Post-Minkowskian expansion of Gravitational scattering from Scattering Amplitudes

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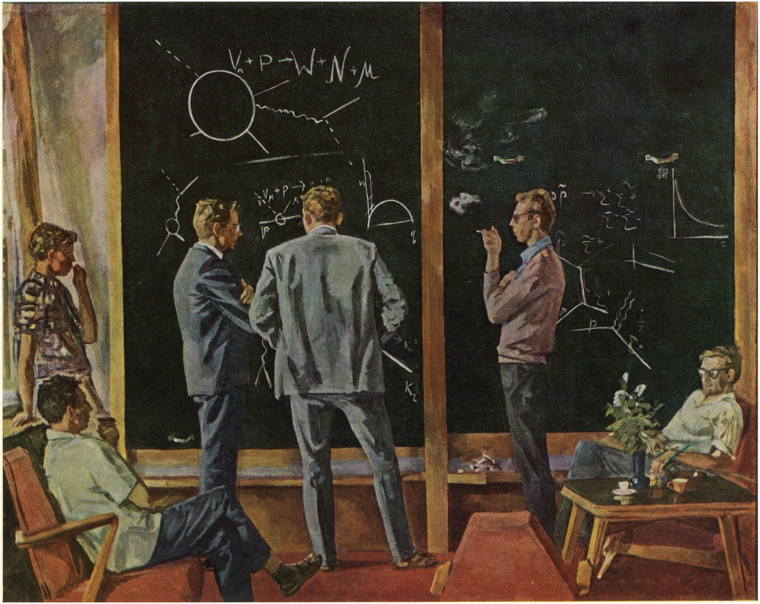
based on 2104.04510, 2105.05218, 2107.12891, 2108.11248
N.E.J. Bjerrum-Bohr, Poul Damgaard, Ludovic Planté,



Happy 70-th anniversary - I wish you many more prolific years



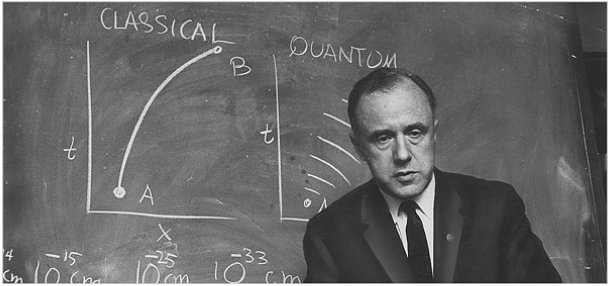
Gravity effective field theories

With the coming age of gravitational physics at various scales, we need to think of **gravity effective field theories** and their connection with observations

$$\mathcal{S}_{eff}^{\text{gravity}} = \frac{1}{16\pi G_N}\int d^4x\,\sqrt{g}\,\left(\mathcal{R} + g^{\mu\nu}T_{\mu\nu}^{\text{matter}}\right) + \mathcal{S}_{\text{EFT}}$$

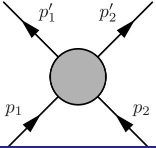
where $\mathcal{S}_{\text{EFT}}$ are corrections to Einstein theory from UV completion (string theory etc.) or various phenomenological modified gravity models

It is important to develop an efficient approach for evaluating post-Minkowskian effects for effective field theory of gravity not just for the plain Einstein-Hilbert action, and understand how to systematically extract the classical physics contributions [Neill, Rotshtein; Bjerrum-Bohr et al.; Damgaard et al.; Bern et al.; Di Vecchia, et al.; Kosower et al.]



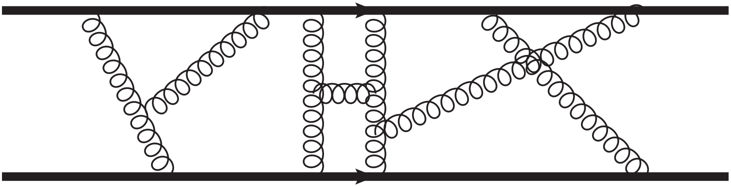
One important **new** insight is that the **classical** gravitational two-body interactions (conservative and radiation) can be extracted from **quantum scattering amplitudes**

$$\mathcal{M}(p_1 \cdot p_2, q^2) =$$



$$= \sum_{L=0}^{+\infty} G_N^{L+1} \mathcal{M}^{\text{L−loop}}$$

Classical Gravity from quantum scattering

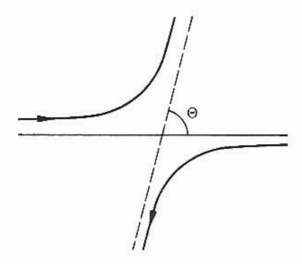


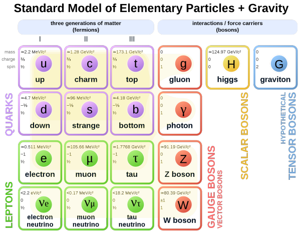
In the limit $\hbar, q^2 \to 0$ with $\underline{q} = q/\hbar$ fixed at each loop order we have a classical contribution ($\gamma = p_1 \cdot p_2/(m_1 m_2)$)

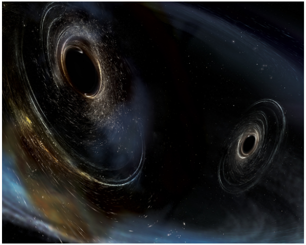
$$\mathcal{M}_L(\gamma, \underline{q}, \hbar) = \frac{\mathcal{M}_L^{(-L-1)}(\gamma, \underline{q}^2)}{\hbar^{L+1}|\underline{q}|^{\frac{L(D-4)}{2}+2}} + \cdots + \frac{\mathcal{M}_L^{(-1)}(\gamma, \underline{q}^2)}{\hbar|\underline{q}|^{\frac{L(D-4)}{2}+2-L}} + O(\hbar^0)$$

In this approach the **classical gravity physics contributions** are determined by **unitarity** of the quantum amplitudes

Nove sed non nova: classical observables

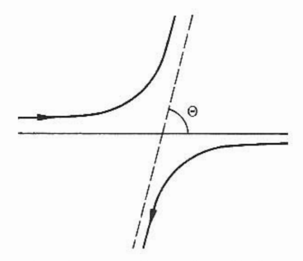






- Classical scattering: scattering angle χ : a lot of physical information for bound orbits
- Quantum scattering for generic EFT of gravity: probability amplitude $\mathcal{M}$
- Spinning black holes as higher-spin massive particles

The scattering angle



The scattering angle is derived from the eikonal phase

$$\sin\left(\frac{\chi}{2}\right) = -\frac{\sqrt{s}}{m_1 m_2 \sqrt{\gamma^2-1}} \frac{\partial \delta(\gamma, b)}{\partial b}$$

$$\widetilde{\mathcal{M}}_L(\gamma, b) = \int_{\mathbb{R}^{D-2}} \frac{d^{D-2}\underline{\vec{q}}}{(2\pi)^{D-2}} \frac{\mathcal{M}_L e^{i\underline{\vec{q}}\cdot\vec{b}}}{4E_{\rm c.m.}P}$$

Eikonalization in b-space

$$1 + i\sum_{L\geqslant 0} \widetilde{\mathcal{M}}_L(\gamma, b) = (1 + 2i\Delta(\gamma, b)) \exp\left(\frac{2i\delta(\gamma, b)}{\hbar}\right).$$

- $\delta(\gamma, b) = \sum_{L\geqslant 0} \delta_L(\gamma, b)$ is the **classical** eikonal phase: no $\hbar$
- $\Delta(\gamma, b)$ contains the quantum corrections: here at the $\hbar$

The scattering angle

One notices that at each loop order there are more and more terms that are more singular than the classical ones - but they are are needed for the exponentiation of the S-matrix [Bjerrum-Bohr et al. ; di Vecchia et al.]

$$\mathcal{M}_L(\gamma, \underline{q}, \hbar) = \frac{\mathcal{M}_L^{(-L-1)}(\gamma, \underline{q}^2)}{\hbar^{L+1}|\underline{q}|^{\frac{L(D-4)}{2}+2}} + \cdots + \frac{\mathcal{M}_L^{(-1)}(\gamma, \underline{q}^2)}{\hbar|\underline{q}|^{\frac{L(D-4)}{2}+2-L}} + O(\hbar^0)$$

This raises a question of the efficiency of the computation : how can we just evaluate $\mathcal{M}_L^{(-1)}(\gamma, \underline{q}^2)$?

Exponentiation of the S-matrix

Using an exponential representation of the $\widehat{S}$ matrix [Damgaard, Planté, Vanhove]

$$\widehat{S} = \mathbb{I} + \frac{i}{\hbar}\widehat{T} = \exp\left(\frac{i\widehat{N}}{\hbar}\right)$$

with the completeness relation that includes all the exchange of gravitons for $n \geqslant 1$ entering the radiation-reaction contributions $\hat{N}^{\rm rad}$

$$\mathbb{I} = \sum_{n=0}^{\infty} \frac{1}{n!} \int \frac{d^{(D-1)}k_1}{(2\pi\hbar)^{(D-1)}} \frac{1}{2E_{k_1}} \frac{d^{(D-1)}k_2}{(2\pi\hbar)^{(D-1)}} \frac{1}{2E_{k_2}}$$
$$\times \frac{d^{(D-1)}\ell_1}{(2\pi\hbar)^{(D-1)}} \frac{1}{2E_{\ell_1}} \cdots \frac{d^{(D-1)}\ell_n}{(2\pi\hbar)^{(D-1)}} \frac{1}{2E_{\ell_n}} \times |k_1, k_2; \ell_1, \ldots \ell_n\rangle\langle k_1, k_2; \ell_1, \ldots \ell_n|,$$

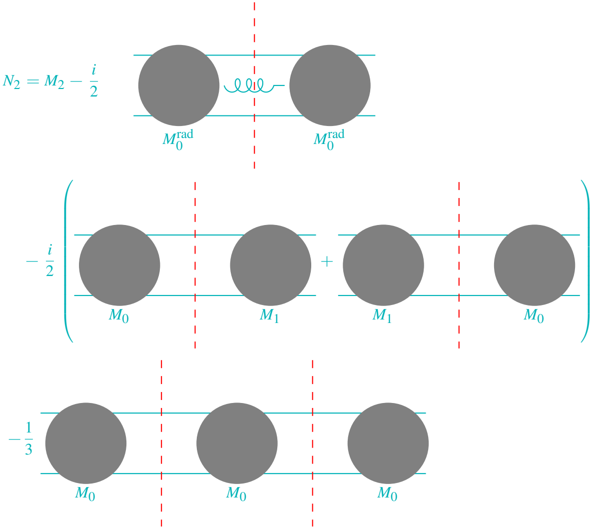
Exponentiation of the S-matrix

For instance at the order G_N^2 (one-loop order) we have

$$
\begin{aligned}
&2\hbar\Im\mathrm{m}[\langle 2|\hat{T}_2|2\rangle] \\
&= \int \prod_{r=1}^{2} \frac{d^{(D-1)}k_r}{(2\pi\hbar)^{(D-1)}} \frac{1}{2E_{k_r}} \frac{d^{(D-1)}\ell}{(2\pi\hbar)^{(D-1)}2E_\ell} \langle 2|\hat{T}_0^{\mathrm{rad}}|k_1,k_2;\ell\rangle\langle k_1,k_2;\ell|\hat{T}_0^{\mathrm{rad}}|2\rangle \\
&+\int \prod_{r=1}^{2} \frac{d^{(D-1)}k_r}{(2\pi\hbar)^{(D-1)}} \frac{1}{2E_{k_r}} \Big(\langle 2|\hat{T}_0|k_1,k_2\rangle\langle k_1,k_2|\hat{T}_1^\dagger|2\rangle + \langle 2|\hat{T}_1|k_1,k_2\rangle\langle k_1,k_2|\hat{T}_0|2\rangle\Big).
\end{aligned}
$$

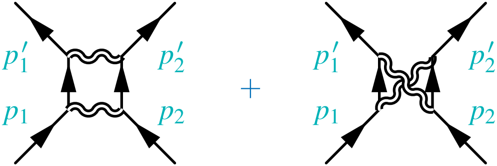
- We have the exchange of the intermediate virtual gravitons
- The conservative part without exchange of virtual gravitons.

Exponentiation of the S-matrix



Velocity cuts

Velocity cuts are unitarity cuts adapted to the post-Minkowskian expansion as introduced in [Bjerrum-Bohr, Damgaard, Planté, Vanhove]



At one-loop order we have the sum of the box and the cross box with $q = |q| u_q$ since $p_i^2 = (p_i')^2 = m_i^2$ with $i = 1, 2$

$$I_\square = -\frac{|\vec{\underline{q}}|^{D-6}}{2\hbar} \int \frac{d^D\ell}{(2\pi)^D} \frac{1}{\ell^2 (\ell + u_q)^2} \left(\frac{1}{(-p_1 + \ell)^2 - m_1^2 + i\epsilon} - \frac{1}{(p_1' + \ell)^2 - m_1^2 - i\epsilon} \right) \times \left(\frac{1}{(-p_2 + \ell)^2 - m_2^2 - i\epsilon} - \frac{1}{(p_2' + \ell)^2 - m_2^2 + i\epsilon} \right)$$

Velocity cuts

Setting $p_1 = \bar{p}_1 + \frac{\hbar}{2}\underline{q}$, $p_1' = \bar{p}_1' - \frac{\hbar}{2}\underline{q}$, $p_2 = \bar{p}_2 - \frac{\hbar}{2}\underline{q}$, $p_2' = \bar{p}_2' + \frac{\hbar}{2}\underline{q}$ we have (neglecting tadpoles)

$$I_{\square} = -\frac{|\vec{\underline{q}}|^{D-6}}{8\hbar}\int \frac{d^D\ell}{(2\pi)^D}\frac{1}{\ell^2(\ell+u_q)^2}\left(\frac{1}{\bar{p}_1\cdot\ell+\frac{\hbar}{2}\ell\cdot\underline{q}+i\epsilon}-\frac{1}{\bar{p}_1\cdot\ell+\frac{\hbar}{2}\ell\cdot\underline{q}-i\epsilon}\right)$$
$$\times\left(\frac{1}{\bar{p}_2\cdot\ell+\frac{\hbar}{2}\ell\cdot\underline{q}-i\epsilon}-\frac{1}{\bar{p}_2\cdot\ell+\frac{\hbar}{2}\ell\cdot\underline{q}+i\epsilon}\right)$$

using the identity

$$\frac{1}{x+i\varepsilon}-\frac{1}{x-i\varepsilon} = -2i\pi\delta(x)$$

we have performing the small $\underline{q}$ expansion and neglecting tadpoless

$$I_{\square} \propto -\frac{|\vec{\underline{q}}|^{D-6}}{8\hbar}\int \frac{d^Dl}{(2\pi)^{D-2}}\frac{\delta(\bar{p}_1\cdot l)\delta(p_2\cdot l)}{\ell^2(\ell+u_q)^2}$$
$$+\frac{|\vec{\underline{q}}|^{D-5}}{16\hbar}\int \frac{d^Dl}{(2\pi)^{D-1}}\frac{1}{\ell^2(\ell+u_q)^2}\left(\frac{\delta(\bar{p}_2\cdot l)}{(\bar{p}_1\cdot\ell)^2}+\frac{\delta(\bar{p}_1\cdot l)}{(\bar{p}_2\cdot\ell)^2}\right)$$

Velocity cuts

The one-loop two-body scattering in general relativity takes the simplified form

$$\mathcal{M}_1(\gamma, \underline{q}^2) = \frac{c_1^{(-2)}}{\hbar^2} \int \frac{\delta(\bar{p}_1 \cdot l)\delta(\bar{p}_2 \cdot l)}{\ell^2(\ell + u_q)^2} \frac{d^D\ell}{(2\pi)^D} + \frac{c_1^{(-1)}}{\hbar} \int \frac{\delta(\bar{p}_1 \cdot l) + \delta(\bar{p}_2 \cdot l)}{\ell^2(\ell + u_q)^2} \frac{d^D\ell}{(2\pi)^D} + O(|\underline{q}|^{D-4})$$

which gives after Fourier transformation to b-space

$$\mathcal{M}_1(\gamma, b) = \frac{i}{2} \underbrace{(\mathcal{M}_0(\gamma, b))^2}_{=O(\hbar^{-2})} + \frac{2\delta_1}{\hbar} + O(\hbar^0)$$

We have the expansion of

$$\exp\left(\frac{2i(\delta_0 + \delta_1 + \cdots)}{\hbar}\right) = 1 + \frac{2i(\delta_0 + \delta_1)}{\hbar} - \frac{2\delta_0{}^2}{\hbar^2} + \cdots$$

Velocity cuts

At two-loop order we have a similar decomposition

$$\mathcal{M}_2(\gamma, b) = \mathcal{M}_2^{(-3)} + \mathcal{M}_2^{(-2)} + \mathcal{M}_2^{(-1)} + O(\hbar^0)$$

with

$$\mathcal{M}_2^{(-3)} = \frac{1}{\hbar^3}\int \delta(\bar{p}_1\cdot l_1)\delta(\bar{p}_1\cdot l_2)\delta(\bar{p}_2\cdot l_1)\delta(\bar{p}_2\cdot l_2) = -\frac{1}{3!}\left(\mathcal{M}_0^{(-1)}\right)^3$$

$$\mathcal{M}_2^{(-2)} = \frac{1}{\hbar^2}\int \left(\delta(\bar{p}_1\cdot l_1)\delta(\bar{p}_1\cdot l_2)\delta(\bar{p}_2\cdot l_1) + \delta(\bar{p}_1\cdot l_1)\delta(\bar{p}_2\cdot l_1)\delta(\bar{p}_2\cdot l_2)\right)$$

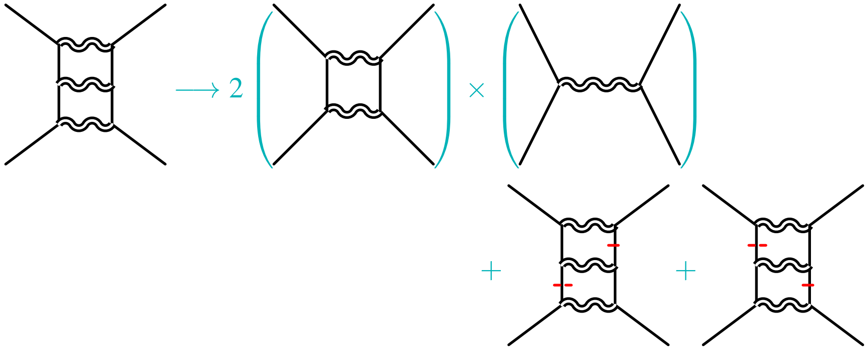
$$= i\mathcal{M}_0^{(-1)}\mathcal{M}_1^{(-1)}$$

$$\mathcal{M}_2^{(-1)} = \frac{1}{\hbar}\int \delta(\bar{p}_1\cdot l_1)\delta(\bar{p}_2\cdot l_1)$$

$$+ \frac{1}{\hbar}\int \left(\delta(\bar{p}_1\cdot l_1)\delta(\bar{p}_1\cdot l_2) + \delta(\bar{p}_1\cdot l_1)\delta(\bar{p}_2\cdot l_2) + \delta(\bar{p}_2\cdot l_1)\delta(\bar{p}_2\cdot l_2)\right)$$

$$= 2i\mathcal{M}_0^{(-1)}\underbrace{\mathcal{M}_1^{(0)}}_{=\Delta_1} + \frac{2\delta_2}{\hbar}$$

Velocity cuts



The cut-integral contribute to the classical part of the eikonal phase δ_2 the rest goes into Δ

$$(1 + i\Delta)\, e^{\frac{2i(\delta_0+\delta_1)}{\hbar}} = (1 + i\Delta_1 + \cdots)\left(1 + \frac{2i(\delta_0 + \delta_1)}{\hbar} - \frac{2\delta_0^2}{\hbar^2} + \cdots\right)$$

Velocity cuts

In practice, we need only evaluate matrix elements in the soft $\underline{q}^2$-expansion, this means that we expand genuine unitarity cuts around the velocity cuts introduced recently [Bjerrum-Boh, Damgaard, Planté, Vanhove]

These velocity cuts seem to provide the most natural way to organise amplitude calculations.

From the complete determination of the two-body amplitude up to 3PM in [Bjerrum-Bohr, Damgaard, Planté, Vanhove] we have a full control of the scattering from the small velocity to the high energy limit

Evaluating the velocity cut integrals

The velocity cut integral are D-dimensional integrals with delta-function insertions

$$\mathcal{D} = \int \frac{d^D l_1 d^D l_2}{(2\pi)^{2D-2}} \frac{\delta(\bar p_1 \cdot l_1)\delta(\bar p_2 \cdot l_2)}{((l_1+l_2+u_q)^2)^{n_7}} \frac{1}{(\bar p_1 \cdot l_2 \pm i\varepsilon)^{n_1} (\bar p_2 \cdot l_1 \pm i\varepsilon)^{n_2} ((u_q+l_1)^2)^{n_3} ((u_q+l_2)^2)^{n_4} (l_1^2)^{n_5} (l_2^2)^{n_6}}$$

which leads to the $D-1$ integrals with $k = \frac{-m_2\gamma \bar p_1 + m_1 \bar p_2}{m_1 m_2 \sqrt{\gamma^2-1}}$

$$\mathcal{D} = \frac{(-1)^{n_1}}{m_1^{n_1+1} m_2^{n_2+1} (\gamma^2-1)^{\frac{n_1+n_2}{2}}} \int \frac{d^{D-1} l_1 d^{D-1} l_2}{(2\pi)^{2D-2}} \frac{1}{((l_1+l_2+u_q)^2 + 2(1-\gamma)(k\cdot l_1)(k\cdot l_2))^{n_7}}$$
$$\times \frac{1}{(k\cdot l_2 \mp i\varepsilon)^{n_1} (k\cdot l_1 \pm i\varepsilon)^{n_2} ((u_q+l_1)^2)^{n_3}} \frac{1}{((u_q+l_2)^2)^{n_4} (l_1^2)^{n_5} (l_2^2)^{n_6}}$$

basis of master integrals

Using `LiteRed` by [Roman Lee] we then determine a basis of master integrals in $D = 4 - 2\epsilon$

$$\frac{d}{d\gamma}\begin{pmatrix} \mathcal{J}_1(\gamma)\\ \mathcal{J}_2(\gamma)\\ \mathcal{J}_3(\gamma)\\ \mathcal{J}_4(\gamma)\\ \mathcal{J}_5(\gamma)\\ \mathcal{J}_6(\gamma)\\ \mathcal{J}_7(\gamma)\\ \mathcal{J}_8(\gamma)\\ \mathcal{J}_9(\gamma)\end{pmatrix} = \epsilon \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & \frac{6\gamma}{\gamma^2-1} & 0 & \frac{1}{\sqrt{\gamma^2-1}} & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & -\frac{2\gamma}{\gamma^2-1} & \frac{2}{\sqrt{\gamma^2-1}} & 0 & 0 & 0 & 0 & 0\\ 0 & -\frac{12}{\sqrt{\gamma^2-1}} & -\frac{2}{\sqrt{\gamma^2-1}} & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & -\frac{2\gamma}{\gamma^2-1} & 0 & 0 & 0 & 0\\ 0 & 0 & -\frac{4\gamma}{\gamma^2-1} & -\frac{2}{\sqrt{\gamma^2-1}} & -\frac{4\gamma}{\gamma^2-1} & -\frac{2\gamma}{\gamma^2-1} & \frac{2}{\sqrt{\gamma^2-1}} & 0 & 0\\ -\frac{4}{\gamma^2-1} & -\frac{12}{\sqrt{\gamma^2-1}} & -\frac{8}{\sqrt{\gamma^2-1}} & 0 & -\frac{8}{\sqrt{\gamma^2-1}} & -\frac{2}{\sqrt{\gamma^2-1}} & \frac{2\gamma}{\gamma^2-1} & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & \frac{1}{\sqrt{\gamma^2-1}} & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} \mathcal{J}_1(\gamma)\\ \mathcal{J}_2(\gamma)\\ \mathcal{J}_3(\gamma)\\ \mathcal{J}_4(\gamma)\\ \mathcal{J}_5(\gamma)\\ \mathcal{J}_6(\gamma)\\ \mathcal{J}_7(\gamma)\\ \mathcal{J}_8(\gamma)\\ \mathcal{J}_9(\gamma)\end{pmatrix}$$

- We need only **9** master integrals

The evaluation of the integrals **depends** on the choice of boundary conditions. The choice of boundary conditions affects the high-energy behaviour $\gamma \gg 1$ of the integrals

Boundary conditions

Let's consider the integral

$$\mathcal{I}_5(\gamma) = \int \frac{d^{3-2\epsilon}\ell_1 d^{3-2\epsilon}\ell_2}{\ell_1^2(\ell_1 - u_q)^2((\ell_1+\ell_2)^2 - 2(\gamma-1)k\cdot\ell_1 k\cdot\ell_2)}$$

it satisfies the differential equation

$$\frac{d\mathcal{I}_5(\gamma)}{d\gamma} = -\epsilon\frac{2\gamma}{\gamma^2-1}\mathcal{I}_5(\gamma) \Longleftrightarrow \mathcal{I}_5(\gamma) = \frac{C_5}{(\gamma^2-1)^\epsilon}$$

A direct evaluation of the integral for $\gamma > 1$ gives

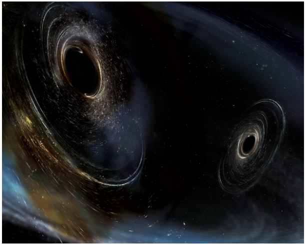
$$\mathcal{I}_5(\gamma) = -\frac{i\epsilon(4\pi e^{-\gamma_E})^{2\epsilon}}{32\pi^3}\left(\frac{-1}{4(\gamma^2-1)}\right)^\epsilon + O(\epsilon)^2$$

But an expansion in the potential region when
$2(\gamma-1)(k\cdot\ell_1)(k\cdot\ell_2) \ll (\ell_1+\ell_2)^2$ leads to $C_5 = 0$ because

$$\mathcal{I}_5(\gamma) = \sum_{n=0}^{\infty}\int d\ell_1 d\ell_2 \frac{(2(\gamma-1)(k\cdot\ell_1)(k\cdot\ell_2))^n}{\ell_1^2(\ell_1-u_q)^2(\ell_1+\ell_2)^{2(n+1)}} = 0$$

which is the potential region boundary conditions used by [Bern et al.]

The scattering angle: conservative part



with $s = (p_1 + p_2)^2 = m_1^2 + m_2^2 + 2m_1 m_2 \gamma$

$$\sin \frac{\chi}{2}\Big|_{\text{conservative}} = \frac{G_N (2\gamma^2 - 1) s^{\frac{1}{2}}}{b(\gamma^2 - 1)} + \frac{G_N^2 3\pi (m_1 + m_2) s^{\frac{1}{2}} (5\gamma^2 - 1)}{8b^2 (\gamma^2 - 1)}$$

$$+ \frac{G_N^3 s^{\frac{1}{2}}}{b^3 (\gamma^2 - 1)} \left(\frac{3(2\gamma^2 - 1)(5\gamma^2 - 1)s}{2(\gamma^2 - 1)} + \frac{m_1^2 + m_2^2}{2} (18\gamma^2 - 1) - \frac{m_1 m_2}{3} \gamma (103 + 2\gamma^2) + \frac{4m_1 m_2 (3 + 12\gamma^2 - 4\gamma^4) \operatorname{arccosh}(\gamma)}{(\gamma^2 - 1)^{\frac{1}{2}}} \right) + \cdots$$

The 1PM (tree-level) and 2PM (one-loop) angle are the same as for test mass in the Schwarzschild black hole of mass $M = m_1 + m_2$.
But this is not true from 3PM (two-loop order) which depends (linearly) on the relative mass $\nu = \frac{m_1 m_2}{m_1 + m_2}$ [Bern et al.; di Vecchia et al.;Bjerrum-Bohr et al.]

The scattering angle: radiation reaction

From 3PM order there is a new contribution from radiation-reaction term

$$\sin\frac{\chi}{2}\Big|_{\rm RR} = \frac{2m_1m_2(2\gamma^2-1)^2 G_N^3\sqrt{s}}{\pi b^3(\gamma^2-1)^{\frac{3}{2}}}\left(-\frac{11}{3}+\frac{d}{d\gamma}\left(\frac{(2\gamma^2-1)\mathrm{arccosh}(\gamma)}{\sqrt{\gamma^2-1}}\right)\right)$$

- The radiation-reaction term is a new ingredient arising in the **full** amplitude from two-loop order arising from the soft boundary conditions [Bjerrum-Bohr et al.]
- The radiation-reaction is needed for restoring a smooth continuity between the non-relativitic, relativistic and ultra-relativistic regimes [di Vecchia et al.]
- This is connected to loss of angular momentum in the collision [Damour]

The Post-Minkowskian result resums the (infinite) post-Newtonian velocity expansion and is valid in the ultra-relativistic regime

Outlook

It is satisfying to be able to embed such classical solutions in the new understanding of the relation between general relativity and the quantum theory of gravity

1. The 2-body gravitational scattering amplitude leads to the classical observables : potential, scattering angle, and radiation
2. The amplitude approach is much simpler that the traditional approach from solving Einstein's equation
3. The velocity cut approach simplifies the extraction of the classical piece and allows to go to high loop order [Bjerrum-Bohr, Planté, Vanhove; to appear]
4. The approach applies to any EFT of gravity where one can compute amplitudes. Therefore this is a power approach to derive new constraints for modified gravity scenarios.
5. **gravity is richer in higher dimensions!** Since the amplitude approach has been validated in higher-dimensions we can explore various interesting classical gravity physics in higher dimensions