Functional reduction of Feynman integrals with masses

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To derive functional equation (FE) for Feynman integrals one can use an algebraic relations for propagators:

$$\boxed{\frac{1}{D_1 D_2 \ldots D_n} = \frac{x_1}{D_0 D_2 \ldots D_n} + \frac{x_2}{D_1 D_0 \ldots D_n} + \ldots + \frac{x_n}{D_1 \ldots D_{n-1} D_0}}$$

where

$$D_j = (k_1 - p_j)^2 - m_j^2 + i\eta.$$

This equation can be fulfilled for arbitrary k_1 by imposing conditions on x_j, m_0, p_0.

$$p_0 = \sum_{j=1}^{n} x_j p_j \qquad \sum_{r=1}^{n} x_r = 1,$$

$$m_0^2 - \sum_{k=1}^{n} x_k m_k^2 + \sum_{j=1}^{n} \sum_{k=1}^{j-1} x_j \, x_k s_{kj} = 0,$$

where

$$s_{ij} = (p_i - p_j)^2.$$

Solutions of this system of equations will depend on $n-2$ *arbitrary parameters* x_i *and one arbitrary mass* m_0.

Integration of the algebraic relation with respect to momentum k_1 yields functional equation for the one-loop n-point integrals

$$I_n^{(d)}(m_1^2, m_2^2, \ldots; s_{12}, s_{23}, \ldots) = \frac{1}{i\pi^{d/2}} \int \frac{d^d k_1}{[(k_1-p_1)^2 - m_1^2 + i\eta] \ldots [(k_1-p_n)^2 - m_n^2 + i\eta]}$$

$$\begin{aligned}
& I_n^{(d)}(m_1^2, m_2^2, \ldots; s_{12}, s_{23}, \ldots, s_{13}, \ldots) = \\
& \; x_1 I_n^{(d)}(m_0^2, m_2^2, \ldots; s_{02}, s_{23}, \ldots, s_{03}, \ldots) \\
& + x_2 I_n^{(d)}(m_1^2, m_0^2, \ldots; s_{01}, s_{03}, \ldots, s_{13}, \ldots) \\
& \qquad \ldots\ldots\ldots\ldots\ldots\ldots \\
& + x_j I_n^{(d)}(\ldots m_j^2 \to m_0^2 \ldots, \ldots s_{jk} \to s_{0k}, \ldots) \\
& \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots
\end{aligned}$$

By choosing arbitrary parameters x_j, m_0 one can try to express initial integral in terms of integrals with fewer variables. If we succeed in finding such parameters we will solve functional equation.
A systematic method for solving functional equations for Feynman integrals was presented in O.V.T. JHEP 02 (2019) 173.
In some sense it is generalization of the method used by mathematicians to solve Sincov's functional equation

$$f(x,y) = f(x,z) - f(y,z).$$

Setting $z = 0$ in this equation, we get the general solution

$$f(x,y) = g(x) - g(y),$$

where

$$g(x) = f(x,0).$$

I.e. the function $f(x,y)$ is a combination of its ' *boundary values*' .

The situation with solution of functional equations for Feynman integrals is much more complicated- too many variables, too many functions are involved.
For this reason we used computer for a systematic search of simple relations between arguments of integrals.
To reduce the number of variables we imposed the following conditions on new variables s_{j0}, m_0^2

$$s_{0j} = 0, \quad s_{0j} - s_{0i} = 0, \quad s_{0j} \pm s_{ik} = 0, \quad s_{0j} \pm m_0^2 = 0,$$
$$s_{0j} \pm m_0^2 \pm m_k^2 = 0, \qquad (i, j, k = 1 \ldots n).$$

Out of these equations we created systems of equations with $2, 3, 4, \ldots$ equations in each system. Solutions of these systems of equations and analysis of these solutions were performed using MAPLE. Depending on n the number of systems varied from 10^3 to 10^6. Execution CPU time varied from several minutes to several hours.

Many solutions of these equations were found. Some of them lead to simultaneous reduction of the number of variables of integrals on the right-hand side of the FE.

Solution of FE for Feynman integrals

Solution of FE will be expressed in terms of ratios of two determinants:

$$\Delta_n \equiv \Delta_n(\{p_1, m_1\}, \ldots \{p_n, m_n\}) = \begin{vmatrix} Y_{11} & Y_{12} & \ldots & Y_{1n} \\ Y_{12} & Y_{22} & \ldots & Y_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ Y_{1n} & Y_{2n} & \ldots & Y_{nn} \end{vmatrix}, \qquad Y_{ij} = m_i^2 + m_j^2 - s_{ij}$$

$$G_{n-1} \equiv G_{n-1}(p_1, \ldots, p_n) = -2 \begin{vmatrix} S_{11} & S_{12} & \ldots & S_{1\ n-1} \\ S_{21} & S_{22} & \ldots & S_{2\ n-1} \\ \vdots & \vdots & \ddots & \vdots \\ S_{n-1\ 1} & S_{n-1\ 2} & \ldots & S_{n-1\ n-1} \end{vmatrix}, \qquad S_{ij} = s_{in} + s_{jn} - s_{ij},$$

We will use also an indexed notation for Δ_n and G_{n-1}

$$\lambda_{i_1 i_2 \ldots i_n} = \Delta_n(\{p_{i_1}, m_{i_1}\}, \{p_{i_2}, m_{i_2}\}, \ldots, \{p_{i_n}, m_{i_n}\}),$$
$$g_{i_1 i_2 \ldots i_n} = G_{n-1}(p_{i_1}, p_{i_2}, \ldots, p_{i_n}).$$

In what follows we will use the following notations

$$r_{ij\ldots k} = -\frac{\lambda_{ij\ldots k}}{g_{ij\ldots k}}, \qquad \kappa_{n\ ij\ldots k} = \frac{\partial\ r_{ij\ldots n\ldots k}}{\partial m_n{}^2}, \qquad r_i = m_i^2.$$

Integrating algebraic relation

$$\frac{1}{D_1 D_2} = \frac{x_1}{D_0 D_2} + \frac{x_2}{D_1 D_0}$$

w.r.t. k_1 yields:

$$I_2^{(d)}(m_1^2, m_2^2, s_{12}) = x_1 I_2^{(d)}(m_2^2, m_0^2, s_{20}) + x_2 I_2^{(d)}(m_1^2, m_0^2, s_{10}),$$

where

$$x_1 = \frac{m_2^2 - m_1^2 + s_{12}}{2s_{12}} \pm \frac{\sqrt{4s_{12}(m_0^2 - r_{12})}}{2s_{12}}, \qquad x_2 = 1 - x_1,$$

$$s_{10} = (p_1 - p_0)^2 = m_1^2 + m_0^2 - 2r_{12} \pm \frac{m_2^2 - m_1^2 - s_{12}}{2s_{12}} \sqrt{4s_{12}(m_0^2 - r_{12})},$$

$$s_{20} = (p_2 - p_0)^2 = m_2^2 + m_0^2 - 2r_{12} \pm \frac{m_2^2 - m_1^2 + s_{12}}{2s_{12}} \sqrt{4s_{12}(m_0^2 - r_{12})},$$

By choosing an arbitrary parameter m_0^2 we get two variants of functional reduction:

$$m_0^2 = r_{12}: \quad I_2(r_{12}, m_2^2, m_2^2 - r_{12}), \quad I_2(r_{12}, m_1^2, m_1^2 - r_{12}),$$

$$m_0^2 = r_{12}: \quad I_2(0, m_2^2, \overline{s}_{02}), \quad I_2(0, m_1^2, \overline{s}_{01}),$$

Integrating algebraic relation for 3 propagators

$$\frac{1}{D_1 D_2 D_3} = \frac{x_1}{D_0 D_2 D_3} + \frac{x_2}{D_1 D_0 D_3} + \frac{x_3}{D_1 D_2 D_0},$$

with respect to the common momentum k_1 yields

$$I_3^{(d)}(m_1^2, m_2^2, m_3^2;\ s_{23}, s_{13}, s_{12}) = x_1\ I_3^{(d)}(m_0^2, m_2^2, m_3^2;\ s_{23}, s_{30}, s_{20})$$
$$+ x_2\ I_3^{(d)}(m_1^2, m_0^2, m_3^2;\ s_{30}, s_{13}, s_{10})$$
$$+ x_3\ I_3^{(d)}(m_1^2, m_2^2, m_0^2;\ s_{20}, s_{10}, s_{12}).$$

Solving different sets of systems of equations with respect to m_0^2, x_k, we found many solutions reducing the number of variables in $I_3^{(d)}$ on the right-hand side. One of the solutions yields:

$$I_3^{(d)}\left(m_1^2, m_2^2, m_3^2, s_{23}, s_{13}, s_{12}\right) =$$
$$\kappa_{123} I_3^{(d)}\left(r_{123}, r_2, r_3, s_{23}, r_3 - r_{123}, r_2 - r_{123}\right)$$
$$+\kappa_{213} I_3^{(d)}\left(r_1, r_{123}, r_3, r_3 - r_{123}, s_{13}, r_1 - r_{123}\right)$$
$$+\kappa_{312} I_3^{(d)}\left(r_1, r_2, r_{123}, r_2 - r_{123}, r_1 - r_{123}, s_{12}\right),$$

Each integral on the right-hand side depends on 4 variables.

All integrals on the right-hand side of this relation can be expressed in terms of integrals depending on 3 variables. For such a reduction the following relation

$$
\begin{aligned}
I_3^{(d)}(r_{123}, r_2, r_3, s_{23}, r_3 - r_{123}, r_2 - r_{123}) = \\
\kappa_{23} I_3^{(d)}(r_{123}, r_{23}, r_3, r_3 - r_{23}, r_3 - r_{123}, r_{23} - r_{123}) \\
+\kappa_{32} I_3^{(d)}(r_{123}, r_2, r_{23}, r_2 - r_{23}, r_{23} - r_{123}, r_2 - r_{123}),
\end{aligned}
$$

was used for the first term. Analogous relations for two other terms were used. Similar formulae were also obtained by A. Davydychev (2017) by other method.

Combining both steps of reductions we get:

$$
\begin{aligned}
&I_3^{(d)}(m_1^2, m_2^2, m_3^2, s_{23}, s_{13}, s_{12}) = \\
&+\kappa_{123}\kappa_{32} I_3^{(d)}(r_{123}, r_{23}, r_2, r_2 - r_{23}, r_2 - r_{123}, r_{23} - r_{123}) \\
&+\kappa_{123}\kappa_{23} I_3^{(d)}(r_{123}, r_{23}, r_3, r_3 - r_{23}, r_3 - r_{123}, r_{23} - r_{123}) \\
&+\kappa_{213}\kappa_{31} I_3^{(d)}(r_{123}, r_{13}, r_1, r_1 - r_{13}, r_1 - r_{123}, r_{13} - r_{123}) \\
&+\kappa_{213}\kappa_{13} I_3^{(d)}(r_{123}, r_{13}, r_3, r_3 - r_{13}, r_3 - r_{123}, r_{13} - r_{123}) \\
&+\kappa_{312}\kappa_{12} I_3^{(d)}(r_{123}, r_{12}, r_2, r_2 - r_{12}, r_2 - r_{123}, r_{12} - r_{123}) \\
&+\kappa_{312}\kappa_{21} I_3^{(d)}(r_{123}, r_{12}, r_1, r_1 - r_{12}, r_1 - r_{123}, r_{12} - r_{123}).
\end{aligned}
$$

Thus the integral $I_3^{(d)}$ depending on 6 variables was presented in terms of integrals depending on 3 dimensionful variables.
This is not the only possible reduction. For example, in (O.V.T. Phys.Lett. B670 (2008) 67-72) an expression for $I_3^{(d)}$ in terms of integrals:

$$
I_3^{(d)}(m^2, 0, 0; q_{23}, 0, q_{12}), \; I_3^{(d)}(0, 0, 0, q_{23}, q_{13}, q_{12})
$$

was presented.

Integrating algebraic relation for 4 propagators

$$\frac{1}{D_1D_2D_3D_4} = \frac{x_1}{D_0D_2D_3D_4} + \frac{x_2}{D_1D_0D_3D_4} + \frac{x_3}{D_1D_2D_0D_4} + \frac{x_4}{D_1D_2D_4D_0},$$

with respect to the common momentum k_1 yields

$$\begin{aligned}
I_4^{(d)}(m_1^2, m_2^2, m_3^2, m_4^2; s_{12}, s_{23}, s_{34}, s_{14}, s_{24}, s_{13}) = {} & x_1 I_4^{(d)}(m_0^2, m_2^2, m_3^2, m_4^2; s_{02}, s_{23}, s_{34}, s_{04}, s_{24}, s_{03}) \\
& + x_2 I_4^{(d)}(m_1^2, m_0^2, m_3^2, m_4^2; s_{01}, s_{03}, s_{34}, s_{14}, s_{04}, s_{13}) \\
& + x_3 I_4^{(d)}(m_1^2, m_2^2, m_0^2, m_4^2; s_{12}, s_{02}, s_{04}, s_{14}, s_{24}, s_{01}) \\
& + x_4 I_4^{(d)}(m_1^2, m_2^2, m_3^2, m_0^2; s_{12}, s_{23}, s_{03}, s_{01}, s_{02}, s_{13})
\end{aligned}$$

Solving different sets of systems of equations with respect to m_0^2, x_k, we get:

$$\begin{aligned}
& I_4^{(d)}(m_1^2, m_2^2, m_3^2, m_4^2; s_{12}, s_{23}, s_{34}, s_{14}, s_{24}, s_{13}) \\
&= \kappa_{1234} I_4^{(d)}(r_{1234}, r_2, r_3, r_4; r_2 - r_{1234}, s_{23}, s_{34}, r_4 - r_{1234}, s_{24}, r_3 - r_{1234}) \\
&\quad + \kappa_{2134} I_4^{(d)}(r_1, r_{1234}, r_3, r_4; r_1 - r_{1234}, r_3 - r_{1234}, s_{34}, s_{14}, r_4 - r_{1234}, s_{13}) \\
&\quad + \kappa_{3124} I_4^{(d)}(r_1, r_2, r_{1234}, r_4^2; s_{12}, r_2 - r_{1234}, r_4 - r_{1234}, s_{14}, s_{24}, r_1 - r_{1234}) \\
&\quad + \kappa_{4123} I_4^{(d)}(r_1, r_2, r_3, r_{1234}; s_{12}, s_{23}, r_3 - r_{1234}, r_1 - r_{1234}, r_2 - r_{1234}, s_{13})
\end{aligned}$$

Each integral on the right-hand side depends on 7 variables.

Integrals depending on 7 variables can be expressed in terms of integrals depending on 5 variables. The following relation was found for the first integral

$$
\begin{aligned}
&I_4(r_{1234}, r_2, r_3, r_4, r_2 - r_{1234}, s_{23}, s_{34}, r_4 - r_{1234}, s_{24}, r_3 - r_{1234}) = \\
&+\kappa_{234} I_4(r_{1234}, r_{234}, r_3, r_4, r_{234} - r_{1234}, r_3 - r_{234}, s_{34}, r_4 - r_{1234}, r_4 - r_{234}, r_3 - r_{1234}) \\
&+\kappa_{324} I_4(r_{1234}, r_2, r_{234}, r_4, r_2 - r_{1234}, r_2 - r_{234}, r_4 - r_{234}, r_4 - r_{1234}, s_{24}, r_{234} - r_{1234}) \\
&+\kappa_{423} I_4(r_{1234}, r_2, r_3, r_{234}, r_2 - r_{1234}, s_{23}, r_3 - r_{234}, r_{234} - r_{1234}, r_2 - r_{234}, r_3 - r_{1234}),
\end{aligned}
$$

and similar relations for other integrals on the right-hand side.
To express integrals depending on 5 variables in terms of integrals depending on 4 variables we used relations like this one

$$
\begin{aligned}
&I_4(r_{1234}, r_{234}, r_3, r_4, r_{234} - r_{1234}, r_3 - r_{234}, s_{34}, r_4 - r_{1234}, r_4 - r_{234}, r_3 - r_{1234}) = \\
&+\kappa_{34} I_4(r_{1234}, r_{234}, r_{34}, r_4, r_{234} - r_{1234}, r_{34} - r_{234}, r_4 - r_{34}, r_4 - r_{1234}, r_4 - r_{234}, r_{34} - r_{1234}) \\
&+\kappa_{43} I_4(r_{1234}, r_{234}, r_3, r_{34}, r_{234} - r_{1234}, r_3 - r_{234}, r_3 - r_{34}, r_{34} - r_{1234}, r_{34} - r_{234}, r_3 - r_{1234}),
\end{aligned}
$$

Combining relations of 3 steps yields 24 terms expression for the $I_4^{(d)}$ integral with arbitrary masses and momenta:

$$I_4^{(d)}(m_1^2, m_2^2, m_3^2, m_4^2, s_{12}, s_{23}, s_{34}, s_{14}, s_{24}, s_{13}) =$$

$$\kappa_{1234}\kappa_{234}\kappa_{34} I_4^{(d)}(r_{1234}, r_{234}, r_{34}, r_4,$$

$$r_{234} - r_{1234}, r_{34} - r_{234}, r_4 - r_{34}, r_4 - r_{1234}, r_4 - r_{234}, r_{34} - r_{1234})$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$+\kappa_{4123}\kappa_{312}\kappa_{21} I_4^{(d)}(r_{1234}, r_{123}, r_{12}, r_1,$$

$$r_{123} - r_{1234}, r_{12} - r_{123}, r_1 - r_{12}, r_1 - r_{1234}, r_1 - r_{123}, r_{12} - r_{1234})$$

Each integral on the right-hand side depends only on 4 variables.

Integrating algebraic relation for 5 propagators

$$\frac{1}{D_1 D_2 D_3 D_4 D_5} = \frac{x_1}{D_0 D_2 D_3 D_4 D_5} + \frac{x_2}{D_1 D_0 D_3 D_4 D_5} + \ldots + \frac{x_5}{D_1 D_2 D_3 D_4 D_0},$$

with respect to the common momentum k_1 yields

$$I_5^{(d)}(m_1^2, m_2^2, m_3^2, m_4^2, m_5^2; s_{12}, s_{23}, s_{34}, s_{45}, s_{15}, s_{13}, s_{14}, s_{24}, s_{25}, s_{35})$$
$$= x_1 I_5^{(d)}(m_0^2, m_2^2, m_3^2, m_4^2, m_5^2; s_{02}, s_{23}, s_{34}, s_{45}, s_{05}, s_{03}, s_{04}, s_{24}, s_{25}, s_{35})$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

Solving different sets of systems of equations with respect to m_0^2, x_k, we obtained 4 steps reduction procedure allowing to express $I_5^{(d)}$ depending on 15 variables as a combination of 120 terms with $I_5^{(d)}$ depending on 5 variables.

Integrating algebraic relation for 6 propagators

$$\frac{1}{D_1 D_2 D_3 D_4 D_5 D_6} = \frac{x_1}{D_0 D_2 D_3 D_4 D_5 D_6} + \frac{x_2}{D_1 D_0 D_3 D_4 D_5 D_6} + \ldots + \frac{x_6}{D_1 D_2 D_3 D_4 D_0},$$

with respect to the common momentum k_1 yields

$$I_6^{(d)}(m_1^2, m_2^2, m_3^2, m_4^2, m_5^2, m_6^2; s_{12}, s_{23}, s_{34}, s_{45}, s_{56}, s_{16}, s_{13}, s_{14}, s_{15}, s_{24}, s_{25}, s_{26}, s_{35}, s_{36}, s_{46}) =$$

$$x_1 I_6^{(d)}(m_0^2, m_2^2, m_3^2, m_4^2, m_5^2, m_6^2; s_{02}, s_{23}, s_{34}, s_{45}, s_{56}, s_{06}, s_{03}, s_{04}, s_{05}, s_{24}, s_{25}, s_{26}, s_{35}, s_{36}, s_{46})$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

Solving different sets of systems of equations with respect to m_0^2, x_k, we obtained 5 steps reduction procedure allowing to express $I_6^{(d)}$ depending on 21 variables as a combination of 720 terms with $I_6^{(d)}$ depending on 6 variables.

Final functional reduction formulae for $I_2^{(d)}, \ldots\ I_6^{(d)}$ can be obtained exploiting the following algorithm:

- write the term

$$\kappa_{1\ldots n}\kappa_{2\ldots n}\ldots\kappa_{n-1\ \ n} \quad I_n^{(d)}(m_1^2, m_2^2, \ldots m_n^2; s_{12}, s_{23}, \ldots) \qquad (1)$$

- replace $s_{ij} \to m_j^2 - m_i^2 \quad (j > i)$
- replace $m_1^2 \to r_{1\ldots n},\ m_2^2 \to r_{2\ldots n}, \ldots, m_n^2 \to r_n$
- replace $\kappa_{ij\ldots} \to \frac{\partial r_{ij\ldots}}{\partial m_i^2}$
- generate $n!$ terms by symmetrizing the term (1) with respect to the indices $1, 2, \ldots n$ and add all these terms

This algorithm perfectly works for $I_2^{(d)},\ I_3^{(d)},\ I_3^{(d)},\ I_4^{(d)}, I_5^{(d)}, I_6^{(d)}$. We checked numerically that this algorithm is also valid for integrals $I_7^{(d)},\ I_8^{(d)}$.

Parametric representation for the elementary integrals

All integrals in the final formula of the functional reduction have the form

$$I_n^{(d)}(...) = (-1)^n \Gamma\left(n - \frac{d}{2}\right) (r_{1\ldots n})^{d/2-n}$$
$$\times \int_0^1 \frac{dt_1}{\sqrt{t_1}} \int_0^{t_1} \frac{dt_2}{\sqrt{t_2}} \ldots \int_0^{t_{n-2}} \frac{dt_{n-1}}{\sqrt{t_{n-1}}} \; h_n^{\frac{d}{2}-n}$$

where h_n is a simple polynomial

$$h_n = 1 + a_1 t_1 + a_2 t_2 + \ldots + a_{n-1} t_{n-1}.$$

This polynomial has no extremum inside integration region that may be useful for numerical integration.

All dimensional recurrence relations have only one inhomogeneous term.

$I_2^{(d)}$:

$$(d-1)I_2^{(d+2)}(r_{12},r_2,r_2-r_{12}) = -2r_{12}I_2^{(d)}(r_{12},r_2,r_2-r_{12}) - I_1^{(d)}(r_2).$$

$I_3^{(d)}$:

$$\begin{aligned}(d-2)I_3^{(d+2)}&(r_{123},r_{23},r_3,r_3-r_{23},r_3-r_{123},r_{23}-r_{123}) = \\ &-2r_{123}I_3^{(d)}(r_{123},r_{23},r_3,r_3-r_{23},r_3-r_{123},r_{23}-r_{123}) \\ &- I_2^{(d)}(r_{23},r_3,r_3-r_{23}).\end{aligned}$$

$I_4^{(d)}$:

$$\begin{aligned}(d-3)I_4^{(d+2)}&(r_{1234},r_{234},r_{34},r_4, \\ &r_{234}-r_{1234},r_{34}-r_{234},r_4-r_{34},r_4-r_{1234},r_4-r_{34},r_{34}-r_{1234}) =\end{aligned}$$

$$\begin{aligned}-2r_{1234}I_4^{(d)}&(r_{1234},r_{234},r_{34},r_4, \\ &r_{234}-r_{1234},r_{34}-r_{234},r_4-r_{34},r_4-r_{1234},r_4-r_{34},r_{34}-r_{1234})\end{aligned}$$

$$- I_3^{(d)}(r_{234},r_{34},r_4,r_4-r_{34},r_4-r_{234},r_{34}-r_{234})$$

$I_5^{(d)}$:

$$(d-4)I_5^{(d+2)}(r_{12345}, r_{2345}, r_{345}, r_{45}, r_5, r_{2345}-r_{12345}, r_{345}-r_{2345}, r_{45}-r_{345},$$
$$r_5-r_{45}, r_5-r_{12345}, r_{345}-r_{12345}, r_{45}-r_{12345}, r_{45}-r_{2345}, r_5-r_{2345}, r_5-r_{345}) =$$

$$-2r_{12345}I_5^{(d)}(r_{12345}, r_{2345}, r_{345}, r_{45}, r_5, r_{2345}-r_{12345}, r_{345}-r_{2345}, r_{45}-r_{345},$$
$$r_5-r_{45}, r_5-r_{12345}, r_{345}-r_{12345}, r_{45}-r_{12345}, r_{45}-r_{2345}, r_5-r_{2345}, r_5-r_{345})$$

$$-I_4^{(d)}(r_{2345}, r_{345}, r_{45}, r_5;$$
$$r_{345}-r_{2345}, r_{45}-r_{345}, r_5-r_{45}, r_5-r_{2345}, r_5-r_{345}, r_{45}-r_{2345}).$$

At $d=4$ we get simple relation:

$$2r_{12345}I_5^{(4)}(r_{12345}, r_{2345}, r_{345}, r_{45}, r_5, r_{2345}-r_{12345}, r_{345}-r_{2345}, r_{45}-r_{345},$$
$$r_5-r_{45}, r_5-r_{12345}, r_{345}-r_{12345}, r_{45}-r_{12345}, r_{45}-r_{2345}, r_5-r_{2345}, r_5-r_{345}) =$$

$$-I_4^{(4)}(r_{2345}, r_{345}, r_{45}, r_5;$$
$$r_{345}-r_{2345}, r_{45}-r_{345}, r_5-r_{45}, r_5-r_{2345}, r_5-r_{345}, r_{45}-r_{2345})$$

- An algorithm of functional reduction and explicit formulae for one-loop integrals $I_n^{(d)}$, (n=2,..,6) with arbitrary masses and external momenta were presented.
- Based on these reductions a systematic algorithm for obtaining final formula was presented. We expect it is also working for integrals with $n > 6$.
- Integrals emerging in final formula have rather simple parametric representation. Arguments of these integrals are ratios of modified Cayley and Gram determinants. This parametric representation can be used for deriving hypergeometric representation of integrals.
- We plan to investigate these parametric integrals for numerical evaluation and expect better convergence of the procedure of numerical integration.
- Dimensional recurrence relations for integrals depending on minimal number of variables are simple. They also can be used for numerical evaluation of $I_n^{(d)}$ integrals.

Summary and outlook

To compare theoretical predictions with experimental data obtained on the LHC and other colliders one should calculate radiative corrections with very high precision.

Characteristic features of these corrections:

- masses of many particles must be taken into account
- diagrams with many external lines, i.e.many kinematical variables must be calculated

Therefore one should know how to calculate analytically and (or) numerically with very high precision functions of many variables. Experimentators working on the LHC collider presented famous ″wishlist‘ – a list of physical processes where next to leading order radiative corrections are needed. Practically all these corrections require evaluation of radiative corrections with 5-, 6- and more external legs.

2009 status of NLO wishlist for LHC

$pp \to W\,W$ jet	Denner/Dittmaier/Kallweit/Uwer,
	Ellis/Campbell/Zanderighi
$pp \to Z\,Z$ jet	Binoth/Guillet/Karg/Kauer/Sanguinetti
$pp \to t\bar{t}\,b\bar{b}$	Bredenstein/Denner/Dittmaier/Pozzorini
$pp \to t\bar{t} + 2$ jets	
$pp \to Z\,Z\,Z$	Lazopoulos/Melnikov/Petriello, Hankele/Zeppenfeld
$pp \to V\,V\,V$	Binoth/Ossola/Papadopoulos/Pittau, Zeppenfeld et al.
$pp \to V\,V\,b\bar{b}$	
$pp \to V\,V + 2$ jets	VBF: Bozzi/Jäger/Oleari/Zeppenfeld, VBFNLO coll.
$pp \to W + 3$ jets	BlackHat coll.; Ellis/Giele/Kunszt/Melnikov/Zanderighi*
$pp \to b\bar{b}b\bar{b}$	Binoth/Guffanti/Guillet/Reiter/Reuter
$pp \to t\,\bar{t}$ jet	Dittmaier/Uwer/Weinzierl
$pp \to t\,\bar{t}\,Z$	Lazopoulos/McElmurry/Melnikov/Petriello
$pp \to b\,\bar{b}\,Z\,,b\,\bar{b}\,W$	Febres Cordero/Reina/Wackeroth

• done • partial results * leading colour only