Quantization of 2+1 dimensional gravity and problem of central singularity in the BTZ black hole

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Introduction

- General relativity in 2+1 dimensions coupled to point particles is exactly solvable at the classical and quantum levels (G. 't Hooft).
- We perform canonical analysis of a model in which gravity in 2+1 spacetime dimensions with a negative cosmological constant is coupled to a cylindrically symmetric dust shell.
- We find canonical variables providing the global chart for the reduced phase space of the model.
- We perform quantization of the model in the vicinity of the horizon using momentum (Euler angle) representation and find the spectrum of shell radius.
- We also calculate quantum transition amplitudes between different regions of the Penrose diagram.

BTZ Black Hole

- The BTZ black hole, Bañados, Teitelboim, Zanelli 1992, in "Schwarzschild" coordinates is described by the metric

$$ds^2 = -(N)^2 dt^2 + N^{-2} dr^2 + R^2 d\phi, \tag{1}$$

with lapse function

$$N = \left(1 - 2m + \frac{R^2}{\ell^2}\right)^{1/2}. \tag{2}$$

The parameters m is the ADM mass, which is related to the mass M in original BTZ conventions as $M = 2m - 1$.

- The metric 1 satisfies the ordinary vacuum field equations of (2+1)-dimensional general relativity with a cosmological constant $\Lambda = -1/\ell^2$.
- BTZ black holes are locally isometric to anti-de Sitter space ADS^2.

Action principle

- The basic variable is SO(2,2)-connection A_μ^{AB}, where $A, B = 0..3$. Here $A_\mu^{3a} = e_\mu^a / l$ is the triad, where $l = 1/\sqrt{\Lambda}$, and $A_\mu^{ab} = \omega_\mu^{ab}$ is the Lorentzian connection, where $a, b = 0..2$.
- The total action consists of gravity action in the Chern-Simons form and the shell action

$$S = \frac{l}{8\pi} \int_M d^3x \epsilon^{\mu\nu\rho} \langle A_\mu, (\partial_\nu A_\rho + \frac{2}{3} A_\nu A_\rho) \rangle + S_{shell}, \tag{3}$$

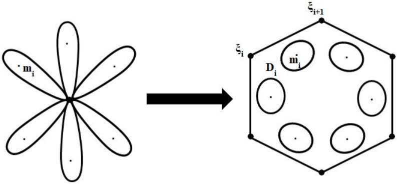
where $A_\mu = \Gamma_{AB} A_\mu^{AB}$ is so(2,2) connection, and $\langle , \rangle$ is a bilinear form on so(2,2) algebra, the Newton constant G is taken to be 1.
- The shell is discretized (represented as an ensemble of N particles)

$$S_{shell} = \sum_i^N \int_{l_i} Tr(K_i A_\mu) dx^\mu, \tag{4}$$

where l_i is i-th particle worldline and $K_i = m_i \Gamma_{03}$ – a fixed element of so(2,2)-algebra, M_i is the mass of i-th particle.

Phase space reduction

- Cut spacetime into N regions (discs) each containing one particle and an outer region, containing no particles (Alekseev, Malkin)



- Apply the results of 't Hooft, Matschull, Welling for each particle: solve the constraints, plug the solution back into the action. Thus the symplectic form for the disk collapses to its boundary:

$$\Omega_{D_i} = \int_{\partial D_i} d\phi \langle \nabla_\phi (\delta g_i g_i^{-1}) \wedge \delta g_i g_i^{-1} \rangle. \tag{5}$$

Phase space reduction

- For cylindrically symmetric arrangement of the particles, the sum of the symplectic form for each particle is combined into a single form

$$\Omega_{full} = \langle \delta h_0 h_0^{-1}, \wedge U^{-1} \delta U \rangle, \quad (6)$$

where $U = \prod_{j=0}^{N} u_i$, the holonomy around the full shell, is composed from holonomies around particles, $u_i = g_i(0) \exp(M_i \Gamma_{12}) g_i(0)^{-1}$, and h_0 is the translational part of $g_0(0)$.

- In the neighborhood of the horizon where $\delta h_0 h_0^{-1} \ll 1$ the symplectic form simplifies further:

$$\Omega_{full} = \frac{1}{l} \langle \delta X, \wedge U^{-1} \delta U \rangle, \quad (7)$$

where $X = l \log(h_0)$.

- The holonomy U plays the role of momentum of the shell. It has two independent components: spacial rotation and boost in radial direction. It has ADS^2-geometry.
- U is subject to the Hamiltonian constraint.

Momentum space

- The holonomy U provides a global chart for the entire momentum space. It has ADS^2 geometry.

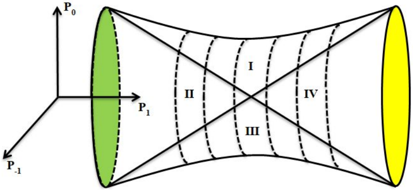


Figure: ADS-momentum space and its four regions. (p_{-1}, p_0, p_1 are coordinates of three dimensional flat space in which ADS^2 is embedded)

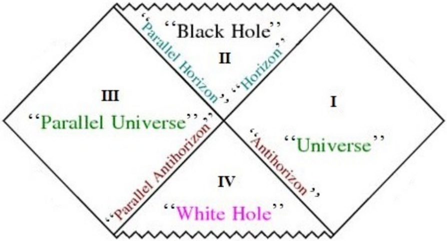


Figure: Corresponding four regions on the Penrose diagram

Hamiltonian constraint

- Deriving the constraint for U by making use of cylindrical symmetry:

$$U = \prod_{i=0}^{N} \underbrace{C_i^{-1}}_{transport\ \ to\ \ the\ \ origin} \underbrace{\exp(\frac{2\pi M \Gamma^{12}}{N})}_{particle\ \ holonomy} \underbrace{C_i}_{transport\ \ back} \quad (8)$$

Here M is the overall bare mass of the shell, i is (discrete) angular variable, and

$$C_i = \exp(-\frac{2\pi i}{N}\Gamma_{12})(\sqrt{1 + R^2/l^2} I + R/l \Gamma_{13}) \exp(\bar{\chi}\Gamma_{10}) \exp(\frac{2\pi i}{N}\Gamma_{12}) \quad (9)$$

- From the condition $TrU = \cos(\sqrt{1 - 2m})$, where m is the ADM mass, one gets the Israel equation

$$M = \sqrt{1 + R^2/l^2 + \sinh^2\bar{\chi}} \pm \sqrt{1 - 2m + R^2/l^2 + \sinh^2\bar{\chi}}. \quad (10)$$

Hamiltonian constraint

- The holonomy $U \in ADS^2$ can give a global chart in terms of the Euler angles ρ, χ (rotation and boost). In terms of these variables we have two independent equations on the components of U:

$$\cos(\rho)\cosh(\chi) = \cos(\pi Q), \quad \sinh(\chi) = M\frac{\sinh\bar{\chi}\sin(\pi Q)}{\pi Q}, \tag{11}$$

where

$$Q = \sqrt{\left(1 - M\sqrt{\cosh^2(\bar{\chi}) + \frac{R^2}{l^2}}\right)^2 - \left(M\sinh(\bar{\chi})\right)^2}. \tag{12}$$

- In the limit $|Q| \ll 1$, which corresponds to $2m - 1 \ll 1$, $\bar{\chi}$ can be explicitly excluded to give the single Hamiltonian constraint

$$\cos(\rho)\cosh(\chi) = \cos\left[\pi\sqrt{\left(1 - M\sqrt{1 + \sinh^2(\chi)/M^2 + \frac{R^2}{l^2}}\right)^2 - \left(M\sinh(\chi)\right)^2}\right]. \tag{13}$$

Quantization

- In the vicinity of the horizon, one can neglect curvature of coordinate space and hence momentum non-commutativity. The states in momentum representation are given by commutative functions of U:

$$\Psi(U) = \Psi(\rho, \chi). \tag{14}$$

- The scalar product is

$$\langle \Phi, \Psi \rangle = \frac{1}{\pi} \int \sinh(2\chi) d\rho d\chi \Phi(\rho, \chi)^* \Psi(\rho, \chi). \tag{15}$$

- Time is canonically conjugate to ρ, and the corresponding operator is

$$\hat{T}|\rho, \chi\rangle = i\hbar \frac{\partial}{\partial \rho} |\rho, \chi\rangle, \tag{16}$$

  its eigenstates are

$$|t; \psi\rangle = \frac{1}{\pi} \int \sinh(2\chi) d\rho d\chi \exp(it\rho) \psi(\chi) |\rho, \chi\rangle, \tag{17}$$

  where t is an integer. Time operator has a discrete spectrum:

$$\hat{T}|t; \psi\rangle = t\hbar |t; \psi\rangle. \tag{18}$$

Quantization

- The perimeterial radius of the shell, R, and coordinates of the shell w.r.t. the horizon, X^a, are related as $R = \sqrt{(2m-1)(X^a X_a + l^2)}$. The invariant distance to the horizon, $X^a X_a$ is represented as the Beltrami–Laplace operator on our momentum space:

$$\hat{X}^2 |t; \psi\rangle = \hbar^2 |t; \Delta\psi\rangle, \tag{19}$$

where

$$\Delta = \left(\frac{1}{\sinh(2\chi)} \frac{\partial}{\partial \chi} \sinh(2\chi) \frac{\partial}{\partial \chi} + \frac{t^2}{\cosh^2(\chi)} \right). \tag{20}$$

- For positive, i.e. spacelike X^2 (outside the horizon) the spectrum is continuous, but separated from zero

$$\hat{X}^2 |t, \lambda\rangle = 2\pi(\lambda^2 + 1)\hbar^2 |t, \lambda\rangle, \tag{21}$$

where λ is a real number.

- For negative, i.e. timelike X^2 (inside the horizon) the spectrum is discrete and includes zero

$$\hat{X}^2 |t, l\rangle = -2\pi l(l+1)\hbar^2 |t, l\rangle, \tag{22}$$

where l is a non-negative integer.

Eigenstates

- The eigenfunctions for the discrete spectrum are obtained from ladder operators

$$\mathscr{L}_\pm = \pm \exp(i\rho)\Big(\frac{\partial}{\partial\chi} \pm \mathrm{i} \tanh\chi \frac{\partial}{\partial\rho}\Big), \tag{23}$$

which satisfy

$$[\mathscr{L}_+, \mathscr{L}_-] = 2i\frac{\partial}{\partial\rho}, \quad [\mathscr{L}_\pm, i\frac{\partial}{\partial\rho}] = \mathscr{L}_\pm. \tag{24}$$

- And the wavefunction is obtained as $\mathscr{L}_-\psi_{t,t} = 0$

$$\psi_{t,\ell} = (\mathscr{L}_+)^{t-\ell}\psi_{t,t} = (\mathscr{L}_+)^{t-\ell}\frac{\exp(it\rho)}{(cosh\chi)^{t+\frac{1}{2}}}, \tag{25}$$

- For a continuous spectrum, the wave solution has an asymptotic of a plane wave. The linear combination of the solution is chosen from the condition that there is a balance between incoming and outgoing waves.

Quantization (dynamics)

- The quantum version of the Hamiltonian constraint is a finite difference equation

$$\psi(t-1,\chi)+\psi(t+1,\chi)=H(\chi)\psi(t,\chi), \tag{26}$$

where

$$H(\chi)=\frac{\cos\left(\pi\sqrt{1+M^2-M\sqrt{1+\sinh(\chi)^2/M^2+R^2/l^2}}\right)}{\cosh(\chi)}. \tag{27}$$

This is Klein-Gordon-like equation for discrete time.

- The solution is

$$\psi(t+1,\chi)=\mathbf{U}(\chi)\psi(t,\chi), \tag{28}$$

where $\mathbf{U}=H+\sqrt{H-1}$ – evolution operator.

Quantization (dynamics)

The transition amplitude from outside the horizon to inside the horizon, I → II, and back, II → I, in one step in time is calculated numerically.

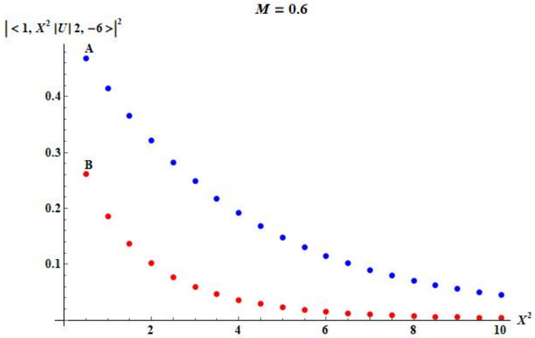


Figure: II → I (curve B) vs. I → II (curve A) relative transition rate.

II → I transition rate is non-zero, but exponentially damped away from the horizon.

Conclusions (intermediate)

- We have quantum description of BTZ Black holes formation due to collapse of a dust shell in a near-horizon region.
- The shell radius has continuous spectrum outside the horizon and discrete inside.
- We evaluate transition amplitude between different regions of the Penrose diagram and find a possibility for quantum tunneling out of the black hole
- Away from the horizon, including central singularity point, momentum non-commutativity cannot be neglected. This will require a full-fledged quantum group technique (see below).

$\Lambda < 0$, non-linear coordinate space, non-commutative momentum space.

Deformation with $q = \exp(\pi G \sqrt{|\Lambda|}\hbar)$
Coordinate space: $U_q(su(1,1))$: $X_\pm, H$, where H is time coordinate

$$q^{H/2} X_\pm q^{-H/2} = q^{\pm 1} X_\pm, \qquad [X_+, X_-] = \frac{q^H - q^{-H}}{q - q^{-1}} \tag{29}$$

Its Casimir, $C_2 = (q - q^{-1})^2 X^+ X^- + q^{-1} q^H + q q^{-H}$ is the radial distance from the shell to the horizon.
Momentum space: $Fun(SU_q(1,1))$, a, b, c, d

$$a = d^*, \qquad b = qc^*, \qquad ad - qbc = 1 \tag{30}$$

$$ab = qba, \quad ac = qca, \quad bd = qdb, \quad cd = qdc$$

$$bc = cb, \qquad ad - da = (q - q^{-1})bc$$

With cylindrical symmetry $b = b^*$, $c = c^*$, $b = qc$.

Momentum representation

$$\Psi(a, b, c, d) = \sum \psi_{mn} d^m b^n \tag{31}$$

(only two matrix entries are independent in cylindrically symmetric setting)
Scalar product

$$\langle \Psi | \Psi' \rangle = \int_q \Psi(a, b, c, d)^* \Psi'(a, b, c, d) \tag{32}$$

It is convergent if $m < 0$ (correspondes to unitary representations of a non-compact group)

Coordinate operators and their spectra:

$$q^H d^{-m} b^n = q^m d^{-m} b^n, \tag{33}$$

$$X^+ d^{-m} b^n = -q^{-m+n} d^{-m} n(d^*) b^{n-1} - \frac{q^n - 1}{q - 1} d^{-m-1} b^{n+1} \tag{34}$$

$$X^- d^{-m} b^n = -q^{-m+n} d^{-m} n(d^*) b^{n-1}$$

Eigenstates of q^H and C_2:

$$\Psi_{m,m} = d^{-m}, \quad \Psi_{m,m+n} = (X^+)^n \Psi_{m,m} \tag{35}$$

$$q^H \Psi_{m,m+n} = q^{m+n} \Psi_{m,m+n}, \, C_2 \Psi_{m,m+n} = \frac{q^m - q^{-m}}{q - q^{-1}} \frac{q^{m+1} - q^{-m-1}}{q - q^{-1}} \Psi_{m,m+n}$$

$$m = 0 ... N, \quad n = 0 ... N - m, \quad N = 1/(G\sqrt{|\Lambda|}\hbar) \tag{36}$$

Inside the BH n and m vary within a finite range, $\rightarrow$ Hilbert space is finite-dimensional

Dynamics

- Hamiltonian near $R = 0$ singularity

$$\sqrt{ad^{-1}} + \sqrt{da^{-1}} = H = 2\frac{\cos(\pi\sqrt{\left(1 - M\sqrt{1+b^2}\right)^2 - M^2(b^2)})}{\sqrt{1+b^2}} \quad (37)$$

- Evolution operator

$$U = H \pm \sqrt{(H^2 - 1)} \quad (38)$$

- Transition amplitudes between singularity and near singularity states

$$\langle N - j, N - 1|U|N, N\rangle \quad (39)$$

where $N = 1/(G\sqrt{|\Lambda|}\hbar)$.

- Calculated numerically: *This work is in progress*

Conclusions (final)

- The Hilbert space of the shell inside the black hole is finite-dimensional, the spectrum of the shell radius is discrete and bounded
- Transition amplitudes between different shell radii, including $R = 0$ singularity are everywhere finite