THREE-LOOP SCATTERING AMPLITUDES IN QCD

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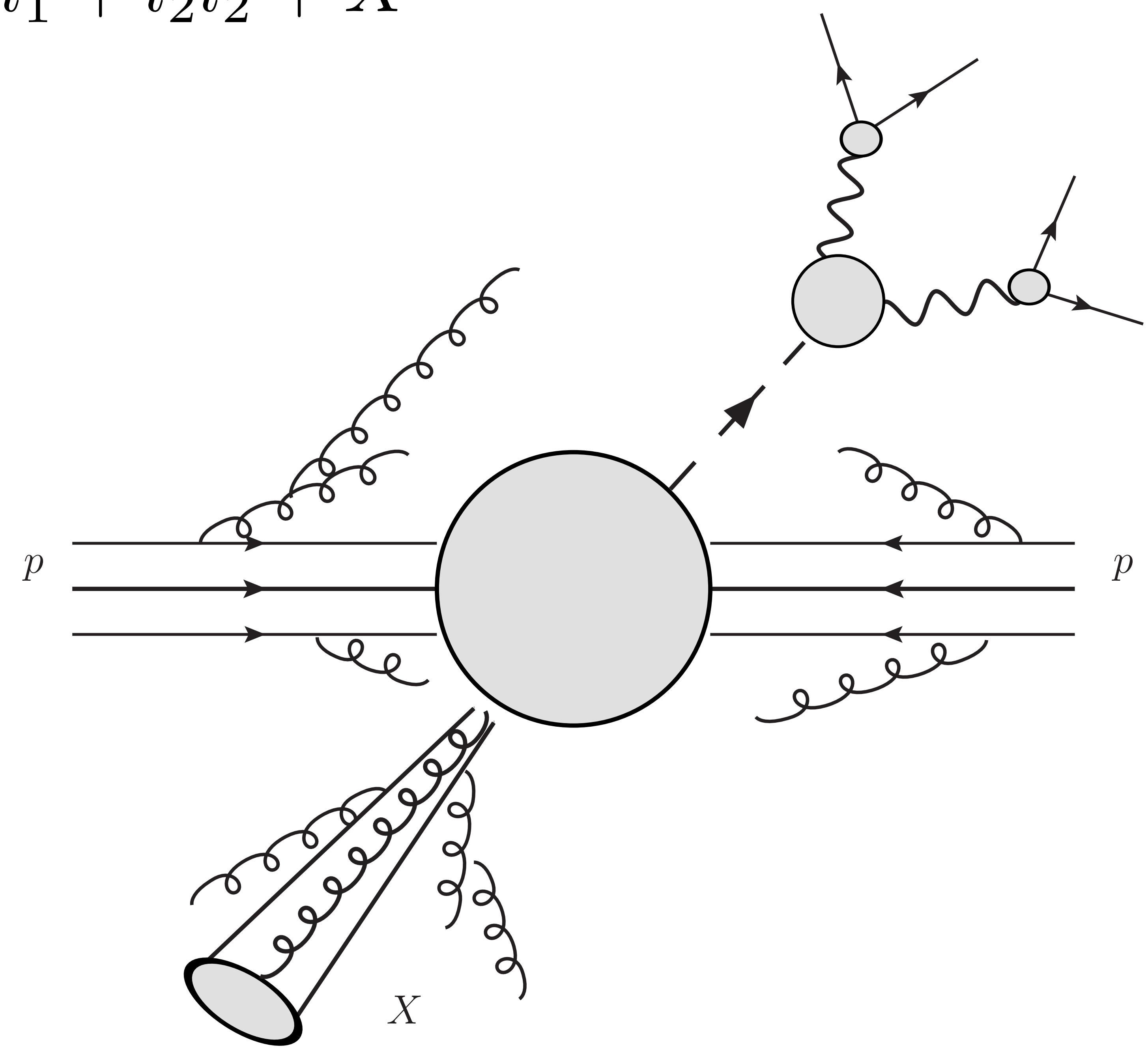
Congratulations!

And thank you, in particular **V. Smirnov** and **K. Chetyrkin,** to for paving the way for much of the research I am doing today…!

LOOPS ~ PRECISION PHYSICS….: HOW FAR CAN/SHALL WE GO?

PRECISION PHYSICS AT THE LHC: HOW FAR CAN WE GO?

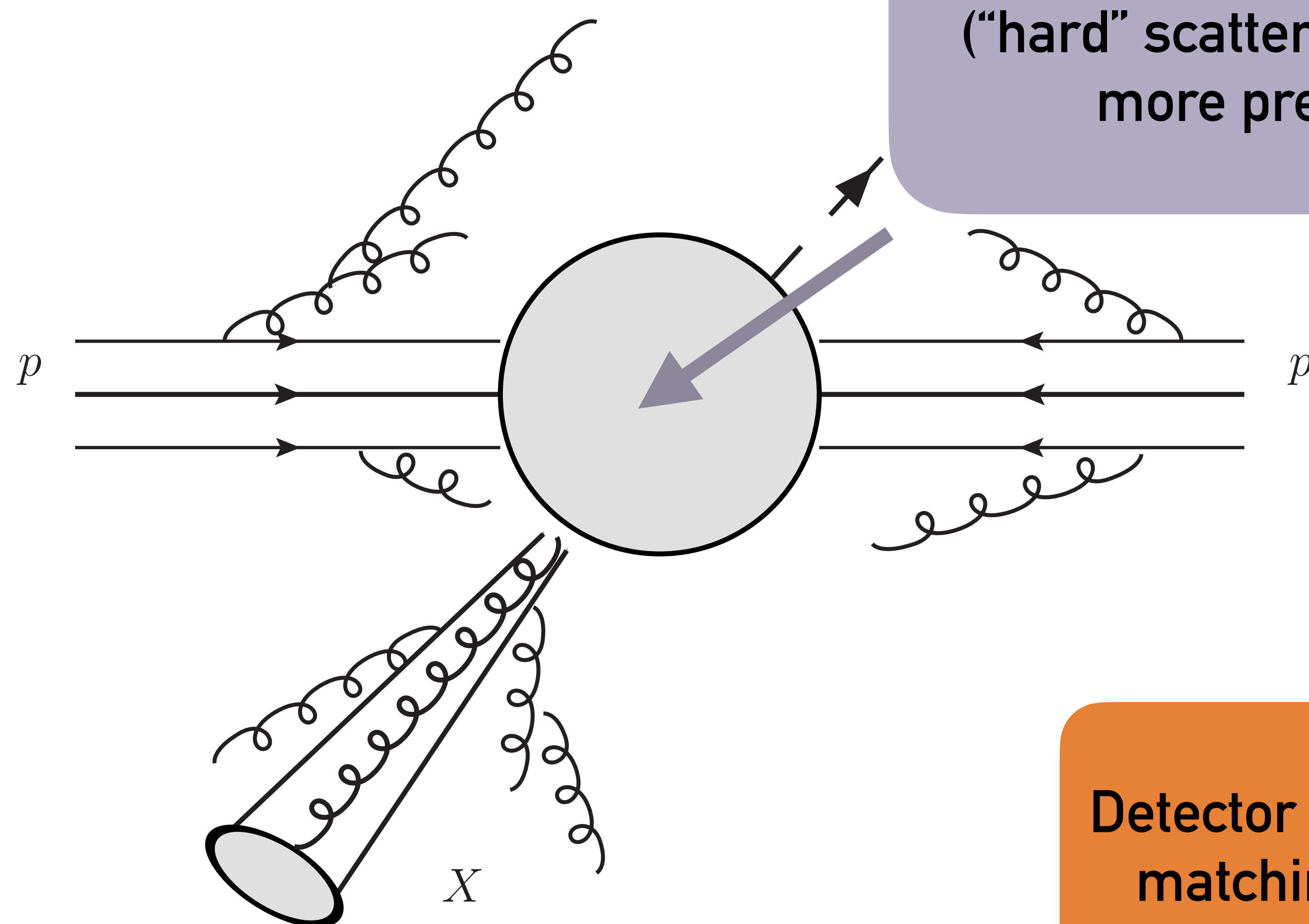
$$pp \to HX \to l_1\bar{l}_1 + l_2\bar{l}_2 + X$$



PRECISION PHYSICS AT THE LHC: HOW FAR CAN WE GO?

$$pp \to HX \to l_1\bar{l}_1 + l_2\bar{l}_2 + X$$

Factorisation theorems, PDFs…?



Loops allow us to compute central part of this picture (“hard” scattering) more and more precisely…

Parton Shower, Hadronisation, Fragmentation …

Detector simulation matching, etc…

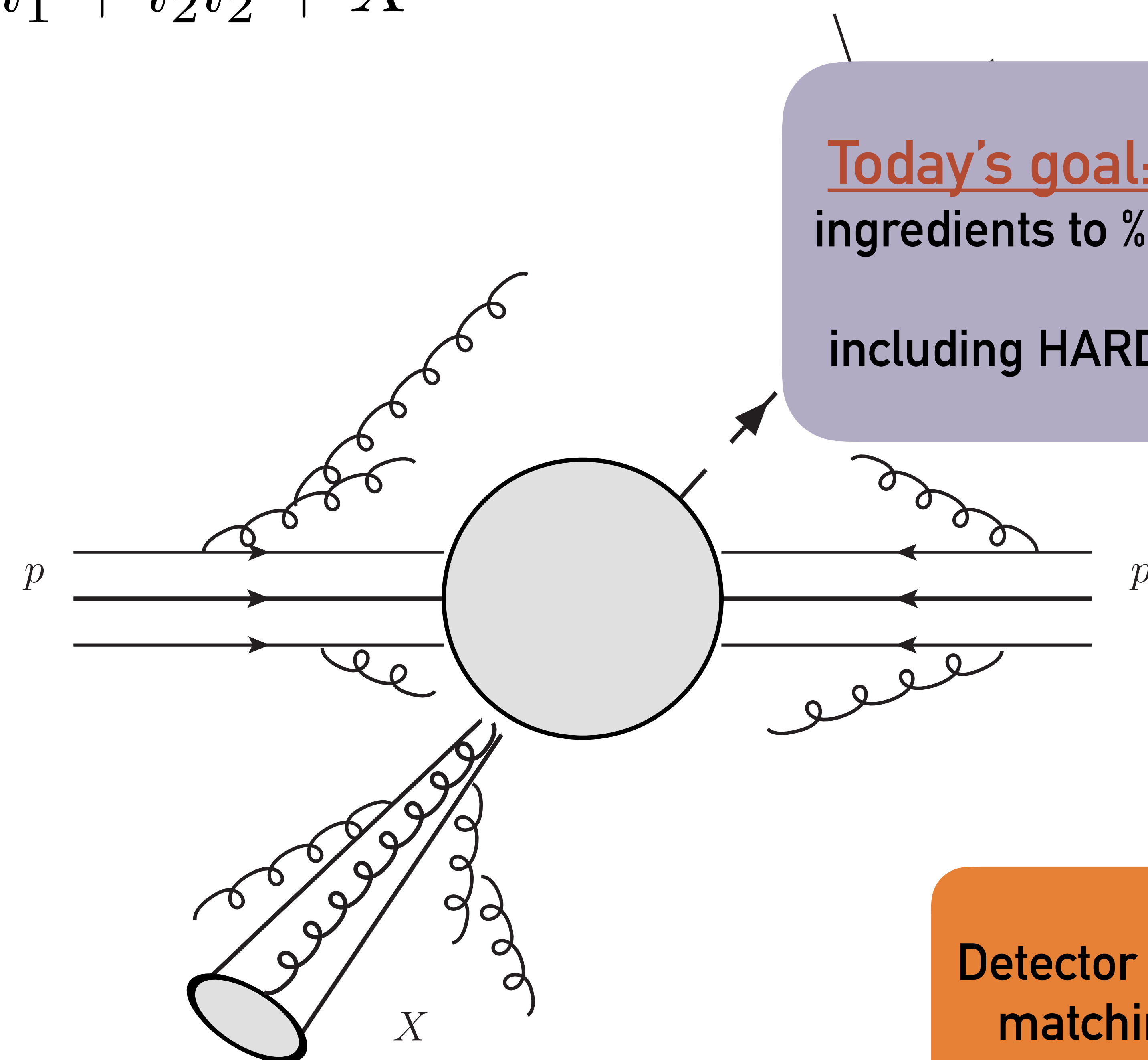
PRECISION PHYSICS AT THE LHC: HOW FAR CAN WE GO?

$$pp \to HX \to l_1\bar{l}_1 + l_2\bar{l}_2 + X$$

Today's goal: push all these ingredients to % level precision!

including HARD SCATTERING!

Factorisation theorems, PDFs…?



Parton Shower, Hadronisation, Fragmentation …

Detector simulation matching, etc…

FIXED ORDER CALCULATIONS

$$\sigma_{q\bar{q}\to gg} = \int [\mathrm{dPS}] \left|\mathcal{M}_{q\bar{q}\to gg}\right|^2$$

$$\left|\mathcal{M}_{q\bar{q}\to gg}\right|^2 = \left|\mathcal{M}^{LO}_{q\bar{q}\to gg}\right|^2 + \left(\frac{\alpha_s}{2\pi}\right)\left|\mathcal{M}^{NLO}_{q\bar{q}\to gg}\right|^2 + \left(\frac{\alpha_s}{2\pi}\right)^2\left|\mathcal{M}^{NNLO}_{q\bar{q}\to gg}\right|^2 + \ldots$$

FIXED ORDER CALCULATIONS

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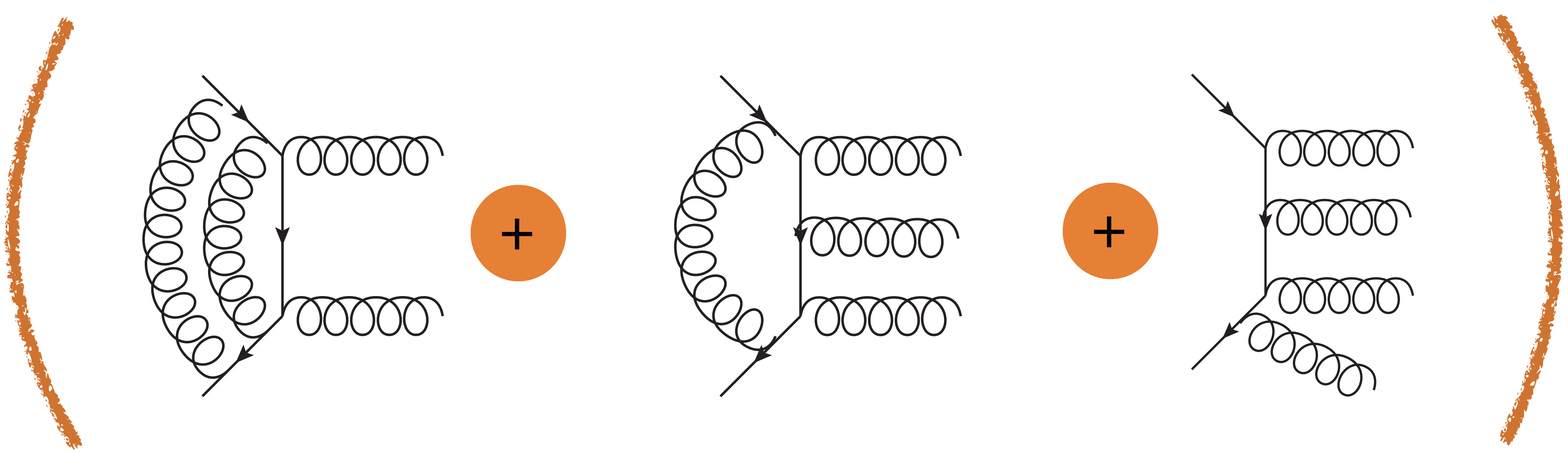
Past two decades have seen impressive effort to reach NNLO for $2 \to \{2,3\}$

FIXED ORDER CALCULATIONS

$$\sigma_{q\bar{q}\to gg} = \int [\mathrm{dPS}]\, |\mathcal{M}_{q\bar{q}\to gg}|^2$$

$$|\mathcal{M}_{q\bar{q}\to gg}|^2 = |\mathcal{M}^{LO}_{q\bar{q}\to gg}|^2 + \left(\frac{\alpha_s}{2\pi}\right) |\mathcal{M}^{NLO}_{q\bar{q}\to gg}|^2 + \left(\frac{\alpha_s}{2\pi}\right)^2 \boxed{|\mathcal{M}^{NNLO}_{q\bar{q}\to gg}|^2} + \ldots$$

Past two decades have seen impressive effort to reach NNLO for $2 \to \{2,3\}$



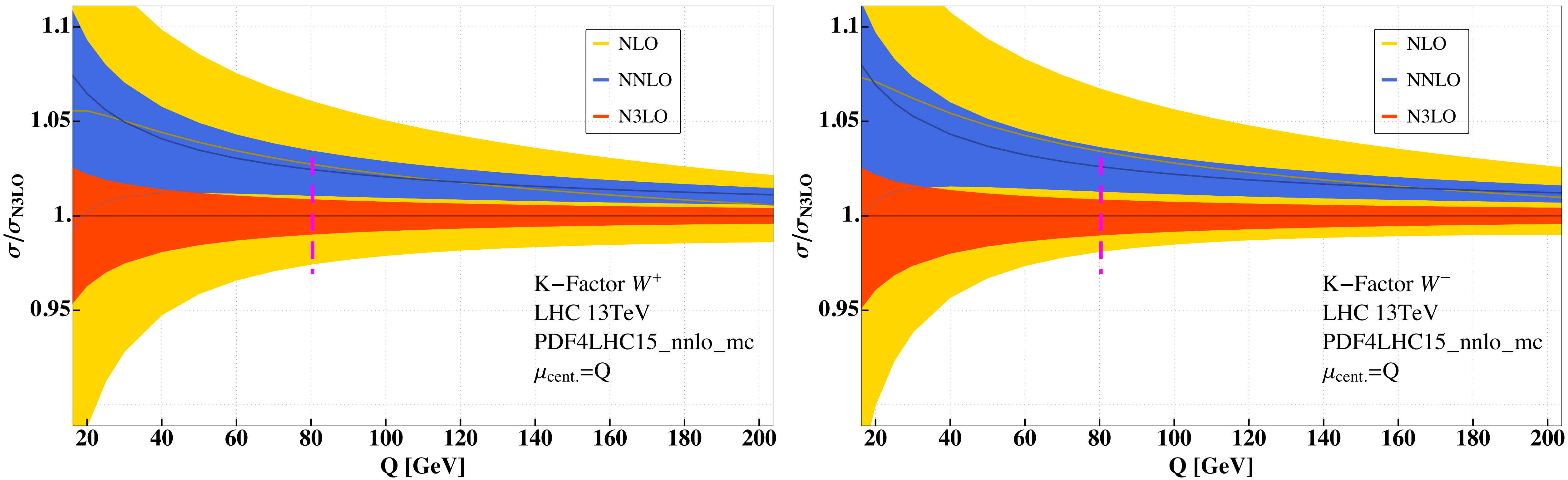
Double Virtual — Real Virtual — Double Real

~ O(5%) precision

- complex integrals
- involved **IR structure**

BEYOND NNLO FOR 2->2 THERE IS STILL A LOT TO LEARN

We are just scratching the surface…!



[Duhr, Dulat, Mistlberger ’20]

Non trivial uncertainty patterns observed going from NNLO to N3LO for W,γ Drell-Yan

We are far from being able to do N^{3}LO pheno for generic processes…

BEYOND NNLO FOR 2->2 THERE IS STILL A LOT TO LEARN

We are just scratching the surface…!

We are far from being able to do N^3LO pheno for generic processes…

New challenges from pushing methods to compute **scattering amplitudes** from two to **three loops**:

Higher combinatorial complexity, *new special functions and new geometries*, discontinuities (bootstrap?)…

IR singularities and new sources for possible **factorisation breaking** (di-jet / $t\bar{t}$ @ N^3LO…)

Particularly interesting

di-jet production @ N3LO!

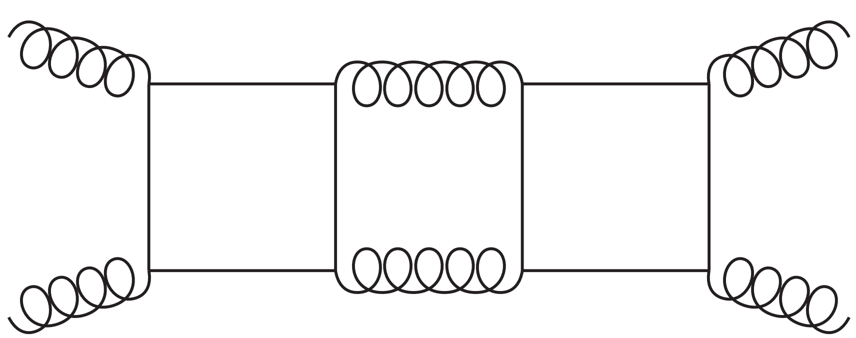
TOWARDS DI-JET AT N3LO

First step is **3 loop scattering amplitudes**:

- Informs on **complexity of functions involved**
- Informs on **IR structure** in three-loop QCD: **quadrupole correlations!**

3 main channels: $gg \to gg\,,\, q\bar{q} \to gg\,,\, q\bar{q} \to Q\bar{Q}$

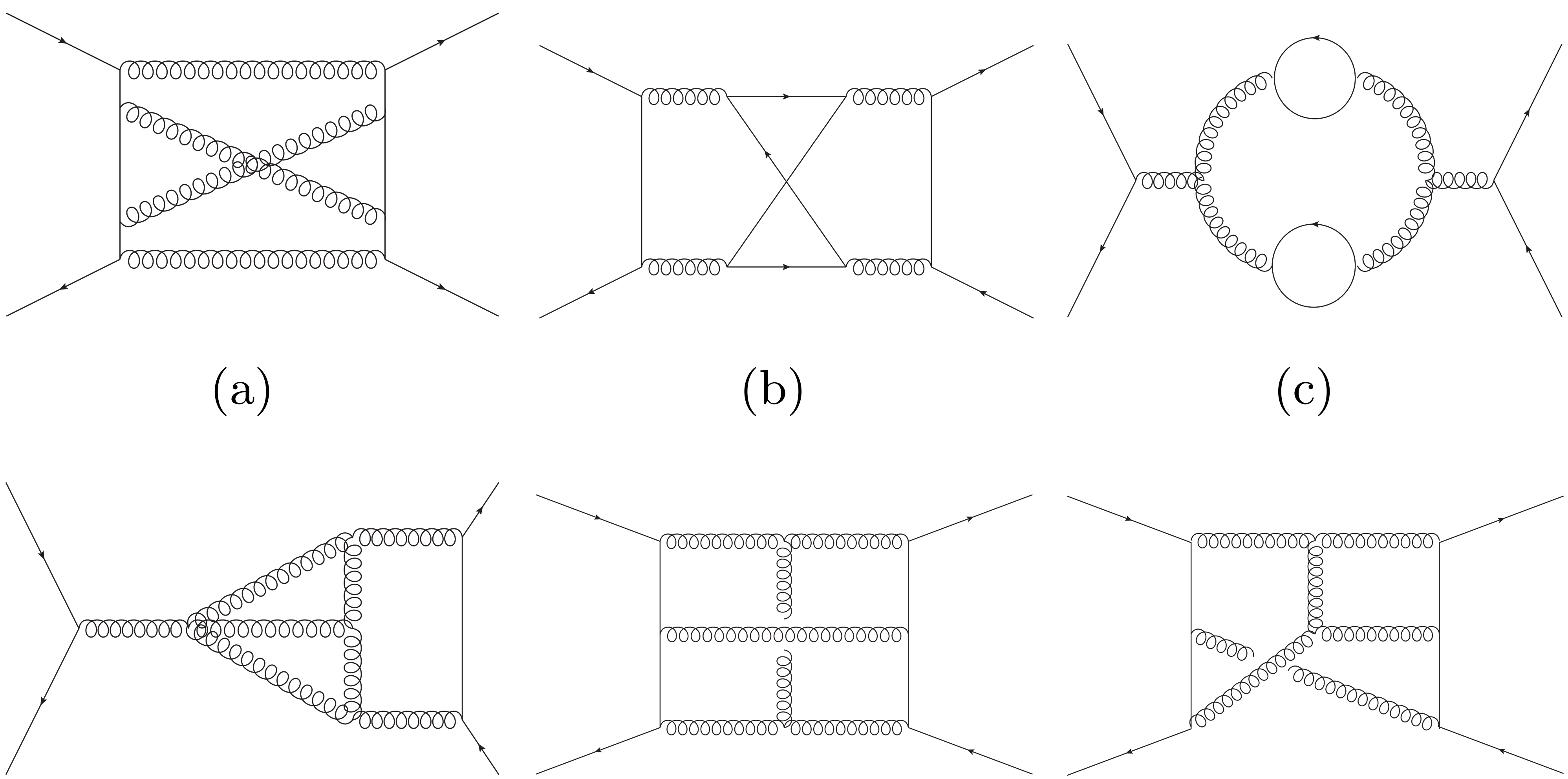
- Number of Feynman diagrams ***explodes***: $gg \to gg$ **@ 3 loops** $\sim$ **50k**



- Each diagram produces ***thousands of terms***, compute $\sim \mathcal{O}(10^6)$ **integrals**

QQQQ SCATTERING UP TO 3 LOOPS

$$q(p_1) + \bar{q}(p_2) \to Q(p_3) + \bar{Q}(p_4)\,, \quad \text{with} \quad p_i^2 = 0$$

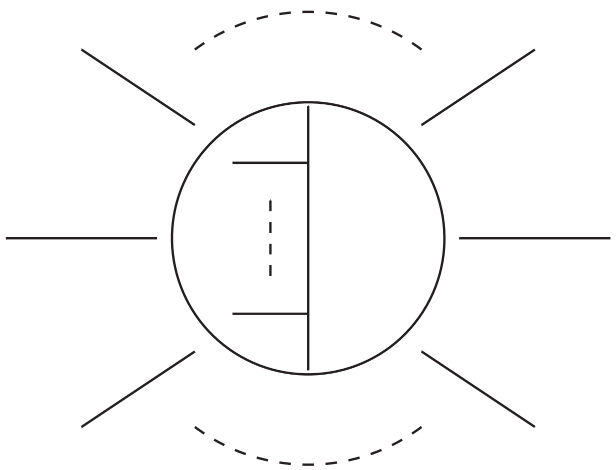


(a) (b) (c)

(d) (e) (f)

MULTILOOP SCATTERING AMPLITUDES: THE STANDARD WAY

One way to go about it: **standard approach** ***(divide et impera)***



$$= \sum_{i=1}^{N} R_i(x_1, \ldots, x_r)\, \mathcal{I}_i(x_1, \ldots, x_n)$$

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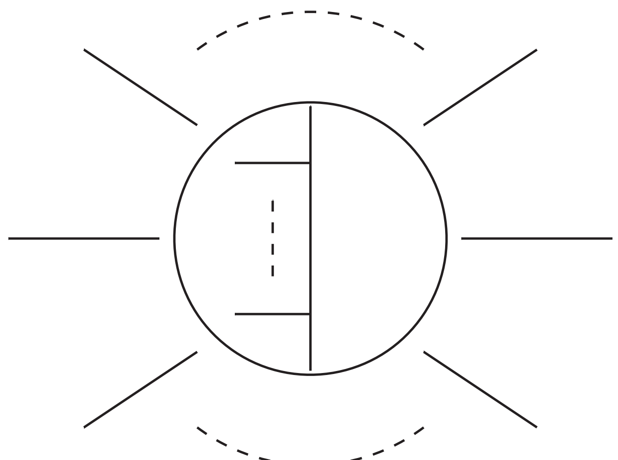

$$= \sum_{i=1}^{N} R_i(x_1,\ldots,x_r)\,\mathcal{J}_i(x_1,\ldots,x_n)$$

Standard steps:

1) Obtain the **integrand** (From Feynman diagrams, Unitarity, …)

2) Reduce this integrand to a **basis** of **master integrals** (IBPs, Finite Fields etc…)

3) Compute the **master integrals** (Diff Equations, Canonical bases, polylogs etc…)

THE HELICITY AMPLITUDES IN 'THV

Helicity amplitudes for $q\bar{q} \to Q\bar{Q}$, start from **generic tensor decomposition in d-dim**

$$\mathscr{A}(s,t) = \sum_{i=1}^{N} \mathscr{F}_i(s,t) D_i$$

$$\begin{aligned}
\mathcal{D}_1 &= \bar{u}(p_1)\gamma_{\mu_1} u(p_2)\ \bar{u}(p_3)\gamma_{\mu_1} u(p_4), \\
\mathcal{D}_2 &= \bar{u}(p_1)\not{p}_3 u(p_2)\ \bar{u}(p_3)\not{p}_1 u(p_4), \\
\mathcal{D}_3 &= \bar{u}(p_1)\gamma_{\mu_1}\gamma_{\mu_2}\gamma_{\mu_3} u(p_2)\ \bar{u}(p_3)\gamma_{\mu_1}\gamma_{\mu_2}\gamma_{\mu_3} u(p_4), \\
\mathcal{D}_4 &= \bar{u}(p_1)\gamma_{\mu_1}\not{p}_3\gamma_{\mu_3} u(p_2)\ \bar{u}(p_3)\gamma_{\mu_1}\not{p}_1\gamma_{\mu_3} u(p_4), \\
\mathcal{D}_5 &= \bar{u}(p_1)\gamma_{\mu_1}\gamma_{\mu_2}\gamma_{\mu_3}\gamma_{\mu_4}\gamma_{\mu_5} u(p_2)\ \bar{u}(p_3)\gamma_{\mu_1}\gamma_{\mu_2}\gamma_{\mu_3}\gamma_{\mu_4}\gamma_{\mu_5} u(p_4), \\
\mathcal{D}_6 &= \bar{u}(p_1)\gamma_{\mu_1}\gamma_{\mu_2}\not{p}_3\gamma_{\mu_4}\gamma_{\mu_5} u(p_2)\ \bar{u}(p_3)\gamma_{\mu_1}\gamma_{\mu_2}\not{p}_1\gamma_{\mu_4}\gamma_{\mu_5} u(p_4).
\end{aligned}$$

- …. when do I stop ? $\to$ in **d-dimensions** it depends on the perturbative order!
- the **γ-algebra in d-dimensions is not closed**

THE HELICITY AMPLITUDES IN 'THV

To compute helicity amplitudes, start from generic tensor decomposition in d-dim

$$\mathcal{A}(s,t) = \sum_{i=1}^{N} \mathcal{F}_i(s,t) D_i$$

Helicity amplitudes in tHV imply external states in d=4

N-2 of N tensors in d-dim are
not independent in d=4

$$\lim_{d\to 4}\left(D_3 - \frac{24}{s_{12}} D_2 + \left(8 - 3d + \frac{12\, s_{13}}{s_{12}}\right) D_1\right) = 0$$

And similarly for others

THE HELICITY AMPLITUDES IN 'THV

Let then pick first 2: $\overline{T}_j = D_j\,,\ \ j = 1{,}2$ Define 2 projectors $\overline{P}_i \cdot \overline{T}_j = \delta_{ij}$

$$M^{2\times 2}_{ij} = T_i^{\dagger} T_j\,, \qquad \overline{P}_i = \sum_{j=1}^{2} \left(M^{(2\times 2)}_{ij}\right)^{-1} \overline{T}_j^{\dagger}$$

$$\left(M^{2\times 2}\right)^{-1}_{ij} = \boxed{\frac{1}{d-3}} X_{ij} \quad \text{with} \quad X_{ij} = \frac{1}{4\,s_{12}^2}\begin{pmatrix} 1 & \dfrac{s_{12}+2s_{23}}{s_{23}(s_{12}+s_{23})} \\ \dfrac{s_{12}+2s_{23}}{s_{23}(s_{12}+s_{23})} & \dfrac{(d-2)s_{12}^2+4s_{23}(s_{12}+s_{23})}{s_{23}^2(s_{12}+s_{23})^2} \end{pmatrix}$$

the matrix is smooth in $d \to 4$

and perform a rotation in the space of the remaining tensors:

$$\overline{T}_i = D_i - \sum_{j=1}^{2} (\overline{P}_j D_i)\, \overline{T}_j\,, \quad i = 3,\ldots,N$$

PROJECTORS IN 'T HOOFT-VELTMAN

In new basis of tensors, **by definition** only first two contribute to hel amplitudes

$$\mathscr{A}_{\lambda_q \lambda_Q}(s,t) = \sum_{i=1}^{N} \overline{\mathscr{F}}_i(s,t) \left[\overline{T}_i\right]_{\lambda_q \lambda_Q, d=4} \quad = \quad \sum_{i=1}^{2} \overline{\mathscr{F}}_i(s,t) \left[\overline{T}_i\right]_{\lambda_q \lambda_Q, d=4} + \mathcal{O}(\epsilon)$$

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Natural basis to derive helicity amplitudes

$$\bar{\boldsymbol{\mathcal{A}}} = \sum_{i=1}^{2} \boldsymbol{\mathcal{F}}_i \, T_i$$

$$T_1 = \bar{u}(p_2)\, \gamma_\alpha \, u(p_1) \times \bar{u}(p_4)\, \gamma^\alpha \, u(p_3)$$

$$T_2 = \bar{u}(p_2)\, \not{p}_3 \, u(p_1) \times \bar{u}(p_4)\, \not{p}_2 \, u(p_3)$$

$$\bar{\boldsymbol{\mathcal{A}}}^{q\bar{q}\to\bar{Q}Q}_{+-+-} = \boldsymbol{\mathcal{H}}_1 \frac{\langle 13 \rangle}{\langle 24 \rangle}, \quad \bar{\boldsymbol{\mathcal{A}}}^{q\bar{q}\to\bar{Q}Q}_{+--+} = \boldsymbol{\mathcal{H}}_2 \frac{\langle 14 \rangle}{\langle 23 \rangle},$$

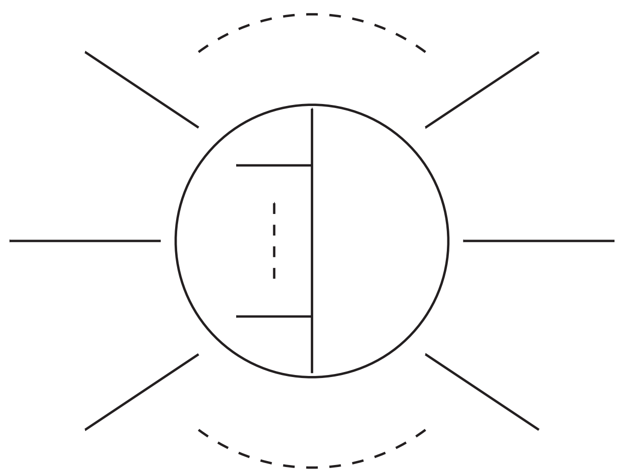
$$\boldsymbol{\mathcal{H}}_1 = 2t\boldsymbol{\mathcal{F}}_1 - tu\boldsymbol{\mathcal{F}}_2, \quad \boldsymbol{\mathcal{H}}_2 = 2u\boldsymbol{\mathcal{F}}_1 + tu\boldsymbol{\mathcal{F}}_2.$$

SIMPLIFYING THE INTEGRAND

Apply this to $q\bar{q} \to Q\bar{Q}$, write scattering amplitudes as $\sim \mathcal{O}(10^5)$ **Feynman integrals!**

Thanks, in particular, to **Chetyrkin** and **Tkachov** we know that these integrals are not independent:

Integration by parts identities augmented by Laporta Algorithm



$$= \sum_{i=1}^{N} R_i(x_1, \ldots, x_r)\, \mathcal{I}_i(x_1, \ldots, x_n)$$

From $\sim 10^5$ down to $\sim 10^2$ independent integrals !

Completed thanks to various (new) techniques:

- extensive use of **symmetry relations** to simplify amplitude before reduction
- finred by *A. von Manteuffel*, **Finite Fields and Syzygy techniques**

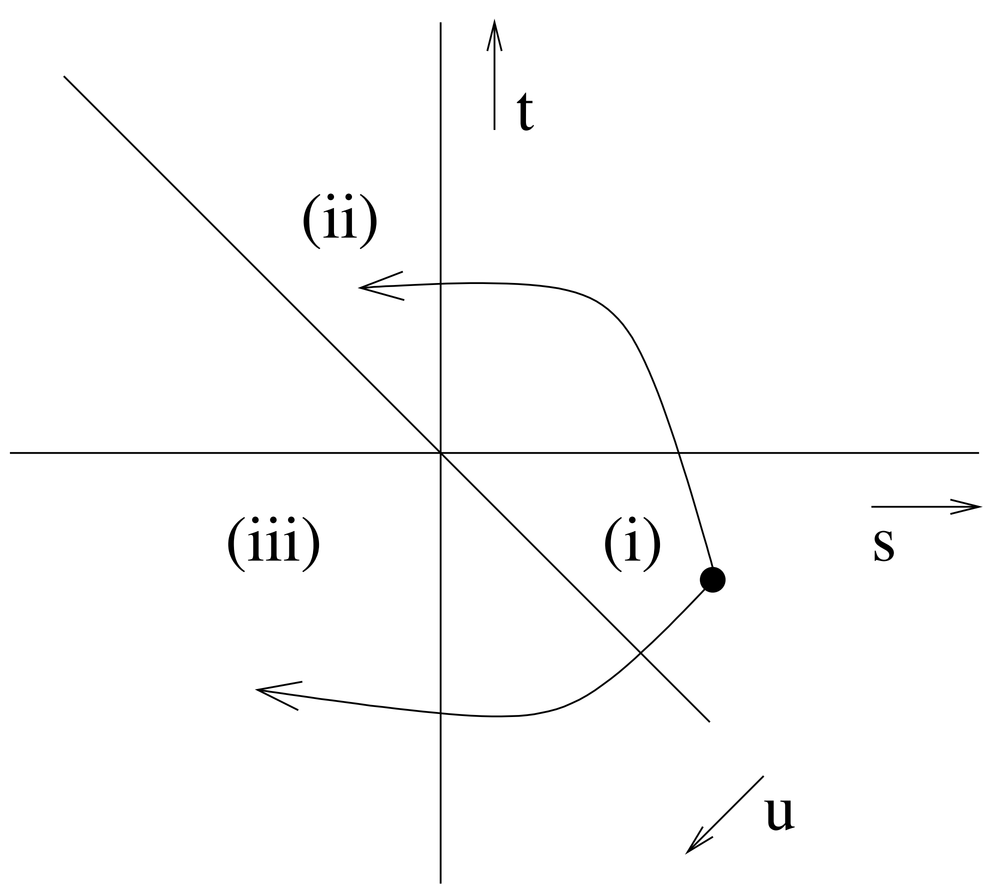
[Manteuffel, Schabinger 2014]
[Peraro 2015]

COMPUTING THE INTEGRALS

Consider the production of 2 quarks in quark-antiquark annihilation

$$q(p_1) + \bar{q}(p_2) \to Q(p_3) + \bar{Q}(p_4)\,, \quad \text{with} \quad p_i^2 = 0$$

$$s = (p_1 + p_2)^2\,, \quad t = (p_1 - p_3)^2\,, \quad \text{and} \quad x = -\,t/s \quad \longrightarrow \quad \boxed{s > 0\,, \quad t < 0 \quad 0 < x < 1}$$



Interesting analytic structure, no Euclidean region

V. Smirnov one of the pioneers in studying the analytic properties of these objects:

[Smirnov '99; Smirnov, Veretin '00; Tausk '00]

[Anastasiou, Gehrmann, Oleari, Remiddi, Tausk '00]

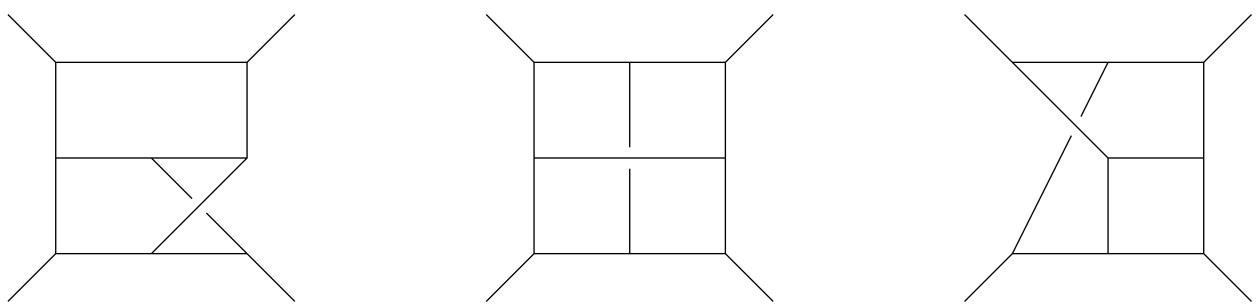
COMPUTING THE INTEGRALS

Master integrals very non-trivial, despite "just" HPLs

$$\begin{aligned} \mathrm{H}(0;x) &= \ln x \,, \\ \mathrm{H}(1;x) &= \int_0^x \frac{dx'}{1-x'} = -\ln(1-x) \end{aligned}$$

[Remiddi, Vermaseren '99]

- **Many master integrals** ($\sim$ 500), single sectors with $\sim \mathcal{O}(10)$ MIs



Approached by **differential equations method** [Kotikov '97; Remiddi '99; Gehrmann Remiddi '00]

$$d\vec{I} = \epsilon A(x)\vec{I}$$ [Arkani-Hamed '10; Kotikov '07 '10; Henn '13, Lee '15]

- Finding a canonical basis is ***very non-trivial.*** Solved by [Henn, Mistlberger, Smirnov, Wasser, 2020]
- Boundaries ***trivialised by regularity conditions and UV properties***: [Henn, Mistlberger, Smirnov, Wasser, 2020]

RESULTS AND IR STRUCTURE FOR $q\bar{q} \to Q\bar{Q}$ @ 3 LOOPS

$$\boldsymbol{\mathcal{H}}_i = \boldsymbol{\mathcal{H}}_i^{(0)} + \left(\frac{\alpha_{\mathrm{s,b}}}{4\pi}\right) \boldsymbol{\mathcal{H}}_i^{(1)} + \left(\frac{\alpha_{\mathrm{s,b}}}{4\pi}\right)^2 \boldsymbol{\mathcal{H}}_i^{(2)} + \left(\frac{\alpha_{\mathrm{s,b}}}{4\pi}\right)^3 \boxed{\boldsymbol{\mathcal{H}}_i^{(3)}} + \mathcal{O}\left(\alpha_{\mathrm{s,b}}^4\right)$$

UV renormalisation is straightforward

$$\begin{aligned}
\boldsymbol{\mathcal{H}}_{i,\mathrm{ren}}^{(0)} &= \boldsymbol{\mathcal{H}}_i^{(0)} \\
\boldsymbol{\mathcal{H}}_{i,\,\mathrm{ren}}^{(1)} &= \boldsymbol{\mathcal{H}}_i^{(1)} - \frac{\beta_0}{\epsilon}\boldsymbol{\mathcal{H}}_i^{(0)} \\
\boldsymbol{\mathcal{H}}_{i,\,\mathrm{ren}}^{(2)} &= \boldsymbol{\mathcal{H}}_i^{(2)} - \frac{2\beta_0}{\epsilon}\left(\frac{\alpha_{\mathrm{s}}}{4\pi}\right)\boldsymbol{\mathcal{H}}_i^{(1)} + \frac{\left(2\beta_0^2 - \beta_1\epsilon\right)}{2\epsilon^2}\boldsymbol{\mathcal{H}}_i^{(0)} \\
\boldsymbol{\mathcal{H}}_{i,\,\mathrm{ren}}^{(3)} &= \boldsymbol{\mathcal{H}}_i^{(3)} - \frac{3\beta_0}{\epsilon}\boldsymbol{\mathcal{H}}_i^{(2)} + \frac{\left(3\beta_0^2 - \beta_1\epsilon\right)}{\epsilon^2}\boldsymbol{\mathcal{H}}_i^{(1)} + \frac{\left(7\beta_1\beta_0\epsilon - 6\beta_0^3 - 2\beta_2\epsilon^2\right)}{6\epsilon^3}\boldsymbol{\mathcal{H}}_i^{(0)}
\end{aligned}$$

$$\begin{aligned}
\beta_0 &= \frac{11}{3}C_A - \frac{2}{3}\,n_f\;, \\
\beta_1 &= \frac{1}{3}\left(34\,C_A^2 - 10\,C_A\,n_f\right) - 2\,C_F\,n_f\;, \\
\beta_2 &= -\frac{1415\,C_A^2\,n_f}{54} + \frac{2857\,C_A^3}{54} - \frac{205\,C_A\,C_F\,n_f}{18} + \frac{79\,C_A\,n_f^2}{54} + C_F^2\,n_f + \frac{11\,C_F\,n_f^2}{9}
\end{aligned}$$

RESULTS AND IR STRUCTURE FOR $q\bar{q} \to Q\bar{Q}$ @ 3 LOOPS

IR structure instead is very non trivial: first verification of **quadrupole structure in QCD**

$$\boldsymbol{\mathcal{H}}_{i,\,\mathrm{fin}}(\epsilon, \{p\}) = \lim_{\epsilon \to 0} \boldsymbol{\mathcal{Z}}^{-1}(\epsilon, \{p\}, \mu)\, \boldsymbol{\mathcal{H}}_{i,\,\mathrm{ren}}(\epsilon, \{p\})$$

$$\boldsymbol{\mathcal{Z}}(\epsilon, \{p\}, \mu) = \mathbb{P} \exp \left[\int_\mu^\infty \frac{\mathrm{d}\mu'}{\mu'} \boldsymbol{\Gamma}(\{p\}, \mu') \right] = \sum_{n=0}^{\infty} \left(\frac{\alpha_\mathrm{s}}{4\pi} \right)^n \boldsymbol{\mathcal{Z}}_n$$

$$\boldsymbol{\Gamma}(\{p\}, \mu) = \boldsymbol{\Gamma}_{\mathrm{dipole}}(\{p\}, \mu) + \boxed{\boldsymbol{\Delta}_4(\{p\})}$$

$$\boldsymbol{\Delta}_4(\{p\}) = \sum_{L=3}^{\infty} \left(\frac{\alpha_\mathrm{s}}{4\pi} \right)^L \boldsymbol{\Delta}_4^{(L)}(\{p\})$$

$$\begin{aligned}
\boldsymbol{\mathcal{H}}^{(0)}_{i,\,\mathrm{fin}} &= \boldsymbol{\mathcal{H}}^{(0)}_{i}\,, \\
\boldsymbol{\mathcal{H}}^{(1)}_{i,\,\mathrm{fin}} &= \boldsymbol{\mathcal{H}}^{(1)}_{i,\,\mathrm{ren}} - \boldsymbol{\mathcal{I}}_1\, \boldsymbol{\mathcal{H}}^{(0)}_{i,\,\mathrm{ren}}\,, \\
\boldsymbol{\mathcal{H}}^{(2)}_{i,\,\mathrm{fin}} &= \boldsymbol{\mathcal{H}}^{(2)}_{i,\,\mathrm{ren}} - \boldsymbol{\mathcal{I}}_2\, \boldsymbol{\mathcal{H}}^{(0)}_{i,\,\mathrm{ren}} - \boldsymbol{\mathcal{I}}_1\, \boldsymbol{\mathcal{H}}^{(1)}_{i,\,\mathrm{ren}}\,, \\
\boldsymbol{\mathcal{H}}^{(3)}_{i,\,\mathrm{fin}} &= \boldsymbol{\mathcal{H}}^{(3)}_{i,\,\mathrm{ren}} - \boldsymbol{\mathcal{I}}_3\, \boldsymbol{\mathcal{H}}^{(0)}_{i,\,\mathrm{ren}} - \boldsymbol{\mathcal{I}}_2\, \boldsymbol{\mathcal{H}}^{(1)}_{i,\,\mathrm{ren}} - \boldsymbol{\mathcal{I}}_1\, \boldsymbol{\mathcal{H}}^{(2)}_{i,\,\mathrm{ren}}\,,
\end{aligned}$$

$$\begin{aligned}
\boldsymbol{\mathcal{I}}_1 &= \boldsymbol{\mathcal{Z}}_1\,, \\
\boldsymbol{\mathcal{I}}_2 &= \boldsymbol{\mathcal{Z}}_2 - \boldsymbol{\mathcal{Z}}_1^2\,, \\
\boldsymbol{\mathcal{I}}_3 &= \boldsymbol{\mathcal{Z}}_3 - 2\boldsymbol{\mathcal{Z}}_1\boldsymbol{\mathcal{Z}}_2 + \boldsymbol{\mathcal{Z}}_1^3 + \boxed{\boldsymbol{\Delta}_4^{(3)}}
\end{aligned}$$

NUMERICAL RESULTS

Extremely compact results ($\sim$ 100 kb)

Expressed in terms of **simple functions**

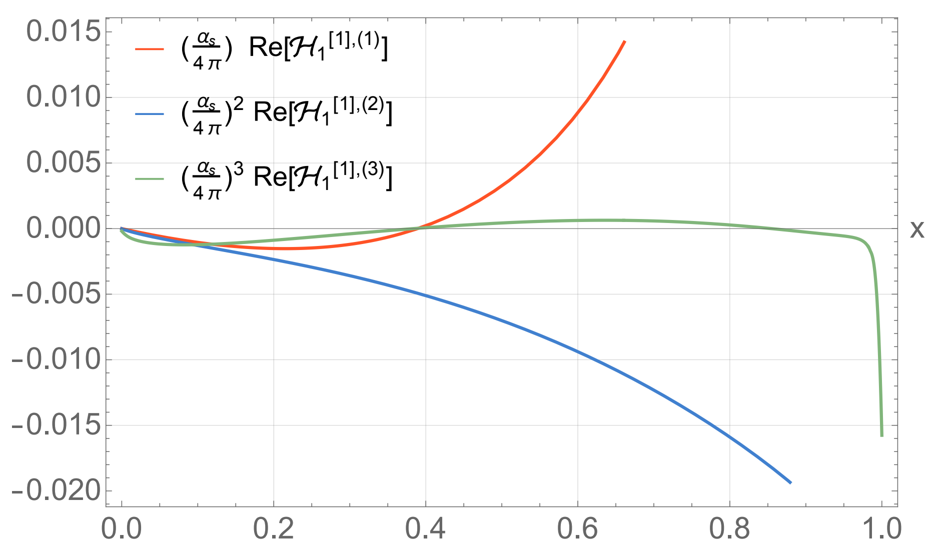
Evaluated numerically fast and reliably

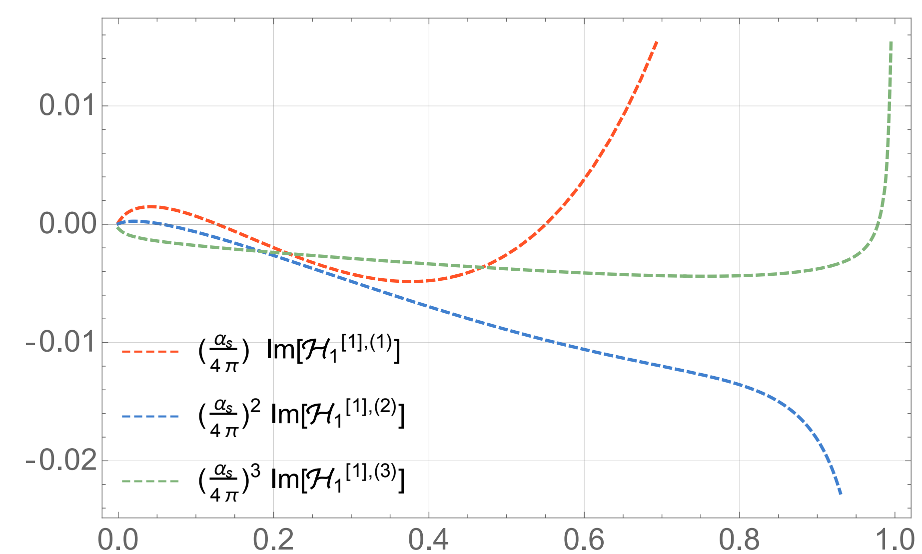
14 classical polylogs

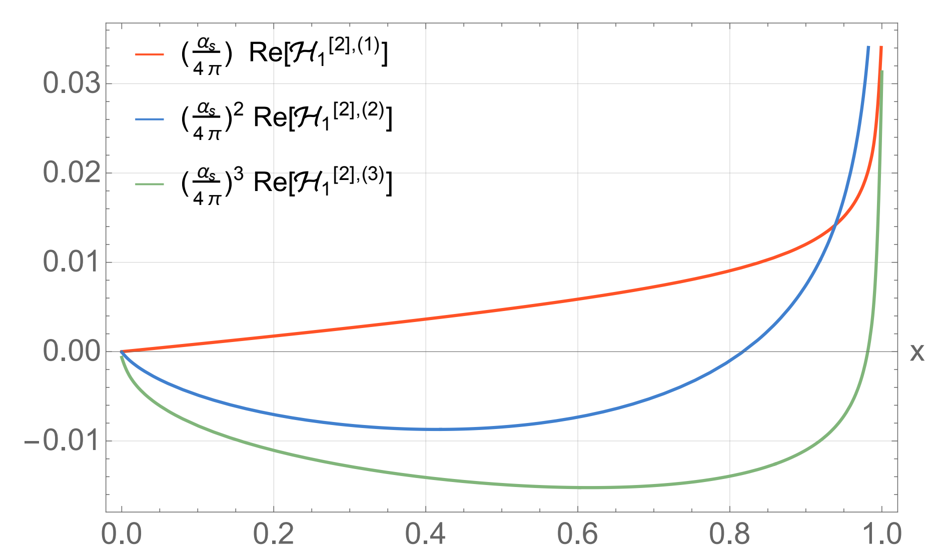
$$\mathrm{Li}_n(r(x))$$

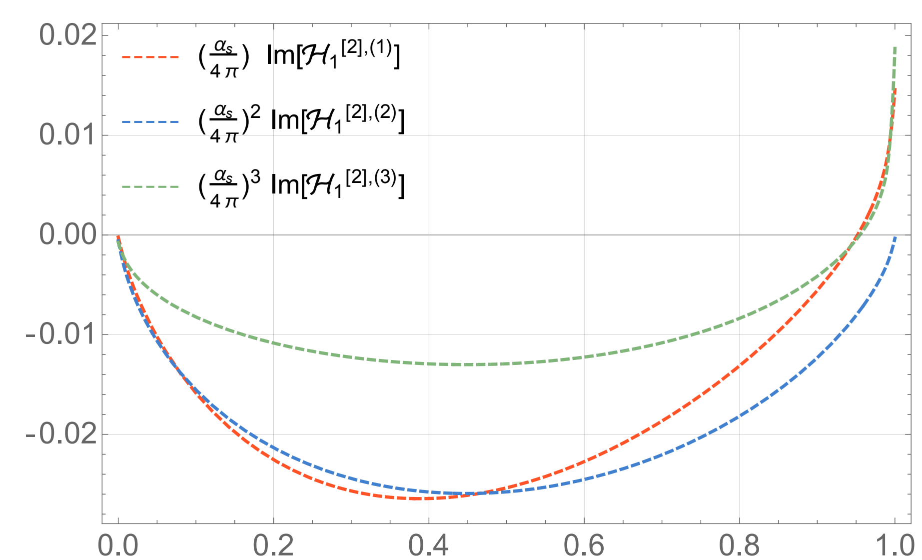
9 extra MPLs

$$\mathrm{Li}_{3,2}(x,1),\ \mathrm{Li}_{3,2}(1-x,1),\ \mathrm{Li}_{3,2}(1,x),$$
$$\mathrm{Li}_{3,3}(x,1),\ \mathrm{Li}_{3,3}(1-x,1),\ \mathrm{Li}_{3,3}(x/(x-1),1),$$
$$\mathrm{Li}_{4,2}(x,1),\ \mathrm{Li}_{4,2}(1-x,1),\ \mathrm{Li}_{2,2,2}(x,1,1),$$









CONCLUSIONS

- N3LO for 2->2 processes is the frontier in QCD
- **Handling complexity** requires advances in all directions: **IR and Amplitudes**
- **First step is results for 3 loop amplitudes**. Di-jet production provides us also with first glimpse on new IR structures in QCD
- **Require many new tools**: new methods to extract helicity amplitudes, finite field arithmetic, modern techniques to manipulate special functions
- Results are extremely simple!
- N3LO journey ready to begin?