1-3\alpha_3/2\pi)}\Big[3+\sum_{I=1}^{3}\Big(\gamma_{Q_I}(\alpha,\lambda)+\frac{1}{2}\gamma_{U_I}(\alpha,\lambda)+\frac{1}{2}\gamma_{D_I}(\alpha,\lambda)\Big)\Big];$$

$$\frac{\beta_2(\alpha,\lambda)}{\alpha_2^2} = -\frac{1}{2\pi(1-\alpha_2/\pi)}\Big[-1+\sum_{I=1}^{3}\Big(\frac{3}{2}\gamma_{Q_I}(\alpha,\lambda)+\frac{1}{2}\gamma_{L_I}(\alpha,\lambda)\Big)+\frac{1}{2}\gamma_{H_u}(\alpha,\lambda)+\frac{1}{2}\gamma_{H_d}(\alpha,\lambda)\Big];$$

$$\frac{\beta_1(\alpha,\lambda)}{\alpha_1^2} = -\frac{3}{5}\cdot\frac{1}{2\pi}\Big[-11+\sum_{I=1}^{3}\Big(\frac{1}{6}\gamma_{Q_I}(\alpha,\lambda)+\frac{4}{3}\gamma_{U_I}(\alpha,\lambda)+\frac{1}{3}\gamma_{D_I}(\alpha,\lambda)+\frac{1}{2}\gamma_{L_I}(\alpha,\lambda)+\gamma_{E_I}(\alpha,\lambda)\Big)+\frac{1}{2}\gamma_{H_u}(\alpha,\lambda)+\frac{1}{2}\gamma_{H_d}(\alpha,\lambda)\Big]$$

and correctly reproduce the (scheme-independent) two-loop contributions.

The exact NSVZ β-functions for the flipped $SU(5)$ model

As another example we consider the flipped $SU(5)$ Grand Unification Theory.

S. M. Barr, Phys. Lett. B **112** (1982), 219; I. Antoniadis, J. R. Ellis, J. S. Hagelin and D. V. Nanopoulos, Phys. Lett. B **194** (1987), 231.

The quark and lepton superfields belong to the representation $3 \times \big(\overline{10}(1) + 5(-3) + 1(5)\big)$ of the gauge group $SU(5) \times U(1)$. Also the theory includes Higgs superfields H and $\widetilde{H}$ in $10(-1)$ and $\overline{10}(1)$; h and $\widetilde{h}$ in $5(2)$ and $\bar{5}(-2)$, and four singlets ϕ. The exact NSVZ β-functions for this model are

$$\frac{\beta_5(\alpha,\lambda)}{\alpha_5^2} = -\frac{1}{2\pi(1-5\alpha_5/2\pi)}\Big[5 + \sum_{I=1}^{3}\Big(\frac{3}{2}\gamma_{\overline{10}_I}(\alpha,\lambda) + \frac{1}{2}\gamma_{5_I}(\alpha,\lambda)\Big)$$

$$+\frac{3}{2}\gamma_H(\alpha,\lambda) + \frac{3}{2}\gamma_{\widetilde{H}}(\alpha,\lambda) + \frac{1}{2}\gamma_h(\alpha,\lambda) + \frac{1}{2}\gamma_{\widetilde{h}}(\alpha,\lambda)\Big];$$

$$\frac{\beta_1(\alpha,\lambda)}{\alpha_1^2} = \frac{1}{8}\cdot\frac{1}{2\pi}\Big[60 - \sum_{I=1}^{3}\Big(2\gamma_{\overline{10}_I}(\alpha,\lambda) + 9\gamma_{5_I}(\alpha,\lambda) + 5\gamma_{E_I}(\alpha,\lambda)\Big)$$

$$-2\gamma_H(\alpha,\lambda) - 2\gamma_{\widetilde{H}}(\alpha,\lambda) - 4\gamma_h(\alpha,\lambda) - 4\gamma_{\widetilde{h}}(\alpha,\lambda)\Big].$$

D.Korneev, D.Plotnikov, K.S., N.Tereshina, JHEP **10** (2021) 046.

The problem of constructing an NSVZ scheme

The NSVZ relation holds only in some special renormalization schemes, which are usually called "NSVZ schemes", and the $\overline{\text{DR}}$-scheme (i.e. dimensional reduction supplemented by modified minimal subtraction) is not NSVZ.

L.V.Avdeev, O.V.Tarasov, Phys.Lett. **112 B** (1982) 356; I.Jack, D.R.T.Jones, C.G.North, Phys.Lett **B386** (1996) 138; Nucl.Phys. **B 486** (1997) 479; R.V.Harlander, D.R.T.Jones, P.Kant, L.Mihaila, M.Steinhauser, JHEP **0612** (2006) 024.

However, the all-loop prescription giving some NSVZ schemes can be constructed with the help of the higher covariant derivative regularization

A.A.Slavnov, Nucl.Phys. **B31**, (1971), 301; Theor.Math.Phys. **13** (1972) 1064.

which, by construction, includes the insertion of the Pauli–Villars determinants for removing residual one-loop divergences

A.A.Slavnov, Theor.Math.Phys. **33**, (1977), 977.

It is self-consistent and can be formulated in a manifestly supersymmetric way

V.K.Krivoshchekov, Theor.Math.Phys. **36** (1978) 745; P.West, Nucl.Phys. B268, (1986), 113.

Renormalizable non-Abelian $\mathcal{N}=1$ supersymmetric gauge theories with matter superfields at the classical level are described by the action

$$S = \frac{1}{2e_0^2}\,\mbox{Re}\,\mbox{tr}\int d^4x\, d^2\theta\, W^a W_a + \frac{1}{4}\int d^4x\, d^4\theta\, \phi^{*i}(e^{2V})_i{}^j\phi_j$$
$$+\Big\{\int d^4x\, d^2\theta\,\Big(\frac{1}{4}m_0^{ij}\phi_i\phi_j + \frac{1}{6}\lambda_0^{ijk}\phi_i\phi_j\phi_k\Big) + \mbox{c.c.}\Big\}.$$

For quantizing the theory it is convenient to use the background field method. It is introduced by the replacement $e^{2V}\to e^{2\mathcal{F}(V)}e^{2\boldsymbol{V}}$, where the function $\mathcal{F}(V)$ includes an infinite set of parameters needed for describing the nonlinear renormalization of the quantum gauge superfield

O. Piguet and K. Sibold, Nucl.Phys. **B197** (1982) 257; 272;
I.V.Tyutin, Yad.Fiz. **37** (1983) 761.

To construct the regularized theory, we first add to the action terms with higher covariant derivatives

$$\nabla_a = D_a;\qquad \bar\nabla_{\dot a} = e^{2\mathcal{F}(V)}e^{2\boldsymbol{V}}\bar D_{\dot a}e^{-2\boldsymbol{V}}e^{-2\mathcal{F}(V)}.$$

Then the regularized action takes the form

$$S_{\text{reg}} = \frac{1}{2e_0^2}\,\text{Re}\,\text{tr}\int d^4x\,d^2\theta\,W^a\left(e^{-2\boldsymbol{V}}e^{-2\mathcal{F}(V)}\right)_{Adj}R\Big(-\frac{\bar\nabla^2\nabla^2}{16\Lambda^2}\Big)_{Adj}$$
$$\times\left(e^{2\mathcal{F}(V)}e^{2\boldsymbol{V}}\right)_{Adj}W_a + \frac{1}{4}\int d^4x\,d^4\theta\,\phi^{*i}\Big[F\Big(-\frac{\bar\nabla^2\nabla^2}{16\Lambda^2}\Big)e^{2\mathcal{F}(V)}e^{2\boldsymbol{V}}\Big]_i{}^j\phi_j$$
$$+\Big[\int d^4x\,d^2\theta\left(\frac{1}{4}m_0^{ij}\phi_i\phi_j + \frac{1}{6}\lambda_0^{ijk}\phi_i\phi_j\phi_k\right) + \text{c.c.}\Big],$$

where the regulator functions $R(x)$ and $F(x)$ should rapidly increase at infinity and satisfy the condition $R(0) = F(0) = 1$. Also it is necessary to fix a gauge and add the Faddeev–Popov and Nielsen–Kallosh ghosts.

For regularizing residual one-loop divergences and subdivergences we insert into the generating functional the Pauli–Villars determinants,

$$Z = \int D\mu\,\text{Det}(PV, M_\varphi)^{-1}\text{Det}(PV, M)^{T(R)/T(R_{\text{PV}})}$$
$$\times\exp\Big\{i\Big(S_{\text{reg}} + S_{\text{gf}} + S_{\text{FP}} + S_{\text{NK}} + S_{\text{sources}}\Big)\Big\},$$

where $M_\varphi = a_\varphi\Lambda$ and $M = a\Lambda$.

It is important to distinguish renormalization group functions (RGFs) defined in terms of the bare couplings α_0 and λ_0,

$$\beta(\alpha_0,\lambda_0) \equiv \frac{d\alpha_0}{d\ln\Lambda}\Big|_{\alpha,\lambda=\text{const}}; \qquad \gamma_x(\alpha_0,\lambda_0) \equiv -\frac{d\ln Z_x}{d\ln\Lambda}\Big|_{\alpha,\lambda=\text{const}},$$

and RGFs standardly defined in terms of the renormalized couplings α and λ,

$$\widetilde{\beta}(\alpha,\lambda) \equiv \frac{d\alpha}{d\ln\mu}\Big|_{\alpha_0,\lambda_0=\text{const}}; \qquad \widetilde{\gamma}_x(\alpha,\lambda) \equiv \frac{d\ln Z_x}{d\ln\mu}\Big|_{\alpha_0,\lambda_0=\text{const}}.$$

A.L.Kataev and K.S., Nucl.Phys. **B875** (2013) 459.

RGFs defined in terms of the bare couplings do not depend on a renormalization prescription for a fixed regularization, but depend on a regularization.

RGFs defined in terms of the renormalized couplings depend both on regularization and on a renormalization prescription.

The HD+MSL scheme. The main statements.

Both definitions of RGFs give the same functions in the HD+MSL-scheme, when a theory is regularized by Higher Derivatives, and divergences are removed by Minimal Subtractions of Logarithms. This means that the renormalization constants include only powers of $\ln\Lambda/\mu$, where μ is a renormalization point.

$$\widetilde{\beta}(\alpha,\lambda)\Big|_{\text{HD+MSL}} = \beta(\alpha_0\to\alpha,\lambda_0\to\lambda);$$
$$\widetilde{\gamma}_x(\alpha,\lambda)\Big|_{\text{HD+MSL}} = \gamma_x(\alpha_0\to\alpha,\lambda_0\to\lambda).$$

Below we will briefly describe the proof of the following statements:

1. The NSVZ equation is valid for RGFs defined in terms of the bare couplings in the case of using the higher covariant derivative regularization for an arbitrary renormalization prescription.

2. For RGFs defined in terms of the renormalized couplings some NSVZ schemes are given by the HD+MSL prescription. (MSL can supplement various versions of the higher covariant derivative regularization.)

The all-loop derivation of the NSVZ equation: the main steps

1. First, it is necessary to prove the ultraviolet finiteness of the triple vertices with two external lines of the Faddeev–Popov ghosts and one external line of the quantum gauge superfield.
2. Next, it is necessary to rewrite the NSVZ relation in the equivalent form

$$\frac{\beta(\alpha_0,\lambda_0)}{\alpha_0^2} = -\frac{1}{2\pi}\Big(3C_2 - T(R) - 2C_2\gamma_c(\alpha_0,\lambda_0) - 2C_2\gamma_V(\alpha_0,\lambda_0) + C(R)_i{}^j(\gamma_\phi)_j{}^i(\alpha_0,\lambda_0)/r\Big).$$

K.S., Nucl.Phys. **B909** (2016) 316.

3. After this we prove that the β-function $\beta(\alpha_0,\lambda_0)$ is determined by integrals of double total derivatives with respect to loop momenta and present a method for constructing these integrals.

K.S., JHEP **10** (2019) 011.

4. Then the NSVZ equation for RGFs defined in terms of the bare couplings is obtained by summing singular contributions.
5. Finally, an NSVZ scheme for the function $\widetilde{\beta}(\alpha,\lambda)$ is constructed.

K.S., Eur.Phys.J. **C80** (2020) 10, 911.

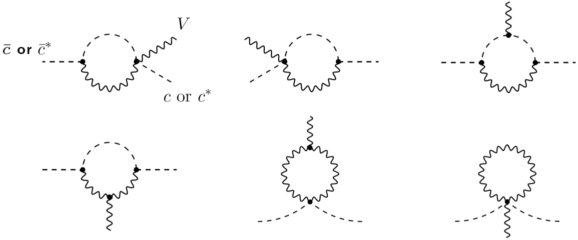
Non-renormalization of the three-point gauge-ghost vertices

The all-order finiteness of the triple vertices in which two external lines correspond to the Faddeev–Popov ghosts and one external line corresponds to the quantum gauge superfield has been proved in the paper

K.S., Nucl.Phys. **B909** (2016) 316.

using the Slavnov–Taylor identities and rules for calculating supergraphs. The result is valid for the superfield formulation of the theory in the general ξ-gauge.

The one-loop contribution to these vertices comes from the superdiagrams presented below. The ultraviolet finiteness of their sum has been verified by an explicit calculation



Non-renormalization of the triple gauge-ghost vertices and the new form of the NSVZ β-function

There are 4 vertices of the considered structure, $\bar c\, V c$, $\bar c^+ V c$, $\bar c\, V c^+$, and $\bar c^+ V c^+$. All of them have the same renormalization constant $Z_\alpha^{-1/2} Z_c Z_V$. Therefore, due to their finiteness

$$\frac{d}{d\ln\Lambda}(Z_\alpha^{-1/2} Z_c Z_V) = 0,$$

where

$$\frac{1}{\alpha_0} = \frac{Z_\alpha}{\alpha}; \qquad \boldsymbol{V} = \boldsymbol{V}_R; \qquad V = Z_V Z_\alpha^{-1/2} V_R; \qquad \bar c c = Z_c Z_\alpha^{-1} \bar c_R c_R.$$

Using this relation after some simple transformations the NSVZ equation can be presented in the equivalent form

$$\frac{\beta(\alpha_0,\lambda_0)}{\alpha_0^2} = -\frac{1}{2\pi}\Big(3C_2 - T(R) - 2C_2\gamma_c(\alpha_0,\lambda_0) - 2C_2\gamma_V(\alpha_0,\lambda_0) + C(R)_i{}^j(\gamma_\phi)_j{}^i(\alpha_0,\lambda_0)/r\Big).$$

It relates the β-function in a certain loop to the anomalous dimensions of quantum superfields in the previous loop, because the right hand side does not contain a denominator depending on couplings.

The β-function of $\mathcal{N}=1$ supersymmetric gauge theories as an integral of double total derivatives

A key observation needed for derivation of the NSVZ relation is that in the case of using the higher covariant derivative regularization the integrals giving the β-function defined in terms of the bare couplings are integrals of double total derivatives in $\mathcal{N}=1$ supersymmetric gauge theories. This was first noted in

> A.A.Soloshenko, K.S., ArXiv: hep-th/0304083v1 (the factorization into total derivatives);
> A.V.Smilga, A.I.Vainshtein, Nucl.Phys. **B 704** (2005) 445 (the factorization into double total derivatives).

The all-loop proof of this statement in the non-Abelian case has been done in

> K.S., JHEP **10** (2019) 011.

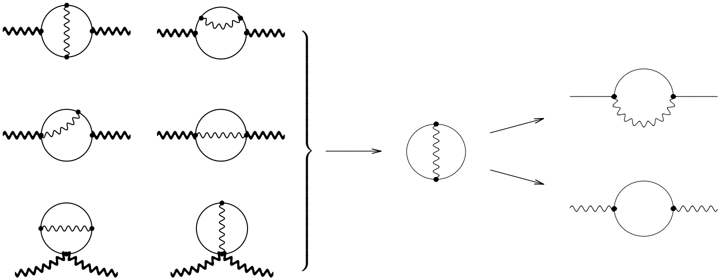
The integrals of double total derivatives do not vanish due to singularities of the integrands. Really, if $f(Q^2)$ is a non-singular function which rapidly decreases at infinity, then

$$\int \frac{d^4Q}{(2\pi)^4} \frac{\partial^2}{\partial Q^\mu \partial Q_\mu} \Big(\frac{f(Q^2)}{Q^2}\Big) = \frac{1}{4\pi^2} f(0) \neq 0.$$

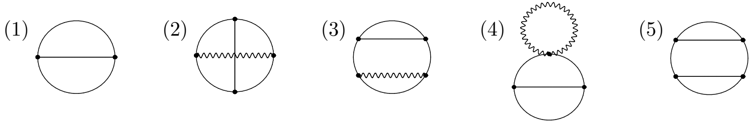
Graphical interpretation of the new form of the NSVZ relation

Due to similar equations the double total derivatives effectively cut internal lines. As a result, we obtain diagrams contributing to various anomalous dimensions of the quantum superfields, in which a number of loops is less by 1. This allows to give a simple qualitative interpretation of the new form of the NSVZ equation:

For each vacuum supergraph the NSVZ equation relates a contribution to the β-function obtained by attaching two external lines of the background gauge superfield to the corresponding contribution to the anomalous dimension of quantum superfields obtained by all various cuts of internal lines:



The two- and three-loop contributions to the β-function which depend on the Yukawa couplings are generated by the vacuum supergraphs



Here we write down the contributions of the supergraphs (1) and (5) which determine the three-loop part of the β-function quartic in the Yukawa couplings

V.Yu.Shakhmanov, K.S., Nucl.Phys., **B920**, (2017), 345;
A.E.Kazantsev, V.Yu.Shakhmanov, K.S., JHEP 1804 (2018) 130.

$$\frac{\Delta\beta(\alpha_0,\lambda_0)}{\alpha_0^2} = -\frac{2\pi}{r}C(R)_i{}^j\frac{d}{d\ln\Lambda}\int\frac{d^4K}{(2\pi)^4}\frac{d^4Q}{(2\pi)^4}\,\lambda_0^{imn}\lambda^*_{0jmn}\frac{\partial}{\partial Q_\mu}\frac{\partial}{\partial Q^\mu}\Big(\frac{1}{K^2}$$
$$\times\frac{1}{F_K\,Q^2F_Q\,(Q+K)^2F_{Q+K}}\Big) + \frac{4\pi}{r}C(R)_i{}^j\frac{d}{d\ln\Lambda}\int\frac{d^4K}{(2\pi)^4}\frac{d^4L}{(2\pi)^4}\frac{d^4Q}{(2\pi)^4}\Big[\lambda_0^{iab}$$
$$\times\lambda^*_{0kab}\lambda_0^{kcd}\lambda^*_{0jcd}\Big(\frac{\partial}{\partial K_\mu}\frac{\partial}{\partial K^\mu}-\frac{\partial}{\partial Q_\mu}\frac{\partial}{\partial Q^\mu}\Big)+2\lambda_0^{iab}\lambda^*_{0jac}\lambda_0^{cde}\lambda^*_{0bde}\frac{\partial}{\partial Q_\mu}\frac{\partial}{\partial Q^\mu}\Big]$$
$$\times\frac{1}{K^2F_K^2\,Q^2F_Q\,(Q+K)^2F_{Q+K}\,L^2F_L\,(L+K)^2F_{L+K}} = -\frac{1}{2\pi r}C(R)_i{}^j(\Delta\gamma_\phi)_j{}^i.$$

The result for these supergraphs (as well for the other ones presented in the previous slide) exactly agrees with the qualitative interpretation of the NSVZ equation.

Thus, all higher order corrections to the β-function turn out to be determined by singular contributions which appear in calculating the integrals of double total derivatives. In all orders of the perturbation theory sums of singular contributions which come from cuts of various propagators have been calculated in

K.S., Eur.Phys.J. **C80** (2020) 10, 911.

The result of this calculation can be written as

$$\frac{\beta(\alpha_0,\lambda_0)}{\alpha_0^2}-\frac{\beta_{\text{1-loop}}(\alpha_0)}{\alpha_0^2}$$
$$=\frac{1}{\pi}C_2\gamma_V(\alpha_0,\lambda_0)+\frac{1}{\pi}C_2\gamma_c(\alpha_0,\lambda_0)-\frac{1}{2\pi r}C(R)_i{}^j(\gamma_\phi)_j{}^i(\alpha_0,\lambda_0).$$

gauge propagators

matter propagators

Faddeev–Popov ghost propagators

Conditions required for the validity of the NSVZ relation in all loops

Substituting the expression for the one-loop β-function we obtain the main result:

The NSVZ relation

$$\frac{\beta(\alpha_0,\lambda_0)}{\alpha_0^2} = -\frac{1}{2\pi}\Big(3C_2 - T(R) - 2C_2\gamma_c(\alpha_0,\lambda_0) -2C_2\gamma_V(\alpha_0,\lambda_0) + C(R)_i{}^j(\gamma_\phi)_j{}^i(\alpha_0,\lambda_0)/r\Big),$$

and, therefore, the NSVZ relation

$$\beta(\alpha_0,\lambda_0) = -\frac{\alpha_0^2\Big(3C_2 - T(R) + C(R)_i{}^j(\gamma_\phi)_j{}^i(\alpha_0,\lambda_0)/r\Big)}{2\pi(1-C_2\alpha_0/2\pi)}$$

are valid in all orders of the perturbation theory for RGFs defined in terms of the bare couplings if a theory is regularized by higher covariant derivatives.

Consequently, for RGFs defined in terms of the renormalized couplings, similar equations hold in the HD+MSL scheme in all orders of the perturbation theory.

Some applications

Now it is clear how one can construct NSVZ schemes and, therefore, we can find a gauge β-function in a certain loop by calculating only an anomalous dimension of the matter superfields in the previous loops. For example, the three-loop β-function for $\mathcal{N}=1$ supersymmetric theories regularized by higher derivatives in an arbitrary subtraction scheme has been calculated in

A.E.Kazantsev, K.S., JHEP **2006** (2020) 108.

starting from the expression for the two-loop anomalous dimension of the matter superfields. Certainly, in general, the NSVZ equation in this approximation is not satisfied.

Here (at the next slide) we only present the result for one particular case, namely, for one-loop finite $\mathcal{N}=1$ supersymmetric theories, see

P.West, Phys.Lett. **B 137** (1984) 371;
A.Parkes, P.West, Phys.Lett. **B 138** (1984) 99.

These theories satisfy the conditions

$$T(R)=3C_2; \qquad \lambda^*_{imn}\lambda^{jmn}=4\pi\alpha C(R)_i{}^j.$$

In this case the two-loop anomalous dimension and the three-loop β-function defined in terms of the renormalized couplings have the form

$$(\widetilde{\gamma}_\phi)_i{}^j(\alpha,\lambda) = -\frac{3\alpha^2}{2\pi^2} C_2 C(R)_i{}^j \Big(\ln\frac{a_\varphi}{a} - b_{11} + b_{12}\Big) - \frac{\alpha}{4\pi^2}\Big(\frac{1}{\pi}\lambda^*_{imn}\lambda^{jml} C(R)_l{}^n$$
$$+2\alpha\left[C(R)^2\right]_i{}^j\Big)\Big(A - B - 2g_{12} + 2g_{11}\Big) + O\Big(\alpha^3,\alpha^2\lambda^2,\alpha\lambda^4,\lambda^6\Big);$$
$$\frac{\widetilde{\beta}(\alpha,\lambda)}{\alpha^2} = \frac{3\alpha^2}{4\pi^3 r} C_2\, \text{tr}\left[C(R)^2\right] \Big(\ln\frac{a_\varphi}{a} - b_{11} + b_{12}\Big) + \frac{\alpha}{8\pi^3 r}\Big(\frac{1}{\pi} C(R)_j{}^i C(R)_l{}^n$$
$$\times\lambda^*_{imn}\lambda^{jml} + 2\alpha\, \text{tr}\left[C(R)^3\right]\Big)\Big(A - B - 2g_{12} + 2g_{11}\Big) + O\Big(\alpha^3,\alpha^2\lambda^2,\alpha\lambda^4,\lambda^6\Big).$$

where

$$A = \int\limits_0^\infty dx\,\ln x\,\frac{d}{dx}\frac{1}{R(x)};\qquad B = \int\limits_0^\infty dx\,\ln x\,\frac{d}{dx}\frac{1}{F^2(x)}\qquad a = \frac{M}{\Lambda};\qquad a_\varphi = \frac{M_\varphi}{\Lambda}.$$

We see that in this case the NSVZ equation is satisfied in the lowest nontrivial approximation for an arbitrary renormalization prescription,

$$\frac{\beta(\alpha,\lambda)}{\alpha^2} = -\frac{1}{2\pi r} C(R)_i{}^j (\gamma_\phi)_j{}^i(\alpha,\lambda) + O(\alpha^3,\alpha^2\lambda^2,\alpha\lambda^4,\lambda^6).$$

The NSVZ equation for theories finite in the lowest loops

For $\mathcal{N} = 1$ supersymmetric theories finite in the one-loop approximation it is possible to tune a subtraction scheme so that the theory will be all-loop finite

D.I.Kazakov, Phys. Lett. B **179** (1986) 352; A.V.Ermushev, D.I.Kazakov, O.V.Tarasov, Nucl.Phys. B **281** (1987) 72; C.Lucchesi, O.Piguet, K.Sibold, Helv.Phys.Acta **61** (1988) 321; Phys.Lett. B **201** (1988) 241.

If a subtraction scheme is tuned in such a way that the gauge β-function vanishes in the first L-loops, while the anomalous dimension of the matter superifelds and the Yukawa β-function vanish in the first $(L-1)$ loops, then

K.S., Eur.Phys.J. C **81** (2021) 571.

for an arbitrary renormalization prescription the $(L+1)$-loop gauge and L-loop Yukawa β-functions satisfy the equations

$$\frac{\beta_{L+1}(\alpha,\lambda)}{\alpha^2} = -\frac{1}{2\pi r} C(R)_i{}^j (\gamma_{\phi,L})_j{}^i(\alpha,\lambda);$$
$$(\beta_{\lambda,L})^{ijk}(\alpha,\lambda) = \frac{3}{2}(\gamma_{\phi,L})_m{}^{(i}(\alpha,\lambda)\lambda^{ij)m}.$$

Conclusion

- In the case of using the regularization by higher covariant derivatives RGFs defined in terms of the bare couplings satisfy the NSVZ equation in all orders for any renormalization prescription.
- RGFs defined in terms of the renormalized couplings satisfy the NSVZ equation in the HD+MSL scheme, when a theory is regularized by higher covariant derivatives, and divergences are removed by minimal subtractions of logarithms.
- The β-function of $\mathcal{N}=1$ supersymmetric gauge theories is determined by integrals of double total derivatives in the momentum space if a theory is regularized by higher covariant derivatives.
- The triple gauge-ghost vertices are UV finite in all orders. This allows to rewrite the NSVZ relation in an equivalent form, which relates the β-function to the anomalous dimensions of the quantum superfields.
- For one-loop finite theories the exact expressions for the gauge and Yukawa β-functions are valid in the first non-trivial approximation in an arbitrary subtraction scheme.

Thank you for the attention!