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Two-loop Rational Terms

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in collaboration with

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based on

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Motivation for two-loop rational terms

- Theory predictions of $\mathcal{O}(1\%)$ precision for LHC processes require NNLO calculations
  $\Rightarrow$ **Automation of numerical two-loop calculations highly desirable**

- Higher-order calculations are usually performed in $D = 4 - 2\varepsilon$ dimensions
  $\rightarrow$ Regularisation of divergences in Feynman integrals

- Numerical tools, such as OpenLoops [Buccioni et al], Recola [Actis et al], MadLoop [Hirschi et al], construct the numerator of loop integrals in 4 dimensions
  $\rightarrow$ The $D - 4$ dimensional numerator part needs to be restored
  $\rightarrow$ One loop: Universal rational counterterms of type R_2 [Ossola, Papadopoulos, Pittau] from the interplay of $(D-4)$-dimensional numerator parts with $\frac{1}{\varepsilon}$ UV poles

  $\rightarrow$ **This talk: Two-loop rational terms of UV origin**

Motivation to present two-loop rational terms here

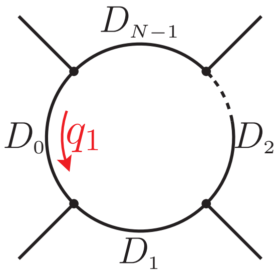
Thanks to Kostja Chetyrkin for many inspiring QFT lessons,
in particular about the R-operation and mass expansions.

Thanks and Congratulations to all the celebrated scientists at this workshop!

I. Rational terms at one loop

Generic one-loop diagram γ in $D = 4 - 2\varepsilon$ dimensions

$$\bar{\mathcal{A}}_{1,\gamma} = \int \mathrm{d}\bar{q}_1 \, \frac{\bar{\mathcal{N}}(\bar{q}_1)}{D_0(\bar{q}_1)\cdots D_{N-1}(\bar{q}_1)} \quad =$$



with $D_k(\bar{q}_1) = (\bar{q}_1 + p_k)^2 - m_k^2$,

$$\int \mathrm{d}\bar{q}_1 = \mu_0^{2\varepsilon} \int \frac{\mathrm{d}^D \bar{q}_1}{(2\pi)^D}$$

(D-dim loop momentum $\bar{q}_1$, masses m_k and 4-dim external momenta p_k)

Split D-dimensional numerator

$$\underbrace{\bar{\mathcal{N}}(\bar{q}_1)}_{D-\text{dim}} = \underbrace{\mathcal{N}(q_1)}_{4-\text{dim}} + \underbrace{\tilde{\mathcal{N}}(\bar{q}_1)}_{(D-4)-\text{dim}} \qquad \text{with} \begin{cases} \bar{q}_i &= q_i + \tilde{q}_i \\ \bar{\gamma}^{\bar{\mu}} &= \gamma^{\mu} + \tilde{\gamma}^{\bar{\mu}} \\ \bar{g}^{\bar{\mu}\bar{\nu}} &= g^{\mu\nu} + \tilde{g}^{\bar{\mu}\bar{\nu}} \end{cases}$$

$$\Rightarrow \bar{\mathcal{A}}_{1,\gamma} = \mathcal{A}_{1,\gamma} + \delta\mathcal{R}_{1,\gamma} \qquad \text{with} \qquad \delta\mathcal{R}_{1,\gamma} = \int \mathrm{d}\bar{q}_1 \, \frac{\tilde{N}(\bar{q}_1)}{D_0(\bar{q}_1)\cdots D_{N-1}(\bar{q})},$$

$\Rightarrow$ Interaction of $\tilde{N}$ with $\frac{1}{\varepsilon}$ **UV** poles leads to a **finite set of universal local counterterms** $\delta\mathcal{R}_{1,\gamma}$ in any renormalisable theory [Ossola, Papadopoulos, Pittau]

IR divergences do not generate rational terms at one loop [Bredensetein, Denner, Dittmaier, Pozzorini]

Tadpole decomposition

Capture UV divergences via **tadpole decomposition** of propagator denominators $D_k = (\bar{q}_1 + p_k)^2 - m_k^2$ with one auxiliary mass scale M^2 [Chetyrkin, Misiak, Münz]

$$\underbrace{\frac{1}{D_k(\bar{q}_1)}}_{\sim\frac{1}{\bar{q}_1^2}} = \underbrace{\frac{1}{\bar{q}_1^2 - M^2}}_{\sim\frac{1}{\bar{q}_1^2}} + \underbrace{\frac{\Delta_k(\bar{q}_1)}{\bar{q}_1^2 - M^2}\,\frac{1}{D_k(\bar{q}_1)}}_{\sim\frac{1}{\bar{q}_1^3}}$$

with polynomial in external momenta and masses $\Delta_k(\bar{q}_1) = -2\bar{q}_1 \cdot p_k - p_k^2 + m_k^2 - M^2$

Recursive application $\rightarrow$ **Tadpole expansion** up to degree of divergence X and UV-finite remainder:

$$\frac{1}{D_k(\bar{q}_1)} = \left(\mathbf{S}_X^{(1)} + \mathbf{F}_X^{(1)}\right)\frac{1}{D_k(\bar{q}_1)} := \sum_{\sigma=0}^{X} \frac{[\Delta_k(\bar{q}_1)]^\sigma}{\left(\bar{q}_1^2 - M^2\right)^{\sigma+1}} + \mathcal{O}\left(\frac{1}{\bar{q}_1^{3+X}}\right)$$

Tadpole expansion for chains of propagators:

$$\mathbf{S}_X^{(i)} \frac{1}{D_0(\bar{q}_i)\cdots D_{N-1}(\bar{q}_i)} = \sum_{\sigma=0}^{X} \frac{\Delta^{(\sigma)}(\bar{q}_i)}{\left(\bar{q}_i^2 - M^2\right)^{N+\sigma}}, \qquad \mathbf{F}_X^{(i)} = 1 - \mathbf{S}_X^{(i)}$$

where $\Delta^{(\sigma)}(\bar{q}_1)$ is a polynomial built from the Δ_k $(k = 0, \ldots, N-1)$.

$\Rightarrow$ **Isolate UV-divergences in pure tadpole integrals** with one scale M^2 and **polynomial dependence on external momenta and masses**.

Computing one-loop UV counterterms

One-loop diagram γ in D dimensions:

$$\bar{\mathcal{A}}_{1,\gamma} = \sum_{r=0}^{R} \Big(\underbrace{\mathcal{N}_{\mu_1\cdots\mu_r}}_{4\text{-dim}} + \underbrace{\tilde{\mathcal{N}}_{\bar{\mu}_1\cdots\bar{\mu}_r}}_{\varepsilon\text{-dim}} \Big) \underbrace{T_N^{\bar{\mu}_1\cdots\bar{\mu}_r}}_{D\text{-dim}} \quad \text{with} \quad T_N^{\bar{\mu}_1\cdots\bar{\mu}_r} = \int \mathrm{d}\bar{q}_1 \frac{\bar{q}_1^{\mu_1}\cdots\bar{q}_1^{\mu_r}}{D_0(\bar{q}_1)\cdots D_{N-1}(\bar{q}_1)}$$

Extract UV counterterm with K-operator (MS-like scheme):

$$f_L(\varepsilon) = \sum_{k=1}^{L} \frac{f_{L,k}}{\varepsilon^k} + f_{L,\text{finite}} \quad \Rightarrow \quad \mathbf{K} f_L(\varepsilon) = \sum_{k=1}^{L} \frac{f_{L,k}}{\varepsilon^k}$$

using the tadpole expansion $\mathbf{K}\,\bar{\mathcal{A}}_{1,\gamma} = \mathbf{K}\,\mathbf{S}_X^{(1)}\bar{\mathcal{A}}_{1,\gamma}$

$$\mathbf{K}\,\bar{\mathcal{A}}_{1,\gamma} = \mathbf{K} \sum_{r=0}^{R} \Big(\mathcal{N}_{\mu_1\cdots\mu_r} + \tilde{\mathcal{N}}_{\bar{\mu}_1\cdots\bar{\mu}_r} \Big) T_N^{\bar{\mu}_1\cdots\bar{\mu}_r} = \sum_{r=0}^{R} \mathcal{N}_{\mu_1\cdots\mu_r}\,\mathbf{K}\,T_N^{\bar{\mu}_1\cdots\bar{\mu}_r} = \mathbf{K}\,\mathcal{A}_{1,\gamma} = -\delta Z_{1,\gamma}$$

$\Rightarrow$ **Extend this operator to also extract the interplay of $\tilde{\mathcal{N}}$ with the same UV poles**

$$\bar{\mathbf{K}}\,\bar{\mathcal{A}}_{1,\gamma} := \sum_{r=0}^{R} \Big(\mathcal{N}_{\mu_1\cdots\mu_r} + \tilde{\mathcal{N}}_{\mu_1\cdots\mu_r} \Big)\,\mathbf{K}\,T_N^{\bar{\mu}_1\cdots\bar{\mu}_r} = -\delta Z_{1,\gamma} + \delta\mathcal{R}_{1,\gamma}$$

$\Rightarrow$ **Capture full UV pole contribution**

One-loop rational terms from tadpole integrals

One-loop master formula for computing a renormalised D-dim amplitude

$$\mathbf{R}\,\bar{\mathcal{A}}_{1,\gamma} = \bar{\mathcal{A}}_{1,\gamma} + \delta Z_{1,\gamma} = \mathcal{A}_{1,\gamma} + \delta Z_{1,\gamma} + \delta\mathcal{R}_{1,\gamma}$$

Generic method to compute $\delta\mathcal{R}_{1,\gamma}$ from tadpole integrals with one (auxiliary) scale M^2:

$$\delta\mathcal{R}_{1,\gamma} = (\bar{\mathbf{K}} - \mathbf{K})\bar{\mathcal{A}}_{1,\gamma} = \sum_{r=0}^{R} \left[\bar{\mathcal{N}}_{\bar{\mu}_1\cdots\bar{\mu}_r} - \mathcal{N}_{\mu_1\cdots\mu_r}\right] \mathbf{K}\, T_N^{\bar{\mu}_1\cdots\bar{\mu}_r}$$

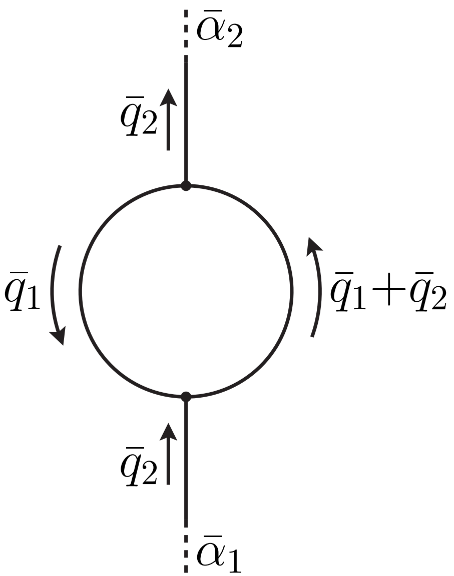
using the tadpole expansion $\mathbf{K}\, T_N^{\bar{\mu}_1\cdots\bar{\mu}_r} = \mathbf{K}\,\mathbf{S}_X^{(1)}\, T_N^{\bar{\mu}_1\cdots\bar{\mu}_r} = \mathbf{K} \sum_{\sigma=0}^{X} \int \mathrm{d}\bar{q}_1 \frac{\bar{q}_1^{\mu_1}\cdots\bar{q}_1^{\mu_r}\Delta^{(\sigma)}}{(\bar{q}_1 - M^2)^{N+\sigma}}$

- Dependence on external momenta and masses resides solely in numerator $\left(\mathcal{N}(q_1) + \tilde{\mathcal{N}}(\bar{q}_1)\right)\Delta^{(\sigma)}$ in polynomial form $\Rightarrow$ **Proof that $\delta\mathcal{R}_{1,\gamma}$ is indeed a rational term**.

- $\delta\mathcal{R}_{1,\gamma}$ stem from same poles as $\delta Z_{1,\gamma}$ $\Rightarrow$ finite set of rational terms in any renormalisable theory.

- Results for $\delta Z_{1,\gamma}$ and $\delta\mathcal{R}_{1,\gamma}$ **independent of** M^2 since $\boxed{\mathbf{K}\, T_N^{\bar{\mu}_1\cdots\bar{\mu}_r} = \mathbf{K}\,\mathbf{S}_X^{(1)}\, T_N^{\bar{\mu}_1\cdots\bar{\mu}_r}}$ is exact, and the l.h.s. (original denominators $D_k(\bar{q}_1)$) is M^2-independent.

One-loop subdiagrams in two-loop diagrams



Numerator in D dimension (including $\bar q_2$ and Lorentz indices $\bar\alpha = (\bar\alpha_1, \bar\alpha_2)$)

$$\begin{aligned}\bar{\mathbf{K}}\, \bar{\mathcal{A}}^{\bar\alpha}_{1,\gamma} &= \sum_{r=0}^{R} \left(\mathcal{N}^{\bar\alpha}_{\mu_1\cdots\mu_r}(\bar q_2) + \tilde{\mathcal{N}}^{\bar\alpha}_{\bar\mu_1\cdots\bar\mu_r}(\bar q_2) \right) \mathbf{K}\, T_N^{\bar\mu_1\cdots\bar\mu_r}(\bar q_2) \\ &= -\delta Z^{\bar\alpha}_{1,\gamma}(\bar q_2) + \delta\mathcal{R}^{\bar\alpha}_{1,\gamma}(\bar q_2)\end{aligned}$$

Numerator in 4 dimensions (but D-dim $\bar q_2$ in denominator)

$$\begin{aligned}\bar{\mathbf{K}}\, \mathcal{A}^{\alpha}_{1,\gamma}(\bar q_2) &= \sum_{r=0}^{R} \mathcal{N}^{\alpha}_{\mu_1\cdots\mu_r}(q_2)\, \mathbf{K}\, T_N^{\bar\mu_1\cdots\bar\mu_r}(q_2 + \tilde q_2) \\ &= -\delta Z^{\alpha}_{1,\gamma}(q_2) - \delta\tilde Z^{\alpha}_{1,\gamma}(\tilde q_2)\end{aligned}$$

- $\delta Z^{\alpha}_{1,\gamma}(q_2) = -\sum_{r=0}^{R} \mathcal{N}^{\alpha}_{\mu_1\cdots\mu_r}(q_2)\, \mathbf{K}\, T_N^{\mu_1\cdots\mu_r}(q_2) \Rightarrow$ **Projection of UV counterterm to 4-dim**
- $\delta\tilde Z^{\alpha}_{1,\gamma}(\tilde q_2) = -\sum_{r=0}^{R} \mathcal{N}^{\alpha}_{\mu_1\cdots\mu_r}(q_2)\mathbf{K} \left(T_N^{\mu_1\cdots\mu_r}(q_2+\tilde q_2)) - T_N^{\mu_1\cdots\mu_r}(q_2) \right) \propto \frac{\tilde q_2^2}{\varepsilon}$

$\Rightarrow$ **New term** stemming from tadpole expansion of $T_N^{\mu_1\cdots\mu_r}(q_2+\tilde q_2))$. For one propagator:

$$\frac{1}{(\bar q_1 + q_2 + \tilde q_2)^2 - m^2} = \frac{1}{\bar q_1^2 - M^2} + \frac{-\left(q_2^2 + \tilde q_2^2\right) - 2\left(q_1\cdot q_2 + \tilde q_1 \cdot \tilde q_2\right) + m^2 - M^2}{(\bar q_1^2 - M^2)^2} + \ldots$$

UV subtracted one-loop subdiagrams

Fully UV subtracted amplitudes in D and 4 dimensions can be identified

$$\underbrace{\bar{\mathcal{A}}^{\bar{\alpha}}_{1,\gamma}(\bar{q}_2) - \bar{\mathbf{K}}\,\bar{\mathcal{A}}^{\bar{\alpha}}_{1,\gamma}(\bar{q}_2)}_{D\text{-dim}} = \underbrace{\mathcal{A}^{\alpha}_{1,\gamma}(q_2) - \bar{\mathbf{K}}\,\mathcal{A}^{\alpha}_{1,\gamma}(q_2)}_{4\text{-dim}} + \mathcal{O}(\varepsilon, \tilde{q}_2)$$

$\Rightarrow$ **Master formula for one-loop subdiagrams:**

$$\underbrace{\bar{\mathcal{A}}^{\bar{\alpha}}_{1,\gamma}(\bar{q}_2) + \delta Z^{\bar{\alpha}}_{1,\gamma}(\bar{q}_2)}_{D\text{-dim renormalisation}} = \underbrace{\mathcal{A}^{\alpha}_{1,\gamma}(q_2) + \delta Z^{\alpha}_{1,\gamma}(q_2) + \delta\tilde{Z}^{\alpha}_{1,\gamma}(\tilde{q}_2)}_{4\text{-dim renormalisation}} + \underbrace{\delta\mathcal{R}^{\alpha}_{1,\gamma}(q_2)}_{(D-4)\text{-dim restored}} + \mathcal{O}(\varepsilon, \tilde{q}_2)\,.$$

Extra UV counterterm $\delta\tilde{Z}^{\alpha}_{1,\gamma}(\tilde{q}_2) \propto \frac{\tilde{q}_2^2}{\varepsilon} = \mathcal{O}(1)$ non-zero only for quadratic divergence

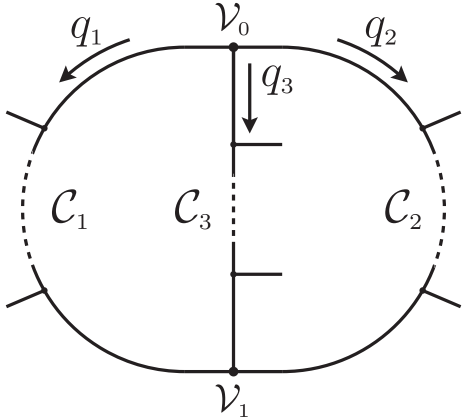
Example: Photon selfenergy (MS scheme)

$$\delta Z^{\bar{\alpha}}_{1,\gamma}(\bar{q}_2) = \left(\frac{\mathrm{i}\alpha}{4\pi}\right)\frac{4}{3\,\varepsilon}\left(\bar{q}_2^2 g^{\bar{\alpha}_1\bar{\alpha}_2} - \bar{q}_2^{\bar{\alpha}_1}\bar{q}_2^{\bar{\alpha}_2}\right), \quad \delta Z^{\alpha}_{1,\gamma}(q_2) = \left(\frac{\mathrm{i}\alpha}{4\pi}\right)\frac{4}{3\,\varepsilon}\left(q_2^2 g^{\alpha_1\alpha_2} - q_2^{\alpha_1}q_2^{\alpha_2}\right),$$

$$\delta\tilde{Z}^{\alpha}_{1,\gamma}(\tilde{q}_2) = \left(\frac{\mathrm{i}\alpha}{4\pi}\right)\frac{2}{3}\frac{\tilde{q}_2^2}{\varepsilon}\,g^{\alpha_1\alpha_2}, \quad \delta\mathcal{R}^{\alpha}_{1,\gamma}(q_2) = \left(\frac{\mathrm{i}\alpha}{4\pi}\right)\frac{2}{3}\,q_2^2\,g^{\alpha_1\alpha_2}$$

II. Rational terms at two loops

Generic irreducible two-loop diagram Γ consists of three chains $\mathcal{C}_i(\bar{q}_i)$ and two vertices $\mathcal{V}_0, \mathcal{V}_1$

$\bar{\mathcal{A}}_{2,\Gamma} =$  $= \int \mathrm{d}\bar{q}_1 \int \mathrm{d}\bar{q}_2 \, \frac{\bar{\mathcal{N}}(\bar{q}_1, \bar{q}_2)}{\mathcal{D}^{(1)}(\bar{q}_1)\,\mathcal{D}^{(2)}(\bar{q}_2)\,\mathcal{D}^{(3)}(\bar{q}_3)}\Big|_{\bar{q}_3 \to -(\bar{q}_1+\bar{q}_2)}$

$$\mathcal{D}^{(i)}(\bar{q}_i) = D_0^{(i)}(\bar{q}_i) \cdots D_{N_i-1}^{(i)}(\bar{q}_i)$$

$\leftarrow$ denominators $D_a^{(i)}(\bar{q}_i) = (\bar{q}_i + p_{ia})^2 - m_{ia}^2$

$$\bar{\mathcal{N}}(\bar{q}_1, \bar{q}_2) = \left[\bar{\mathcal{V}}_0 \bar{\mathcal{V}}_1\right] \cdot \prod_{i=1}^{3} \bar{\mathcal{N}}^{(i)}(\bar{q}_i)\Big|_{\bar{q}_3 \to -(\bar{q}_1+\bar{q}_2)}$$

$\Rightarrow$ **Factorisation of chains**

Three subdiagrams γ_i from chains $\mathcal{C}_j$ and $\mathcal{C}_k$

$\leftarrow$ $(i|jk)$ is a partition of (123)

$$\bar{\mathcal{A}}_{1,\gamma_i}(\bar{q}_i) = \int \mathrm{d}\bar{q}_j \, \frac{[\bar{\mathcal{V}}_0 \bar{\mathcal{V}}_1] \cdot \bar{\mathcal{N}}^{(j)}(\bar{q}_j)\,\bar{\mathcal{N}}^{(k)}(\bar{q}_k)}{\mathcal{D}^{(j)}(\bar{q}_j)\,\mathcal{D}^{(k)}(\bar{q}_k)}\Bigg|_{\bar{q}_k = -\bar{q}_i - \bar{q}_j}$$

$\leftarrow$ Superficial degree of divergence $X(\gamma_i)$

$X(\gamma_i) \geq 0 \;\Rightarrow$ **Subdivergence of Γ**

$X(\Gamma) \geq 0 \;\Rightarrow$ **Global divergence of Γ**

Complements Γ/γ_i

$$\bar{\mathcal{A}}_{2,\Gamma} = \bar{\mathcal{A}}_{1,\gamma_i} \cdot \bar{\mathcal{A}}_{1,\Gamma/\gamma_i} = \int \mathrm{d}\bar{q}_i \, \bar{\mathcal{A}}_{1,\gamma_i}(\bar{q}_i) \cdot \frac{\bar{\mathcal{N}}^{(i)}(\bar{q}_i)}{\mathcal{D}^{(i)}(\bar{q}_i)}$$

$\Rightarrow$ **Factorisation of γ_i and Γ/γ_i**

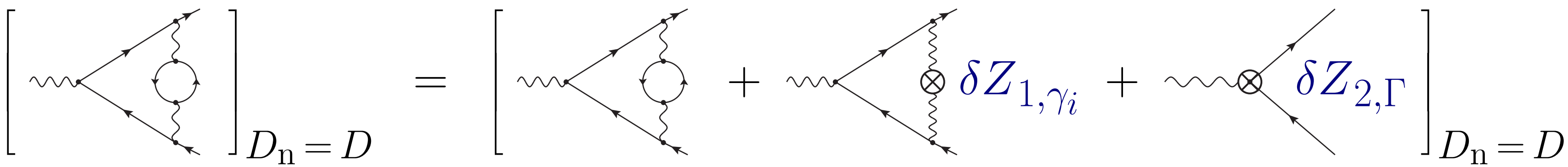
Renormalised D-dimensional amplitude

Renormalisation procedure based on R-operation [Bogoliubov, Parasiuk; Hepp; Zimmermann; Caswell, Kennedy]

$$\mathbf{R}\,\bar{\mathcal{A}}_{2,\Gamma} \;=\; \bar{\mathcal{A}}_{2,\Gamma} \;+\; \sum_{\gamma_i} \underbrace{\delta Z_{1,\gamma_i} \cdot \bar{\mathcal{A}}_{1,\Gamma/\gamma_i}}_{\text{subtract subdivergences}} \;+\; \underbrace{\delta Z_{2,\Gamma}}_{\text{subtract remaining local divergence}}$$

All amplitudes with numerator dimension $D_\mathrm{n} = D$

Example: R



R-operation: $\mathbf{R}\,\bar{\mathcal{A}}_{2,\Gamma} = (1 - \mathbf{K}_\mathrm{sub} - \mathbf{K}_\mathrm{loc})\,\bar{\mathcal{A}}_{2,\Gamma}$

1. **Subdivergences:** $\mathbf{K}_\mathrm{sub}\,\bar{\mathcal{A}}_{2,\Gamma} = \sum_{\gamma_i} (\mathbf{K}\,\bar{\mathcal{A}}_{1,\gamma_i}) \cdot \bar{\mathcal{A}}_{1,\Gamma/\gamma_i}, \qquad \mathbf{K}\,\bar{\mathcal{A}}_{1,\gamma_i} \;=\; -\delta Z_{1,\gamma_i}(\bar{q}_i)$
2. **Remaining divergence:** $\mathbf{K}_\mathrm{loc}\,\bar{\mathcal{A}}_{2,\Gamma} = \mathbf{K}\,(1 - \mathbf{K}_\mathrm{sub})\,\bar{\mathcal{A}}_{2,\Gamma} \;=\; -\delta Z_{2,\Gamma}$ ←local counterterm

Linear operations wrt sums of diagrams or sums of terms in a single diagram, e.g.

$$\mathbf{K}_\mathrm{sub}\left(\sum_\sigma \bar{\mathcal{A}}_{2,\Gamma_\sigma}\right) = \sum_\sigma \mathbf{K}_\mathrm{sub}\,\bar{\mathcal{A}}_{2,\Gamma_\sigma}$$

Goal: Computation of R $\bar{\mathcal{A}}_{2,\Gamma}$ from amplitudes with numerator dimension $D_\mathrm{n} = 4$

Master formula for computation of renormalised D-dimensional amplitude from amplitudes with loop numerators in $D_{\mathrm{n}} = 4$

$$\mathbf{R}\,\bar{\mathcal{A}}_{2,\Gamma} = \mathcal{A}_{2,\Gamma} + \sum_{\gamma}\Big(\underbrace{\delta Z_{1,\gamma} + \delta\tilde{Z}_{1,\gamma}}_{\text{subtract subdivergences}} + \underbrace{\delta\mathcal{R}_{1,\gamma}}_{\text{restore } \tilde{\mathcal{N}}\text{-terms from subdiagrams}} \Big)\cdot\mathcal{A}_{1,\Gamma/\gamma} + \Big(\underbrace{\delta Z_{2,\Gamma}}_{\text{subtract remaining local divergence}} + \underbrace{\delta\mathcal{R}_{2,\Gamma}}_{\text{restore remaining } \tilde{\mathcal{N}}\text{-term}} \Big)$$

- Amplitudes on rhs computed with numerators $\mathcal{N}(q_1,q_2) = \bar{\mathcal{N}}(\bar{q}_1,\bar{q}_2)\Big|_{\bar{g}^{\bar{\mu}\bar{\nu}}\to g^{\mu\nu},\ \bar{\gamma}^{\bar{\mu}}\to\gamma^{\mu},\ \bar{q}_i\to q_i}$
- Subtract (sub-)divergences and restore $\tilde{\mathcal{N}}$-terms (from subdiagrams and a remaining global) stemming from $\bar{\mathcal{N}}(\bar{q}_1,\bar{q}_2) = \mathcal{N}(q_1,q_2) + \tilde{\mathcal{N}}(\bar{q}_1,\bar{q}_2)$

The master formula implicitly defines

$$\delta\mathcal{R}_{2,\Gamma} = \underbrace{\Big(\bar{\mathcal{A}}_{2,\Gamma} + \sum_{\gamma}\delta Z_{1,\gamma}\cdot\bar{\mathcal{A}}_{1,\Gamma/\gamma}\Big)}_{\text{computed in } D_{\mathrm{n}}=D} - \underbrace{\Big(\mathcal{A}_{2,\Gamma} + \sum_{\gamma}\Big(\delta Z_{1,\gamma} + \delta\tilde{Z}_{1,\gamma} + \delta\mathcal{R}_{1,\gamma}\Big)\cdot\mathcal{A}_{1,\Gamma/\gamma}\Big)}_{\text{computed in } D_{\mathrm{n}}=4}$$

To be shown in the following:

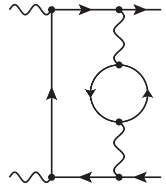
- $\delta\mathcal{R}_{2,\Gamma}$ is a **rational term**.
- In renormalisable theories there is a **finite set** of $\delta\mathcal{R}_{2,\Gamma} \neq 0$.
- **Generic method to compute $\delta\mathcal{R}_{2,\Gamma}$ once and for all.**

Case 1: Two-loop diagrams with no global divergence

Superficial degree of divergence of two-loop diagram $X(\Gamma) < 0$

$\Rightarrow$ **At most one subdivergence**, i.e. one subdiagram γ with $X(\gamma) \geq 0$

$$\mathbf{R}\,\bar{\mathcal{A}}_{2,\Gamma} = \underbrace{(1-\mathbf{K})\,\bar{\mathcal{A}}_{1,\gamma}}_{\text{divergence subtracted}} \cdot \underbrace{\bar{\mathcal{A}}_{1,\Gamma/\gamma}}_{\text{no divergence}} \overset{\text{e.g.}}{=} \Big[$$



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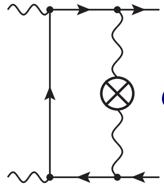
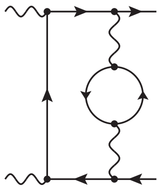


$$\delta Z_{1,\gamma}\Big]_{D_\mathrm{n}=D}$$

$$= \left(\bar{\mathcal{A}}_{1,\gamma} + \delta Z_{1,\gamma}\right) \cdot \mathcal{A}_{1,\Gamma/\gamma} + \mathcal{O}(\varepsilon)$$ $\leftarrow$ Project finite chain to $D_\mathrm{n} = 4$

$$= \left(\mathcal{A}_{1,\gamma} + \delta Z_{1,\gamma} + \delta \tilde{Z}_{1,\gamma} + \delta \mathcal{R}_{1,\gamma}\right) \cdot \mathcal{A}_{1,\Gamma/\gamma} + \mathcal{O}(\varepsilon)$$ $\leftarrow$ Express UV subtracted subdiagram in $D_\mathrm{n} = 4$

$$\overset{\text{e.g.}}{=} \Big[$$



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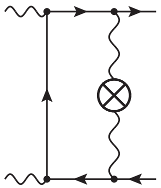


$$\left(\delta Z_{1,\gamma} + \delta \tilde{Z}_{1,\gamma} + \delta \mathcal{R}_{1,\gamma}\right)\Big]_{D_\mathrm{n}=4}$$

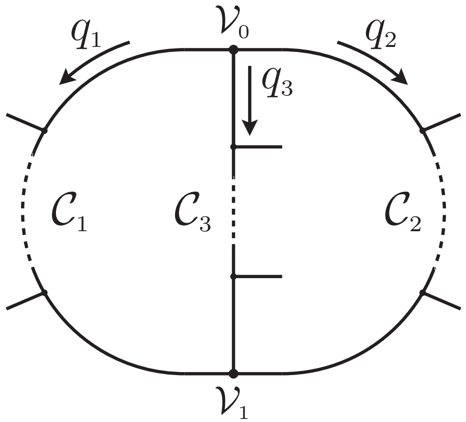
$\Rightarrow$ **Two-loop $\tilde{\mathcal{N}}$-contribution $\delta\mathcal{R}_{2,\Gamma} = 0$ and UV counterterm $\delta Z_{2,\Gamma} = 0$ for $X(\Gamma) < 0$.**

$\Rightarrow$ **Only globally divergent diagrams contribute to $\delta\mathcal{R}_{2,\Gamma}$ and $\delta Z_{2,\Gamma}$**

$\Rightarrow$ **Finite set of $\delta\mathcal{R}_{2,\Gamma}$ and $\delta Z_{2,\Gamma}$ counterterms in any renormalisable theory**

Case 2: Two-loop diagrams with a global divergence

$\bar{\mathcal{A}}_{2,\Gamma} =$



$$= \int \mathrm{d}\bar{q}_1 \int \mathrm{d}\bar{q}_2 \, \frac{\left[\bar{\mathcal{V}}_0 \bar{\mathcal{V}}_1\right] \prod_{i=1}^{3} \bar{\mathcal{N}}^{(i)}(\bar{q}_i)}{\mathcal{D}^{(1)}(\bar{q}_1)\, \mathcal{D}^{(2)}(\bar{q}_2)\, \mathcal{D}^{(3)}(\bar{q}_3)} \Bigg|_{\bar{q}_3 \to -(\bar{q}_1 + \bar{q}_2)}$$

Isolate all (sub)divergences via tadpole decomposition for every chain $\mathcal{C}_i$ ($i = 1, 2, 3$)

- Define **for each chain $\mathcal{C}_i$** the **maximum degree of divergence** of the full diagram ($X(\Gamma) \leq 0$) and the two sub-diagrams γ_j, γ_k involving this chain ← $(i|jk)$ is a partition of (123)

$$X_i = \text{Max}\left\{X(\Gamma), X(\gamma_j), X(\gamma_k)\right\}$$

- Decompose the diagram using the tadpole expansion operators (acting on individual chains)

$$\bar{\mathcal{A}}_{2,\Gamma} = \left(\mathbf{S}^{(1)}_{X_1} + \mathbf{F}^{(1)}_{X_1}\right)\left(\mathbf{S}^{(2)}_{X_2} + \mathbf{F}^{(2)}_{X_2}\right)\left(\mathbf{S}^{(3)}_{X_3} + \mathbf{F}^{(3)}_{X_3}\right)\bar{\mathcal{A}}_{2,\Gamma}$$

Case 2: Two-loop diagrams with a global divergence

$$\bar{\mathcal{A}}_{2,\Gamma} = \underbrace{\mathbf{S}^{(1)}_{X_1}\mathbf{S}^{(2)}_{X_2}\mathbf{S}^{(3)}_{X_3}\bar{\mathcal{A}}_{2,\Gamma}}_{\text{Global divergence}} + \underbrace{\sum_{i=1}^{3}\mathbf{F}^{(i)}_{X_i}\mathbf{S}^{(j)}_{X_j}\mathbf{S}^{(k)}_{X_k}\bar{\mathcal{A}}_{2,\Gamma}}_{\substack{\text{No global and at most}\\ \text{one subdivergence}}} + \underbrace{\left(\sum_{i=1}^{3}\mathbf{S}^{(i)}_{X_i}\mathbf{F}^{(j)}_{X_j}\mathbf{F}^{(k)}_{X_k} + \mathbf{F}^{(1)}_{X_1}\mathbf{F}^{(2)}_{X_2}\mathbf{F}^{(3)}_{X_3}\right)\bar{\mathcal{A}}_{2,\Gamma}}_{\text{No divergences}}$$

Exploit the linearity of the R-operation (i.e. of the operators $\mathbf{K}, \mathbf{K}_{\mathbf{sub}}, \mathbf{K}_{\mathbf{loc}}$) and apply the master formula (implicitly defining $\delta\mathcal{R}_{2,\Gamma}$)

$$\mathbf{R}\,\bar{\mathcal{A}}_{2,\Gamma} = \mathcal{A}_{2,\Gamma} + \sum_{\gamma_i}\Big(\underbrace{\delta Z_{1,\gamma_i} + \delta\tilde{Z}_{1,\gamma_i}}_{\substack{\text{subtract}\\ \text{subdivergences}}} + \underbrace{\delta\mathcal{R}_{1,\gamma_i}}_{\substack{\text{restore } \tilde{\mathcal{N}}\text{-terms}\\ \text{from subdiagrams}}}\Big)\cdot\mathcal{A}_{1,\Gamma/\gamma_i} + \Big(\underbrace{\delta Z_{2,\Gamma}}_{\substack{\text{subtract remaining}\\ \text{local divergence}}} + \underbrace{\delta\mathcal{R}_{2,\Gamma}}_{\substack{\text{restore remaining}\\ \tilde{\mathcal{N}}\text{-term}}}\Big)$$

to each term in

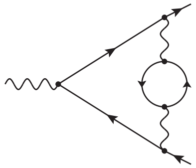
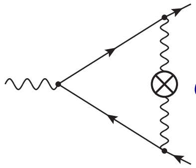
$$\mathbf{R}\bar{\mathcal{A}}_{2,\Gamma} = \mathbf{R}\left(\mathbf{S}^{(1)}_{X_1}\mathbf{S}^{(2)}_{X_2}\mathbf{S}^{(3)}_{X_3}\bar{\mathcal{A}}_{2,\Gamma}\right) + \underbrace{\sum_{i=1}^{3}\mathbf{R}\left(\mathbf{F}^{(i)}_{X_i}\mathbf{S}^{(j)}_{X_j}\mathbf{S}^{(k)}_{X_k}\bar{\mathcal{A}}_{2,\Gamma}\right) + \ldots}_{\text{case 1} \Rightarrow \text{ no contribution to } \delta\mathcal{R}_{2,\Gamma}}$$

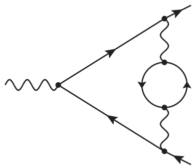
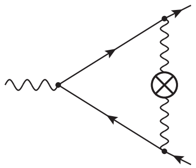
$\Rightarrow$ Only the pure tadpole term $\boxed{\mathbf{S}^{(1)}_{X_1}\mathbf{S}^{(2)}_{X_2}\mathbf{S}^{(3)}_{X_3}\bar{\mathcal{A}}_{2,\Gamma}}$ contributes to $\delta\mathcal{R}_{2,\Gamma}$ and $\delta Z_{2,\Gamma}$

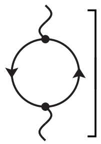
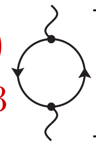
Two-loop rational terms from massive tadpoles with one scale M^2

Generic method to compute $\tilde{N}$-contribution

Example:

$$\delta\mathcal{R}_{2,\Gamma} = \Big[\mathbf{S}^{(1)}_{X_1}\mathbf{S}^{(2)}_{X_2}\mathbf{S}^{(3)}_{X_3}$$  $$+ \mathbf{S}^{(1)}_{X_1}$$  $$\delta Z_{1,\gamma_1} \Big]_{D_\mathrm{n}=D}$$

$$- \Big[\mathbf{S}^{(1)}_{X_1}\mathbf{S}^{(2)}_{X_2}\mathbf{S}^{(3)}_{X_3}$$  $$+ \mathbf{S}^{(1)}_{X_1}$$  $$\left(\delta Z_{1,\gamma_1} + \delta\tilde{Z}_{1,\gamma_1} + \delta\mathcal{R}_{1,\gamma_1}\right) \Big]_{D_\mathrm{n}=4}$$

exploiting $\bar{\mathbf{K}}\Big[\mathbf{S}^{(2)}_{X_2}\mathbf{S}^{(3)}_{X_3}$  $\Big]_{D_\mathrm{n}=D} = -\delta Z_{1,\gamma_1} + \delta\mathcal{R}_{1,\gamma_1}$, $\mathbf{K}\Big[\mathbf{S}^{(2)}_{X_2}\mathbf{S}^{(3)}_{X_3}$  $\Big]_{D_\mathrm{n}=4} = -\delta Z_{1,\gamma_1}(q_2) - \delta\tilde{Z}_{1,\gamma_1}(\tilde{q}_2)$

- **Only numerators depend on external momenta and masses in polynomial form**
  $\Rightarrow$ $\delta\mathcal{R}_{2,\Gamma}$ is indeed a rational term

- **Linearity of the R-operation allows generalisation to sets of diagrams**
  $\Rightarrow$ Compute finite set of rational counterterms $\delta\mathcal{R}_{1,\Gamma}$, $\delta\mathcal{R}_{2,\Gamma}$
  and UV counterterms $\delta\tilde{Z}_{1,\Gamma}$, $\delta Z_{2,\Gamma}$, $\delta Z_{2,\Gamma}$ only for the 1PI UV-divergent vertex functions Γ of any renormalisable model once and for all!

Calculation of two-loop rational terms

- Implementation in the GEFICOM [Chetyrkin, M.Z.] framework: QGRAF [Noguira] → Q2E+EXP [Seidesticker, Harlander, Steinhauser] → FORM [Vermaseren] code → MATAD [Steinhauser]
- Checked with independent in-house calculation using IBP identities [Chetyrkin, Tkachov]

Example: $\mathcal{L}_{\text{QED}} = \bar{\psi}(\mathrm{i}\gamma^\mu D_\mu - m)\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2\,(1-\eta)}(\partial^\mu A_\mu)^2$

Structure of rational term for fermion propagator in the $\overline{\text{MS}}$ scheme :

$$= \ \mathrm{i} \sum_{k=1}^{2} \left(\frac{\alpha}{4\pi}\right)^k \left[\delta\hat{\mathcal{R}}^{(\mathrm{P})}_{k,ee}\ \not{p}_{\alpha\beta} \ + \ \delta\hat{\mathcal{R}}^{(\mathrm{m})}_{k,ee}\ m\,\delta_{\alpha\beta}\right]$$

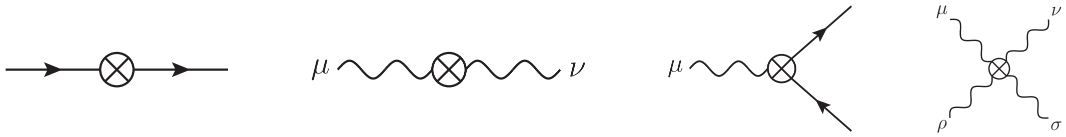
$$\delta\hat{\mathcal{R}}^{(\mathrm{P})}_{1,ee} \ = \ -1 + \frac{2}{3}\eta\,, \qquad \delta\hat{\mathcal{R}}^{(\mathrm{P})}_{2,ee} \ = \ \left(\frac{19}{18} - \frac{143}{72}\eta + \frac{11}{30}\eta^2\right)\frac{1}{\varepsilon} + \left(\frac{247}{108} + \frac{293}{864}\eta + \frac{391}{14400}\eta^2\right)\,,$$

$$\delta\hat{\mathcal{R}}^{(\mathrm{m})}_{1,ee} \ = \ 2 - \frac{1}{2}\eta\,, \qquad \delta\hat{\mathcal{R}}^{(\mathrm{m})}_{2,ee} \ = \ \left(-11 + \frac{41}{9}\eta - \frac{1}{4}\eta^2\right)\frac{1}{\varepsilon} + \left(-\frac{5}{6} - \frac{13}{54}\eta - \frac{7}{288}\eta^2\right)$$

- **Interaction of $\tilde{\mathcal{N}}$ with $\frac{1}{\varepsilon^2}$ poles leads to rational terms $\propto \frac{1}{\varepsilon}$**
- **Rational terms depend on the gauge parameter**
- **In general: Non-trivial dependence on the renormalisation scheme**
  → Can be fully expressed in terms of the one-loop UV counterterms $\delta Z_{1,\alpha}, \delta Z_{1,ee}, \ldots$

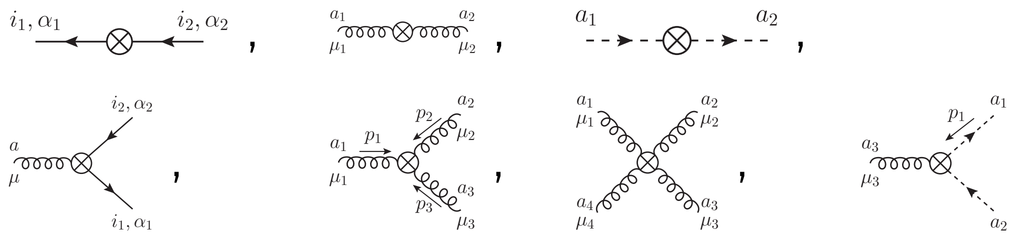
Status of two-loop rational terms

- Complete set of QED rational terms in MS schemes with full gauge parameter dependence



JHEP 05 (2020) 077 [arXiv:2001.11388]
[Pozzorini, Zhang, MZ]

- Renormalisation scheme dependence of rational terms in any model,
complete set of SU(N) and U(1) rational terms in a generic renormalisation scheme



JHEP 10 (2020) 016 [arXiv:2007.03713]
[Lang, Pozzorini, Zhang, MZ]

- Relation between rational terms in spontaneously broken models to the ones in the symmetric phase through vev expansions, complete set of rational terms for QCD corrections to the SM

[arXiv:2107.10288]
[Lang, Pozzorini, Zhang, MZ]

Current and future projects:

- Two-loop rational terms for the full SM

- $\tilde{\mathcal{N}}$-interactions with IR poles

Summary

- Renormalised amplitudes in D-dimensions can be computed from amplitudes with 4-dimensional numerators and a **finite set of universal UV and rational counterterm insertions**:

$$\mathbf{R}\,\bar{\mathcal{A}}_{1,\gamma} = \mathcal{A}_{1,\gamma} + \delta Z_{1,\gamma} + \delta\mathcal{R}_{1,\gamma}$$
$$\mathbf{R}\,\bar{\mathcal{A}}_{2,\Gamma} = \mathcal{A}_{2,\Gamma} + \sum_{\gamma}\left(\delta Z_{1,\gamma} + \delta\tilde{Z}_{1,\gamma} + \delta\mathcal{R}_{1,\gamma}\right)\cdot\mathcal{A}_{1,\Gamma/\gamma} + \left(\delta Z_{2,\Gamma} + \delta\mathcal{R}_{2,\Gamma}\right)$$

  $\Rightarrow$ **Numerical implementation in automated tools, e.g. OpenLoops, possible**

- **Generic method** to compute $\delta\mathcal{R}_{L,\gamma}$, $\delta\tilde{Z}_{1,\gamma}$ and $\delta Z_{L,\gamma}$ from simple tadpole integrals, which also serves as a **proof that they are rational**
- **Complete renormalisation scheme dependence available**
- **Connection between rational terms in symmetric theories and their spontaneously broken counterparts through systematic vev expansions**
- **Full set of rational terms at two-loop level for**
  - QED with full dependence on the gauge parameter
  - $SU(N)$ and $U(1)$ in any renormalisation scheme
  - QCD corrections to the SM

Backup

Details of the calculation

Generic formula for calculation of two-loop rational terms:

$$\delta\mathcal{R}_{2,\Gamma} = \underbrace{\left(\mathbf{S}^{(1)}_{X_1}\mathbf{S}^{(2)}_{X_2}\mathbf{S}^{(3)}_{X_3}\bar{\mathcal{A}}_{2,\Gamma} + \sum_{\gamma_i}\delta Z_{1,\gamma_i}\cdot\mathbf{S}^{(i)}_{X_i}\bar{\mathcal{A}}_{1,\Gamma/\gamma_i}\right)}_{\text{Tadpole expansion in } D_\mathrm{n}=D \text{ with subtracted subdivergences}}$$

$$- \underbrace{\left(\mathbf{S}^{(1)}_{X_1}\mathbf{S}^{(2)}_{X_2}\mathbf{S}^{(3)}_{X_3}\mathcal{A}_{2,\Gamma} + \sum_{\gamma_i}\left(\delta Z_{1,\gamma_i} + \delta\tilde{Z}_{1,\gamma} + \delta\mathcal{R}_{1,\gamma}\right)\cdot\mathbf{S}^{(i)}_{X_i}\mathcal{A}_{1,\Gamma/\gamma}\right)}_{\text{Tadpole expansion in } D_\mathrm{n}=4 \text{ with subtracted subdivergences and restored subdiagram rational terms}}$$

exploiting $\bar{\mathbf{K}}\left(\mathbf{S}^{(j)}_{X_j}\mathbf{S}^{(k)}_{X_k}\bar{\mathcal{A}}_{1,\gamma_i}\right) = -\delta Z_{1,\gamma_i} + \delta\mathcal{R}_{1,\gamma_i}$, $\quad\mathbf{K}\left(\mathbf{S}^{(j)}_{X_j}\mathbf{S}^{(k)}_{X_k}\mathcal{A}_{1,\gamma_i}\right) = -\delta Z_{1,\gamma_i}(q_2) - \delta\tilde{Z}_{1,\gamma_i}(\tilde{q}_2)$

Explicit structure:

$$\delta\mathcal{R}_{2,\Gamma} = \int \mathrm{d}\bar{q}_1 \int \mathrm{d}\bar{q}_2 \underbrace{\left[\bar{\mathcal{N}}(\bar{q}_1,\bar{q}_2) - \mathcal{N}(q_1,q_2)\right]}_{\text{2-loop numerator difference}} \underbrace{\prod_{i=1}^{3}\left[\sum_{\sigma_i=0}^{X_i}\frac{\Delta_i^{(\sigma_i)}(\bar{q}_i)}{(\bar{q}_i^2 - M^2)^{N_i+\sigma_i}}\right]_{q_3=-q_1-q_2}}_{\text{2-loop tadpole integral}} + \ldots$$

$$+ \sum_{i=1}^{3}\int \mathrm{d}\bar{q}_i \underbrace{\left[\delta Z_{1,\gamma_i}\cdot\bar{\mathcal{N}}^{(i)}(\bar{q}_i) - \left(\delta Z_{1,\gamma_i} + \delta\tilde{Z}_{1,\gamma_i}(\tilde{q}_i) + \delta\mathcal{R}_{1,\gamma_i}\right)\cdot\mathcal{N}^{(i)}(q_i)\right]}_{\text{1-loop numerator difference with CT insertions}} \underbrace{\left[\sum_{\sigma_i=0}^{X_i}\frac{\Delta_i^{(\sigma_i)}(\bar{q}_i)}{(\bar{q}_i^2 - M^2)^{N_i+\sigma_i}}\right]}_{\text{1-loop tadpole integral}}$$

Optimisations of the calculation of rational terms

The tadpole expansion of a single propagator

$$\mathbf{S}_X^{(i)} \frac{1}{D_k(\bar{q}_i)} = \sum_{\sigma=0}^{X} \frac{[\Delta_k(\bar{q}_i)]^\sigma}{(\bar{q}_1^2 - M^2)^{\sigma+1}} \quad \text{with} \quad \Delta_k(\bar{q}_i) = -p_k^2 - 2\bar{q}_i \cdot p_k + m_k^2 - M^2$$

is designed such that $(1-\mathbf{S}_X^{(i)})\frac{1}{D_k(\bar{q}_i)} \leq \mathcal{O}\left(\frac{1}{\bar{q}_i^{X+1}}\right)$. But it contains different orders of $\bar{q}_i$

$\Rightarrow$ Potentially many finite terms generated, which cancel in the difference

$$\delta\mathcal{R}_{2,\Gamma} = \mathbf{S}_{X_1}^{(1)} \mathbf{S}_{X_2}^{(2)} \mathbf{S}_{X_3}^{(3)} \left(\bar{\mathcal{A}}_{2,\Gamma} - \mathcal{A}_{2,\Gamma}\right) + \dots$$

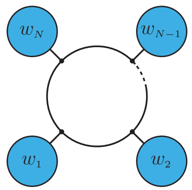
Optimisations (for details see **JHEP 10 (2020) 016** [arXiv:2007.03713]) [Lang, Pozzorini, Zhang, MZ]

- Power counting in external momenta and masses $\Rightarrow$ Restriction to mass dimension of the result
- Power counting in loop momenta $\bar{q}_i \Rightarrow$ Discard terms without UV (sub)divergences
- Taylor expansion trick:
  - Add the auxiliary mass M^2 in every propagator denominator D_k by hand
  - Generate the relevant terms of the tadpole expansion through a Taylor expansion in external masses and propagators
  - Perform a separate M^2-expansion **or** use the M^2-independence of the result to construct auxiliary M^2-counterterms order by order.

Reducible one-loop diagrams

Generic unrenormalised amplitude of a one-loop diagram γ

$$\bar{\mathcal{M}}_{1,\gamma} \quad = $$



$$= \bar{\mathcal{A}}_{1,\gamma}^{\sigma_1 \dots \sigma_N} \prod_{i=1}^{N} [w_i]_{\sigma_i} \,,$$

- Amplitude $\bar{\mathcal{A}}_{1,\gamma}$ of the 1PI amputated one-loop subdiagram of γ
- Factorised subtrees w_i (blue bubbles)

$$\mathbf{R}\,\bar{\mathcal{M}}_{1,\gamma} \quad = \quad \left(\mathbf{R}\,\bar{\mathcal{A}}_{1,\gamma}^{\sigma_1 \dots \sigma_N}\right) \prod_{i=1}^{N} [w_i]_{\sigma_i} \,.$$

In the 't Hooft–Veltman scheme all tree structures w_i are in 4 dimensions.

$\Rightarrow$ External momenta and indices of the 1PI amplitude $\bar{\mathcal{A}}_{1,\gamma}$ handled as 4-dimensional

$\Rightarrow$ **Tree structures do not generate rational terms** (even in other schemes due to factorisation)

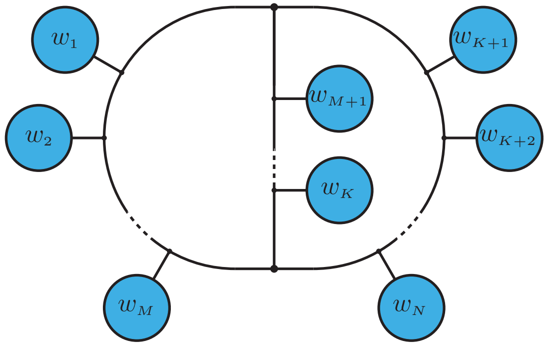
$\Rightarrow$ **Rational terms can be determined at the level of 1PI subdiagrams**

Reducible two-loop amplitudes

$$\mathbf{R}\,\bar{\mathcal{A}}_{2,\Gamma} \;=\; \mathcal{A}_{2,\Gamma} + \sum_{\gamma_i}\left(\delta Z_{1,\gamma_i} + \delta\tilde{Z}_{1,\gamma} + \delta\mathcal{R}_{1,\gamma}\right)\cdot \mathcal{A}_{1,\Gamma/\gamma} + \left(\delta Z_{2,\Gamma} + \delta\mathcal{R}_{2,\Gamma}\right)$$

which was derived for 1PI amplitudes is **applicable to any two-loop process** Γ due to the factorisation of external subtrees. Full diagram or process (sum of diagrams) Γ:

$$\bar{\mathcal{M}}_{2,\Gamma} \;=$$

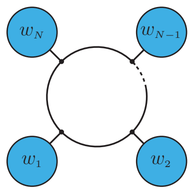


$$= \underbrace{\bar{\mathcal{A}}^{\sigma_1\cdots\sigma_N}_{2,\Gamma}}_{\text{1PI amputated amplitude}} \cdot \underbrace{\prod_{i=1}^{N}[w_i]_{\sigma_i}}_{\text{External subtrees (blue bubbles)}} ,$$

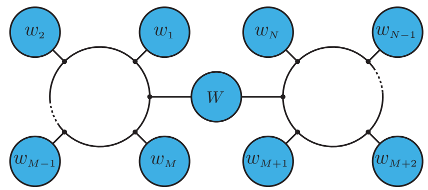
$$\mathbf{R}\,\bar{\mathcal{M}}_{2,\Gamma} \;=\; \left(\mathbf{R}\,\bar{\mathcal{A}}^{\sigma_1\cdots\sigma_N}_{2,\Gamma}\right)\prod_{i=1}^{N}[w_i]_{\sigma_i} .$$

$\Rightarrow$ **Tree structures do not generate rational terms**

Similar for amplitudes composed of 1PI one-loop subdiagrams,



and



$\Rightarrow$ **Finite set of rational counterterms stemming from 1PI UV-divergent vertex functions allow for two-loop computation of all processes**

Proof of master formula for one-loop subdiagrams

Fully UV subtracted amplitude in D dimensions:

$$\bar{\mathcal{A}}^{\bar{\alpha}}_{1,\gamma}(\bar{q}_2) - \bar{\mathbf{K}}\,\bar{\mathcal{A}}^{\bar{\alpha}}_{1,\gamma}(\bar{q}_2) \;=\; \sum_{r=0}^{R} \bar{\mathcal{N}}^{\bar{\alpha}}_{\bar{\mu}_1\cdots\bar{\mu}_r}(\bar{q}_2)\left[T_N^{\bar{\mu}_1\cdots\bar{\mu}_r}(\bar{q}_2) - \mathbf{K}\,T_N^{\bar{\mu}_1\cdots\bar{\mu}_r}(\bar{q}_2)\right].$$

Fully UV subtracted amplitude in 4 dimensions:

$$\mathcal{A}^{\alpha}_{1,\gamma}(q_2) - \bar{\mathbf{K}}\,\mathcal{A}^{\alpha}_{1,\gamma}(q_2) \;=\; \sum_{r=0}^{R} \mathcal{N}^{\alpha}_{\mu_1\cdots\mu_r}(q_2)\left[T_N^{\mu_1\cdots\mu_r}(\bar{q}_2) - \mathbf{K}\,T_N^{\mu_1\cdots\mu_r}(\bar{q}_2)\right].$$

Since all UV poles are cancelled in $[\ldots]$ we find

$$\left[T_N^{\bar{\mu}_1\cdots\bar{\mu}_r}(\bar{q}_2) - \mathbf{K}\,T_N^{\bar{\mu}_1\cdots\bar{\mu}_r}(\bar{q}_2)\right] = \left[T_N^{\mu_1\cdots\mu_r}(\bar{q}_2) - \mathbf{K}\,T_N^{\mu_1\cdots\mu_r}(\bar{q}_2)\right] + \mathcal{O}(\varepsilon, \tilde{q}_2).$$

From this follows

$$\boxed{\bar{\mathcal{A}}^{\bar{\alpha}}_{1,\gamma}(\bar{q}_2) - \bar{\mathbf{K}}\,\bar{\mathcal{A}}^{\bar{\alpha}}_{1,\gamma}(\bar{q}_2) = \mathcal{A}^{\alpha}_{1,\gamma}(q_2) - \bar{\mathbf{K}}\,\mathcal{A}^{\alpha}_{1,\gamma}(q_2) + \mathcal{O}(\varepsilon, \tilde{q}_2)}$$

Practical calculation of two-loop rational terms

$$
\begin{aligned}
\delta\mathcal{R}_{2,\Gamma} &= \left(\bar{\mathcal{A}}_{2,\Gamma} + \sum_{\gamma_i} \delta Z_{1,\gamma_i} \cdot \bar{\mathcal{A}}_{1,\Gamma/\gamma_i}\right) - \left(\mathcal{A}_{2,\Gamma} + \sum_{\gamma_i}\left(\delta Z_{1,\gamma_i} + \delta\tilde{Z}_{1,\gamma} + \delta\mathcal{R}_{1,\gamma}\right) \cdot \mathcal{A}_{1,\Gamma/\gamma}\right) \\
&= \int \mathrm{d}\bar{q}_1 \int \mathrm{d}\bar{q}_2 \left[\bar{\mathcal{N}}(\bar{q}_1,\bar{q}_2) - \mathcal{N}(q_1,q_2)\right] \left[\prod_{i=1}^{3} \mathbf{S}_{X_i}^{(i)} \frac{1}{\mathcal{D}^{(i)}(\bar{q}_i)}\right]_{q_3=-q_1-q_2} \\
&\quad + \sum_{i=1}^{3} \int \mathrm{d}\bar{q}_i \left[\delta Z_{1,\gamma_i}(\bar{q}_i) \cdot \bar{\mathcal{N}}^{(i)}(\bar{q}_i)\right. \\
&\quad \left. - \left(\delta Z_{1,\gamma_i}(q_i) + \delta\tilde{Z}_{1,\gamma_i}(\tilde{q}_i) + \delta\mathcal{R}_{1,\gamma_i}(q_i)\right) \cdot \mathcal{N}^{(i)}(q_i)\right] \mathbf{S}_{X_i}^{(i)} \left(\frac{1}{\mathcal{D}^{(i)}(\bar{q}_i)}\right) \\
&= \int \mathrm{d}\bar{q}_1 \int \mathrm{d}\bar{q}_2 \sum_{r_1=0}^{R_1} \sum_{r_2=0}^{R_2} \left[\bar{\mathcal{N}}_{\bar{\mu}_1\cdots\bar{\mu}_{r_1}\bar{\nu}_1\cdots\bar{\nu}_{r_2}} - \mathcal{N}_{\mu_1\cdots\mu_{r_1}\nu_1\cdots\nu_{r_2}}\right] \times \\
&\quad \times \left[\sum_{\sigma_1=0}^{X_1} \sum_{\sigma_2=0}^{X_2} \sum_{\sigma_3=0}^{X_3} \frac{\bar{q}_1^{\mu_1} \cdots \bar{q}_1^{\mu_{r_1}} \bar{q}_2^{\nu_1} \cdots \bar{q}_2^{\nu_{r_2}} \Delta_1^{(\sigma_1)}(\bar{q}_1) \Delta_2^{(\sigma_2)}(\bar{q}_2) \Delta_3^{(\sigma_3)}(\bar{q}_3)}{(\bar{q}_1^2 - M^2)^{N_1+\sigma_1} (\bar{q}_2^2 - M^2)^{N_2+\sigma_2} (\bar{q}_3^2 - M^2)^{N_3+\sigma_3}}\right]_{q_3=-q_1-q_2} \\
&\quad + \ldots
\end{aligned}
$$

$\Rightarrow$ Computation with D-dim tensor integrals and differences of numerator coefficients in $D_\mathrm{n} = D$ and $D_\mathrm{n} = 4$ dimensions

Renormalisation scheme dependence of two-loop rational counterterms

Master formula for two-loop amplitudes:

$$\mathbf{R}^{(Y)}\,\bar{\mathcal{A}}_{2,\Gamma} \;=\; \mathcal{A}_{2,\Gamma} + \sum_{\gamma}\left(\delta Z^{(Y)}_{1,\gamma} + \delta\tilde{Z}^{(Y)}_{1,\gamma} + \delta\mathcal{R}^{(Y)}_{1,\gamma}\right)\cdot\mathcal{A}_{1,\Gamma/\gamma} + \left(\delta Z^{(Y)}_{2,\Gamma} + \delta\mathcal{R}^{(Y)}_{2,\Gamma}\right)$$

with

$$\delta\mathcal{R}^{(Y)}_{2,\Gamma} \;=\; \underbrace{(t^{\varepsilon}_{Y})^2\,\delta\mathcal{R}^{(\mathrm{MS})}_{2,\Gamma}}_{\substack{\text{rescaling of two-loop}\\ \text{rational term (with } \frac{1}{\varepsilon}\text{-poles)}}} + \underbrace{(t^{\varepsilon}_{Y})^2\,D^{\Delta Y}_{1}\,\delta\mathcal{R}^{(\mathrm{MS})}_{1,\Gamma}}_{\substack{\text{multiplicative renormalisation}\\ \text{of one-loop rational term}}} + \underbrace{\delta\mathcal{K}^{(\Delta Y)}_{2,\Gamma}}_{\substack{\text{non-trivial remainder}\\ \text{from 4-dim numerator}}}$$

where $\delta\mathcal{K}^{(\Delta Y)}_{2,\Gamma}$ stems from the subtlety that the multiplicative renormalisation of a one-loop amplitude $\mathcal{A}_{1,\Gamma}$ **after projection** to numerator dimension $D_{\mathrm{n}} = 4$ does not give the same result as a counterterm insertion with **subsequent projection** to $D_{\mathrm{n}} = 4$:

$$\delta\mathcal{K}^{(\Delta Y)}_{2,\Gamma} \;=\; (t^{\varepsilon}_{Y})\left(D^{(\Delta Y)}_{1}\mathcal{A}_{1,\Gamma} - \sum_{\gamma}\delta Z^{(\Delta Y)}_{1,\gamma}\cdot\mathcal{A}_{1,\Gamma/\gamma}\right) \neq 0$$

But $\delta\mathcal{K}^{(\Delta Y)}_{2,\Gamma}$ can be expressed through one-loop renormalisation constants and a small set of universal scheme-independent counterterms (presented in **JHEP 10 (2020) 016** [arXiv:2007.03713]):

$$\delta\mathcal{K}^{(\Delta Y)}_{2,\Gamma} \;=\; \sum_{\chi}\delta\mathcal{Z}^{(\Delta Y)}_{1,\chi}\,\delta\hat{\mathcal{K}}^{(\chi)}_{1,\Gamma}$$

⇒ **Full renormalisation scheme dependence of two-loop rational terms available**

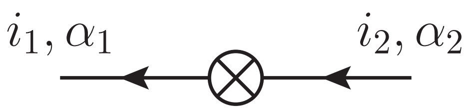
Two-loop rational terms for SU(N) and U(1) in a generic scheme

- Rational terms for a 1PI vertex function Γ depend on the scale factor t^ε and the renormalisation constants $\mathcal{Z}_\chi = 1 + \Sigma_{k=1}^{\infty} \left(\frac{\alpha\, t^\varepsilon}{4\pi}\right)^k \delta\hat{\mathcal{Z}}_{k,\chi}$ for parameters $\chi = \alpha,\, m_f, \lambda$ and fields $\chi = f,\, A,\, u$

- Set the gauge parameter $\lambda = 1$ (Feynman gauge), but keep generic renormalisation $\mathcal{Z}_{\mathrm{gp}} = \frac{\mathcal{Z}_A}{\mathcal{Z}_\lambda}$

- Express result in terms of Casimirs $C_{\mathrm{F}}, C_{\mathrm{A}}$ and fundamental trace T_{F} and dimension N

Two-point function of a fermion f



$$= \; \mathrm{i} \; \underbrace{\delta_{i_1 i_2}}_{\text{gauge group structure}} \left\{ \sum_{k=1}^{2} \left(\frac{\alpha_s\, t^\varepsilon}{4\pi}\right)^k \left[\delta\hat{\mathcal{R}}^{(\mathrm{P})}_{k,\mathrm{ff}}\, \not{p}_{\alpha_1\alpha_2} + \delta\hat{\mathcal{R}}^{(\mathrm{m})}_{k,\mathrm{ff}}\, m_f\, \delta_{\alpha_1\alpha_2} \right] \right\},$$

$$\delta\hat{\mathcal{R}}^{(\mathrm{P})}_{1,\mathrm{ff}} = -C_{\mathrm{F}}\,,$$

$$\delta\hat{\mathcal{R}}^{(\mathrm{P})}_{2,\mathrm{ff}} = \left(\frac{7}{6} C_{\mathrm{F}}^2 - \frac{61}{36} C_{\mathrm{A}}\, C_{\mathrm{F}} + \frac{5}{9} T_{\mathrm{F}}\, n_{\mathrm{f}}\, C_{\mathrm{F}}\right) \frac{1}{\varepsilon} + \left(\frac{43}{36} C_{\mathrm{F}}^2 - \frac{1087}{216} C_{\mathrm{A}}\, C_{\mathrm{F}} + \frac{59}{54} T_{\mathrm{F}}\, n_{\mathrm{f}}\, C_{\mathrm{F}}\right)$$

$$- C_{\mathrm{F}} \left(\underbrace{\delta\hat{\mathcal{Z}}_{1,\alpha_s} + \frac{2}{3}\,\delta\hat{\mathcal{Z}}_{1,f} - \frac{2}{3}\,\delta\hat{\mathcal{Z}}_{1,\mathrm{gp}}}_{\text{Renormalisation scheme dependendent}} \right)$$

Similarly for $\delta\hat{\mathcal{R}}^{(\mathrm{m})}_{1,\mathrm{ff}}, \mathcal{R}^{(\mathrm{m})}_{2,\mathrm{ff}}$