Introduction to Lattice Hadron Spectroscopy

Part II: Tetraquarks

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Helmholtz International Summer School
"Hadron Structure, Hadronic Matter, and Lattice QCD"

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A practical introduction

Disclaimer

The material covered in this lecture is not entirely original, it has been previously shown and has been compiled here by the author for pedagogical presentation. As such figures and slides may have been copied from other sources.

Exercises

A number of home and in class exercises are strewn across the lectures. These are purely voluntary. Solutions in general will not be given. However, we can discuss the solution to the exercises outside of class, or if you do them any time in the future after the school, by email.

Discussions and questions

Please, feel free to ask questions and contribute to the discussion points. There is so much accumulated knowledge in the room, if the author cannot answer a question, there will be someone who can. We are eager to share our experiences!

Outline

A different kind of spectroscopy

Hadron spectra in finite volumes

-
-

Tetraquarks

-
-

Lattice calculation

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-

Implications

-
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- Bound states
- Scattering states

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- Phenomenological intuition of binding
- Setting up a GEVP

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Tetraquarks

- Phenomenological intuition of binding
- Setting up a GEVP

Lattice calculation

- Particulars of contraction
- Analysis and extrapolations
- Discussion of systematics

Implications

- Experimental detection
- Phenomenological intuition

Bound states in lattice QCD

When determining $a_\mu^{had}[LO]$ we were interested in the **total hadronic contribution** to the anomalous magnetic moment.

Phenomenologically this meant we integrated over the e/m spectral function. On the lattice we were instead interested in an integral over a (lattice) correlator.

We were **not** interested in:

- Matrix elements
- Ground state energies
- Excitation spectra
- Decays and form factors

This changes now.

Since all hadron correlators decay exponentially (see Heng-Tong's lectures), the ground state will dominate as slowest decaying exponential when $t \to \infty$:

$$
\begin{aligned}
G_{\mathcal{O}_1\mathcal{O}_2}(\vec{p}, t) &= \langle \sum_x e^{ip\cdot x} \mathcal{O}_1^{s_1}(x,t) \mathcal{O}_2^{s_2}(0,0)^\dagger \rangle \\
&\overset{\vec{p}=0}{=} \sum_n \underbrace{\langle 0|\mathcal{O}_1^{s_1}|n\rangle\langle n|\mathcal{O}_2^{s_2}|0\rangle}_{A_n} e^{-E_n t} \\
&\xrightarrow{t\to\infty} A_0 e^{-E_0 t}
\end{aligned}
$$

Discussion

Let's rethink this statement and expression. What does this mean for any lattice calculated correlator in practice?

With $G(t \gg) \approx A_0 e^{-E_0\, t}$ it is in principle easy to extract the ground state energy. Simply take the log-derivative, otherwise known as: computing the effective mass

$$m_{eff}(t) = \frac{1}{\delta} \log\left(\frac{G(t+\delta)}{G(t)}\right)$$

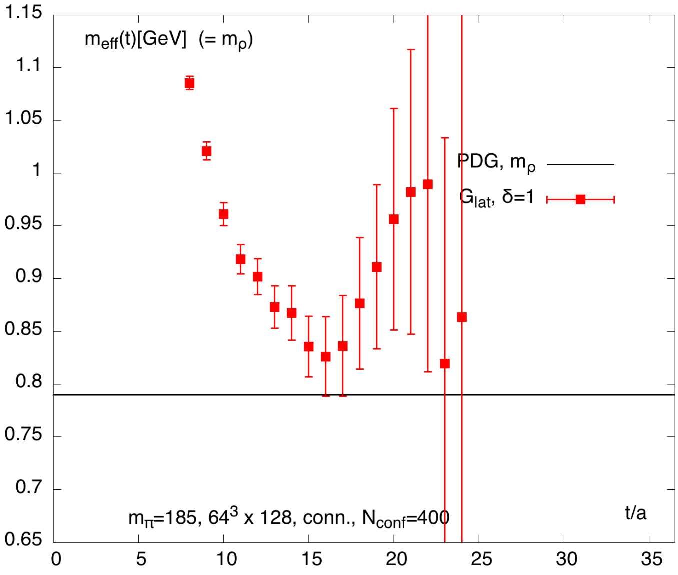
whereby usually $\delta = 1$.

For long distances we then expect the formation of a plateau in $m_{eff}(t)$ that we can fit a constant to:

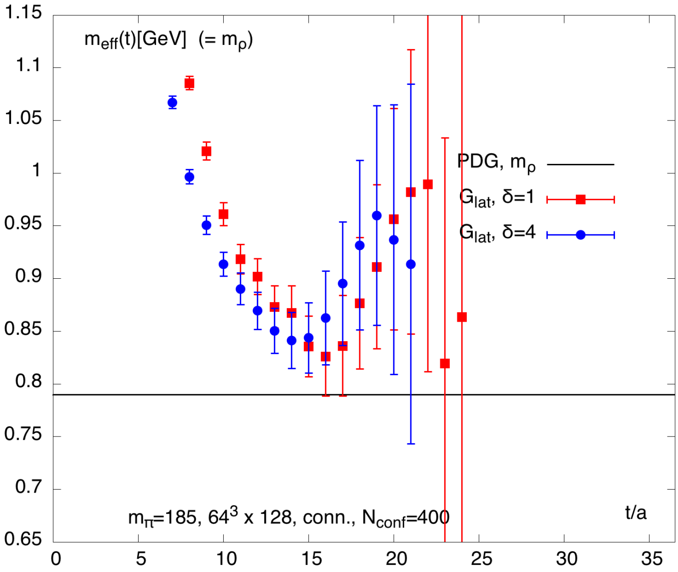
$$m_{eff}(t \gg) \approx E_0$$

Discussion

Incidentally, what do you think will happen for $\delta \neq 1$?



$m_{eff}(t)$[GeV] (= m_ρ)
PDG, m_ρ
G_{lat}, δ=1
m_π=185, 64^3 x 128, conn., N_{conf}=400
t/a
1.15
1.1
1.05
1
0.95
0.9
0.85
0.8
0.75
0.7
0.65
0
5
10
15
20
25
30
35



$m_{eff}(t)$[GeV] (= m_ρ)
PDG, m_ρ
G_{lat}, δ=1
G_{lat}, δ=4
m_π=185, 64^3 x 128, conn., N_{conf}=400
t/a
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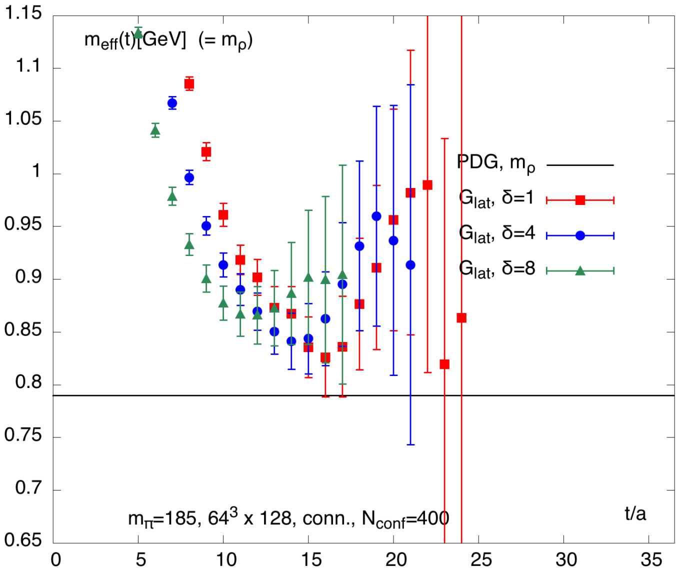
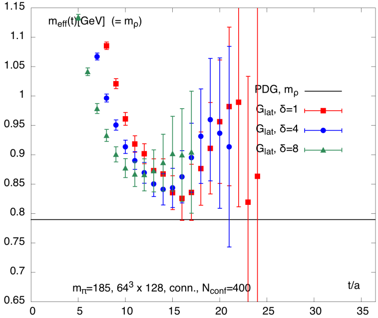
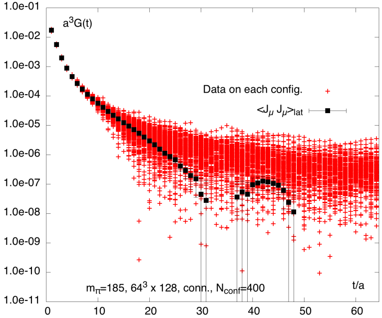


The plateau seems to drift! Example of "self-induced systematic"





Looks can be deceiving! The effective mass with any δ is based on the same correlator data.

Take away: The effective mass is best used for visualisation. It is **often (=check)**, better to directly fit the correlator at large times.

This avoids a possible extra systematic at the cost of an extra fit parameter and more non-trivial fit.

$$G(t \gg) \approx A_0 e^{-E_0 t}$$

Remark:

The correlator contains all states and excitations that have the (lattice) quantum numbers of the interpolating operator $\mathcal{O}(x,t)$ entered in the calculated correlation function

$$G_{\mathcal{O}_1\mathcal{O}_2}(t) = \left\langle \sum_x \mathcal{O}_1(x,t)\mathcal{O}_2(0,0)^\dagger \right\rangle = \sum_n A_n\, e^{-E_n t}$$

Comment: It seems fitting only E_0 from large t is wasteful and it is inefficient to ignore all this extra information on E_n.

Discussion

Is there maybe a way to extract more energies other than the ground state? Any ideas?

GEVP: The generalized eigenvalue problem

Imagine you had a set of interpolating operators of the same quantum numbers, yet constructed from different combinations of γ-matrices and quarks.

For example, you could impose different "smearing" profiles ($\approx$ trial wave functions) onto the quark source when determining the quark propagators.

Then we could compute a complete mixing matrix of these different operators $\mathcal{O}_{1,2,3}$:

$$F(t) = \begin{pmatrix} \langle \mathcal{O}_1 \mathcal{O}_1 \rangle & \langle \mathcal{O}_1 \mathcal{O}_2 \rangle & \langle \mathcal{O}_1 \mathcal{O}_3 \rangle \\ \langle \mathcal{O}_2 \mathcal{O}_1 \rangle & \langle \mathcal{O}_2 \mathcal{O}_2 \rangle & \langle \mathcal{O}_2 \mathcal{O}_3 \rangle \\ \langle \mathcal{O}_3 \mathcal{O}_1 \rangle & \langle \mathcal{O}_3 \mathcal{O}_2 \rangle & \langle \mathcal{O}_3 \mathcal{O}_3 \rangle \end{pmatrix}$$

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Since each of the operators overlaps with the same states, just with different strengths, you would be tempted to think:

$$F(t) = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} E_0 \\ E_1 \\ E_2 \end{pmatrix}$$

i.e. each entry in the correlator matrix is some mixing of the three same states $E_{0,1,2}$.

Diagonalising this matrix will then yield the three energies and the corresponding mixing vectors. Or, as they are better known: the eigenvalues and eigenvectors.

> Solving the **GEVP** defined by the correlator matrix gives a clean way to disentangle the states contained in a set of correlators.

Caveat:

One needs to find a "large" basis of "sensible" operators for this method to be useful.

- **"sensible"**: The operators need to overlap with the same states. I.e. they need to have the same quantum numbers.

  At the same time they all need to be able to mix! Otherwise individual operators will dominate the GEVP and skew the results. Finding trivial eigenvectors is not a good sign for sensible operators!

- **"large"**: The higher eigenvalues will contain all the discretisation effects present in the correlator. With more resolved eigenvalues the contamination is moved to higher states.

  There should be at least **one** *more state resolved than the ones to be studied. For example, in a 2×2 GEVP the discretisation effects are contained in the second eigenvalue, its physical interpretation is lost!*

With the GEVP we have an efficient tool to disentangle and extract the excited and, most importantly, the ground state from a set of lattice correlators.

Question: Given we can only extract energies (and maybe amplitudes) cleanly, what does it mean to predict a bound state from a lattice calculation?

With the GEVP we have an efficient tool to disentangle and extract the excited and, most importantly, the ground state from a set of lattice correlators.

Question: Given we can only extract energies (and maybe amplitudes) cleanly, what does it mean to predict a bound state from a lattice calculation?

Alternative question: What does it mean to have a bound state in a box?

 

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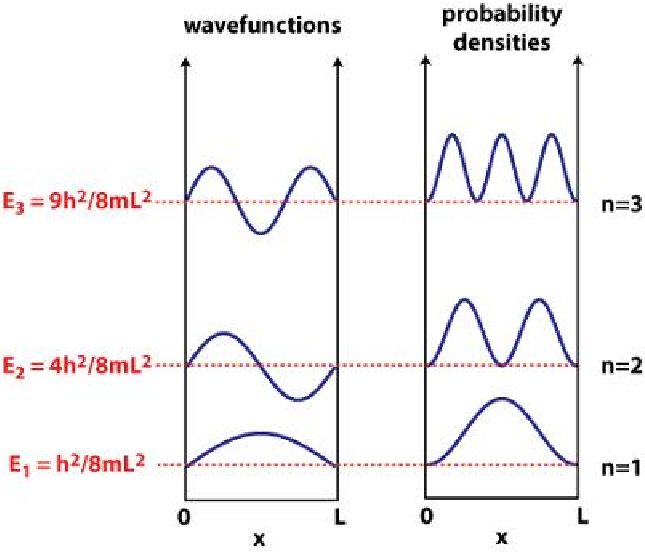
We know already from quantum mechanics that the energy eigenstates in a $1-D$ potential well depend on the size L of the well.



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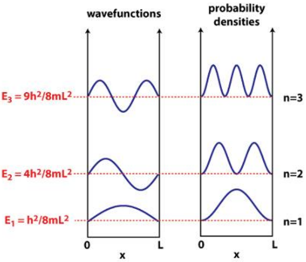
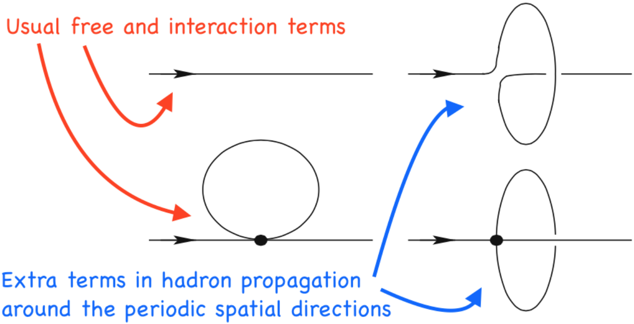


It is important to make sure that the (ground) state found in the box behaves as that in the continuum.

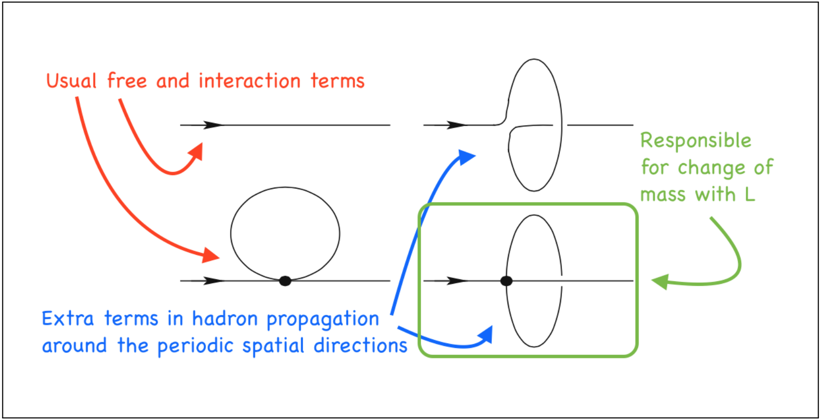
Or: That we can at least extrapolate/estimate whether it behaves as a bound or a scattering state once the box is removed.

Bound state volume dependence:

Let us first examine the case of a bound state in a finite volume and how it reacts to a change in L.



Observation: Extra terms arise because the hadrons may wrap around the periodic spatial directions or interact with mirror charges located there.



In this case, the mass change should be proportional to the loop contribution:

$$m^2 = m_0^2 + \underbrace{\alpha}_{const.} \cdot \underbrace{\overbrace{\int \frac{d^4 p}{(2\pi)^4} G(p)}^{\text{integral over momentum prop}}}_{\text{real space prop: } G(0)} = m_0^2 + \alpha G(0)$$

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In a finite volume ($L =$finite, $T = \infty$) the real space propagator becomes:

$$G(x) \Rightarrow \sum_{s \in \mathbb{Z}^3} G(x + Ls)$$

where s counts the windings around the spatial directions.

Consequently:

$$m^2(L) - m^2 = \alpha \sum_{s \neq 0} G(Ls)$$

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Explicitly, the real space free boson propagator *(pion = boson, good approximation)* is

$$G(x) = \int \frac{d^4 p}{(2\pi)^4} \frac{e^{ipx}}{p^2 + m^2} \sim \frac{m^2 \sqrt{8\pi}}{16\pi^2} \frac{e^{-m|x|}}{(m|x|)^{3/2}}$$

Or more clearly:

We expect the leading size volume dependence to be:

$$m^2(L) - m^2 \propto \frac{\exp(-mL)}{L^{3/2}}$$

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This can be generalized to all order perturbation theory (Lüscher '86), the basic message remains the same:

The leading finite volume effect goes as $\propto \exp[-\kappa_0 L]/L$, *where* κ_0 *is the binding momentum:*

$$B = -\left(\sqrt{m_1^2 - \kappa_0^2} + \sqrt{m_2^2 - \kappa_0^2} - m_1 - m_2\right)$$

with the binding energy B *in infinite volume* $L = \infty$*.*

In class exercise

$$B = -\left(\sqrt{m_1^2 - \kappa_0^2} + \sqrt{m_2^2 - \kappa_0^2} - m_1 - m_2\right)$$

Let's look at this more closely:

1. Assume $m = m_1 = m_2$ and simplify the equation
2. Solve the equation for κ_0, you should find:

$$\kappa_0 = \frac{1}{2}\sqrt{4mB - B^2}$$

3. Using a calculator insert $B = 180\text{MeV}$ and $M = 5300\text{MeV}$ for a hypothetical binding of two B-type mesons. By how many units of B is the volume effect suppressed, if it goes as $\propto \exp[-\kappa_0 L]/L$?

Scattering state volume dependence:

If the finite volume ground state is in fact **not** a bound state but a scattering state of two (or more) particles instead, what happens then?

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If the finite volume ground state is in fact **not** a bound state but a scattering state of two (or more) particles instead, what happens then?

Think of the stable and unstable ρ-particle before, in the first the ρ was stable (=bound) in the other there were $\pi\pi$-scattering states below the ρ-energy.

Scattering state volume dependence:

If the finite volume ground state is in fact **not** a bound state but a scattering state of two (or more) particles instead, what happens then?

In this case the situation is much more complicated. However, it was worked out in a seminal paper by Lüscher:

`M. Lüscher, Commun. Math. Phys. 105, 153 (1986)`

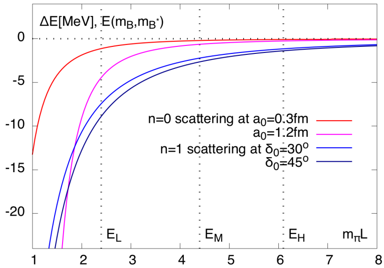
$$\Delta E_0 \approx -\frac{4\pi a_0}{\mu L^3}\left[1+c_1\frac{a_0}{L}+c_2\left(\frac{a_0}{L}\right)^2\right],\ \Delta E_1 \approx -\frac{12 tg(\delta_0)}{\mu L^2}\left[1+c_1' tg(\delta_0)+c_2' tg^2(\delta_0)\right]$$

For a system of two particles, e.g. the $\pi\pi$-system, with $I=0$ and $I=1$ explicit energy shifts due to finite volumes were derived

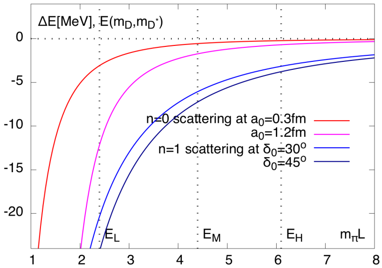
Here a_0 is the scattering length and δ_0 is the scattering angle.

$$\boxed{\Delta E_0 \sim (L^{-3})_{scatt.} \gg (L^{-1}\exp(-\kappa_0 L))_{bound}}$$

In our example of a hypothetical *BB*-system or a *DD*-system, for that matter, we would have:



`BB:` $\Delta E_0^V = 5, < 1$; $\Delta E_1^V = 10, 4, 2$



`DD:` $\Delta E_0^V = 12, 2, 1$; $\Delta E_1^V = 24, 8, 4$

- For a *DD*- vs. a *BB*-system the finite volume induced shift for scattering states is much larger in the lighter system.
- Using the results from the in class exercise (and artificially introducing a lattice spacing of $a^{-1} = 2.194 GeV$) for a bound state the effect would be < 0.1MeV for the *BB*-system. It is significantly suppressed.

Tetraquarks in phenomenology, on the lattice
... and in nature(?)

Before we dive into the lattice calculation, and before you start your own lattice work, ask yourself: Are you ready to commit?

It will take ...

- a significant amount of computing resources. (Will they be available?)
- a great deal of preparation in the sense of setting up an efficient computing paradigm. (Are there better or worse methods? What systematics can be handled easily/with difficulty?)
- an innovation effort to improve the methods or motivate a new observable. (Tetraquarks: diquark-diquark and meson-meson operators?)
- a long time ($\sim$year(s)) to perform a trustable calculation.

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In the previous part of the lecture we already decided on the method: We want to set up a GEVP in order to cleanly determine the ground state.

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In the previous part of the lecture we already decided on the method: We want to set up a GEVP in order to cleanly determine the ground state.

However: We still need to decide on the operators we want to use.

Before we dive into the lattice calculation, and before you start your own lattice work, ask yourself: Are you ready to commit?

It will take ...

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-

Additionally: Phenomenological intuition is often a useful guide to get a feeling what one might expect.

E.g. one might be able to estimate what the phenomenological window for a bound tetraquark might be.
A reasonable expectation of $B \sim 3MeV$ or $B \sim 200MeV$ greatly changes how you approach the lattice calculation.

Community wish-lists

From experiment:

- It should be detectable.
  - the candidate should be as stable as possible (QCD, E/M; EW ok)
  - not too heavy and not too light,
  - preferably with many quantum numbers

$\Rightarrow$ Aim for candidates with c and perhaps b quarks.

From the lattice:

- Multi-quark states are very noisy
  - want good signal-to-noise $\Rightarrow$ typically heavy, c or b, quark systems are beneficial. Best: s quarks
  - operator should not contain disconnected diagrams

$\Rightarrow$ Aim for candidates with s, c, b and no disconnected diagrams.

From phenomenology:

- There should be insight into the binding mechanism
  - the candidate should be motivated with a specific mechanism in mind

$\Rightarrow$ Aim to study the applicability of a mechanism over a candidate

Discussion

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The lattice wish-list immediately eliminates operators of the form:

$$(q\bar{q})(Q\bar{Q})\ ,\ (q\bar{q}')(Q\bar{Q})\ ,\ (q\bar{q})(Q\bar{Q}')\ ,\ (Q\bar{q})(q'\bar{Q})$$

Why? What would be a more viable option?

Tetraquark operators that do not have disconnected diagrams are of the form:

$$(qq')(\bar{Q}\bar{Q})$$

Possible flavour combinations in this category would be:

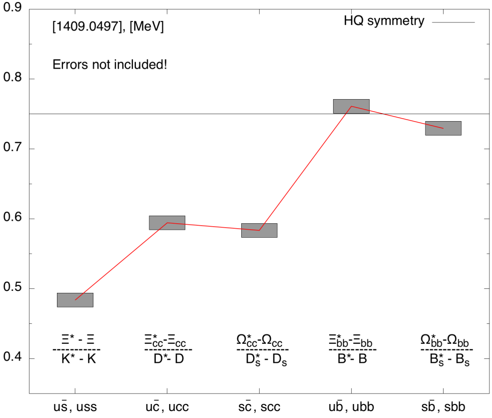
$$(ud)(\bar{b}\bar{b})\ ,\ (us)(\bar{b}\bar{b})$$
$$(ud)(\bar{c}\bar{b})\ ,\ (us)(\bar{c}\bar{b})$$
$$(ud)(\bar{c}\bar{c})\ ,\ (us)(\bar{c}\bar{c})$$

$$(ud)(\bar{s}\bar{s})\ ,\ (ud)(\bar{s}\bar{c})\ ,\ (ud)(\bar{s}\bar{b})$$

Measuring all of these would be quite costly. In addition we still want to fill a GEVP with sensible operators.

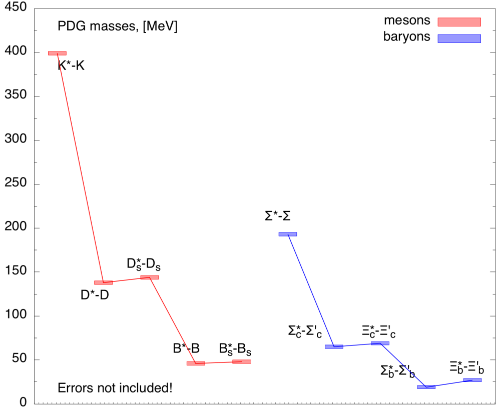
Can phenomenology comment on these options?

Intuition from phenomenology: Heavy quark symmetry



In pheno the notion of approximate/emergent heavy quark symmetry can be motivated from the observation that heavy mesons and baryons approach a limit in which their heavy quark spins decouple.

$\Rightarrow$ *A single anti-quark $\bar{Q}$ behaves very much like a pair of quarks QQ.*



The smallness of the difference of the *experimentally observed* bottom mesons and baryons points to ...

- heavy quark symmetry being active for bottom quarks ...
- and perhaps even for charm quarks

If, heavy quark symmetry is indeed good for bottom quarks and $\bar{b}$ behaves much like bb, or alternatively b like $\bar{b}\bar{b}$

Then, one might be tempted to replace the single b-quark in a bottom baryon, with two anti-quarks to get a modeling of how a tetraquark might look like based on the observed spectrum.

Good example candidates for such a replacement would be:

- $\Lambda_b = [udb]_5$ and $\Sigma_b = [uub]_i$, good
- $\Xi_b = [usb]_5$ and $\Xi'_b = [usb]_i$, even better!

Why even better?

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- $\Xi_b = [usb]_5$ and $\Xi_b' = [usb]_i$, even better!

Why even better?

The second option differs only in the **diquark** in the baryon operator:

$$\mathcal{O}(\Xi_b) = \Big[b\,(u\,C\gamma_5\,s)\Big] \quad , \quad \mathcal{O}(\Xi_b) = \Big[b\,(u\,C\gamma_i\,s)\Big]$$

The first also differs in a flavor $u \leftrightarrow d$:

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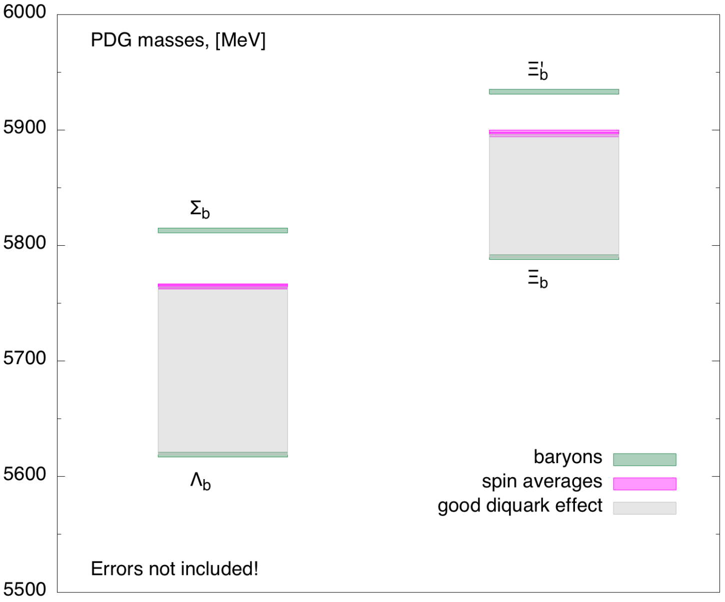
So what?

This motivates a binding mechanism through diquark configurations!

Comparing the individual baryon masses with their corresponding spin averages, e.g. $(3\Xi_b + \Xi_b')/4$ *, one finds a measure of "preference" for either* γ_5*-diquarks* $(S = 0, C = \bar{3}_c)$ *or* γ_j*-diquarks* $(S = 1, C = \bar{3}_c)$.

Taken together with heavy quark symmetry, this preference also poses an energy window for bound tetraquarks.



6000
5900
5800
5700
5600
5500
PDG masses, [MeV]
Ξ'_b
Σ_b
Ξ_b
Λ_b
baryons
spin averages
good diquark effect
Errors not included!

These ideas have been around for a long time ($\sim 80's$). The diquark configurations have been dubbed "good" and "bad" diquarks by Jaffe (2005).

In the plot we saw significant energy windows $\sim$ 100MeV for a bound tetraquark for the $ud\bar{b}\bar{b}$ and $us\bar{b}\bar{b}$ flavor combinations.

Of course, this is somewhat crude and oversimplified; HQ symmetry is not a clean concept and neither is the idea of "good diquarks". However, a window of $\sim$ 100MeV for $(u\, C\gamma_5\, d)$ and $(u\, C\gamma_5\, s)$ is a uniquely large "prediction" from this picture.

More predictions:

- Deeper binding with heavier quarks, $\sim 1/m_Q$
- Deepest binding for pairs $\bar{Q}\bar{Q}$
- Binding set by the lighter of Q, Q' for $\bar{Q}\bar{Q}'$
- Deeper binding for lighter quarks in the qq' component

Predictions:

- Binding mechanism from good diquark configurations
- Binding energies of ~ 100MeV or more for $ud\bar{b}\bar{b}$ and $us\bar{b}\bar{b}$
- Deeper binding with heavier quarks, $\sim 1/m_Q$
- Deepest binding for pairs $\bar{Q}\bar{Q}$
- Binding set by the lighter of Q, Q' for $\bar{Q}\bar{Q}'$
- Deeper binding for lighter quarks in the qq' component

Wish-list:

- No disconnected diagrams and clear lattice interpretation ✓
- Phenomenological guiding principle ✓
- Experimental detectability ✓ (difficult!)

> - Good starting point for a lattice calculation with operators motivated from this mechanism.

A diquark-anti diquark operator

Let's start with quark content $ud\bar{b}\bar{b}$.

- Heavy quark symmetry best realized
- Biggest good diquark effect

The *ud* portion should be Λ-like, not Σ-like. *ud* is *antisymmetric* in both color and flavor:

$$L_a(x) = \epsilon_{abc}(u_b^\alpha)^T(x)(C\gamma_5)^{\alpha\beta}d_c^\beta(x) \,.$$

The $\bar{b}\bar{b}$ portion will be quark-like. To join with *ud*, it must be color antisymmetric but *flavor symmetric*:

$$H_a(x) = \epsilon_{ade}\bar{b}_d^\kappa(x)(C\gamma_i)^{\kappa\rho}(\bar{b}_e^\rho)^T(x) \,.$$

The total **(diquark-anti diquark)** operator then is $J^P = 1^+$ and

$$D(x) = L_a(x)H_a(x) \,.$$

A meson-meson operator

With quark content $ud\bar{b}\bar{b}$ and $J^P = 1^+$ the lightest conventional state would be a meson pair:

$$B(5279) \quad (J^P = 0^-) \quad \text{and} \quad \mathrm{B}^*(5325) \quad (\mathrm{J}^\mathrm{P} = 1^-)\ .$$

A **(meson-meson)** operator with definite isospin is,

$$M(x) = \bar{b}_a^\alpha(x)\gamma_5^{\alpha\beta} u_a^\beta(x)\ \bar{b}_b^\kappa(x)\gamma_i^{\kappa\rho} d_b^\rho(x) - \bar{b}_a^\alpha(x)\gamma_5^{\alpha\beta} d_a^\beta(x)\ \bar{b}_b^\kappa(x)\gamma_i^{\kappa\rho} u_b^\rho(x)\ .$$

It mixes with $D(x)$ but differs in its internal color structure.

Since $D(x)$ and $M(x)$ have the same quantum numbers, they can propagate the same physical states albeit with different overlaps.

$\Rightarrow$ The (diquark-anti diquark) and (meson-meson) operators enable the definition of a 2×2-GEVP.

Procedure:

- Implement Diquark-Diquark operator:

$$D(x) = (u_a^\alpha(x))^T (C\gamma_5)^{\alpha\beta} q_b^\beta(x) \;\times\; \bar{b}_a^\kappa(x)(C\gamma_i)^{\kappa\rho}(\bar{b}_b^\rho(x))^T \;,$$

- Implement Dimeson-Dimeson operator:

$$M(x) = \; \bar{b}_a^\alpha(x)\gamma_5^{\alpha\beta} u_a^\beta(x) \; \bar{b}_b^\kappa(x)\gamma_i^{\kappa\rho} d_b^\rho(x) - \bar{b}_a^\alpha(x)\gamma_5^{\alpha\beta} d_a^\beta(x) \; \bar{b}_b^\kappa(x)\gamma_i^{\kappa\rho} u_b^\rho(x) \;.$$

- Compute the energies from the 2×2 GEVP

$$F(t) = \begin{pmatrix} G_{DD}(t) & G_{DM}(t) \\ G_{MD}(t) & G_{MM}(t) \end{pmatrix}, \quad F(t)\nu = \lambda(t) F(t_0)\nu \;,$$

With the "binding correlator":

$$G_{\mathcal{O}_1\mathcal{O}_2} = \frac{C_{\mathcal{O}_1\mathcal{O}_2}(t)}{C_{PP}(t)C_{VV}(t)} \;,\; \lambda(t) = Ae^{-\Delta E(t-t_0)} \;.$$

Watch out: Possibly ambiguous state identification, if the volume effects are large.

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Watch out: Possibly ambiguous state identification if the volume effects are large.

Recall: From the in class exercise for a hypothetical BB-system we found volume effects of around 5-10MeV for a scattering state in a volume of $m_\pi L = 2.4$. For a bound state the effect was below < 0.1MeV.

Should we find a state significantly below the BB^*-threshold, i.e. $B = \Delta E \gg 5 - 10 MeV$ in a small volume, there is little chance for it to be a scattering state.

Lattice Setup

Basic-Setup	Iwasaki GA	Wilson-Clover FA	CG-wall props
β	1.9	1.9	1.9
c_{SW}	1.715	1.715	1.715
Label	E_H	E_M	E_L
Extent	$32^3 \times 64$	$32^3 \times 64$	$32^3 \times 64$
a^{-1} [GeV]	2.194(10)	2.194(10)	2.194(10)
κ_l	0.13754	0.13770	0.13781
κ_s	0.13666	0.13666	0.13666
am_π	0.18928(36)	0.13618(46)	0.07459(54)
am_K	0.27198(28)	0.25157(30)	0.23288(25)
$m_\pi L$	6.1	4.4	2.4
M_Υ [GeV]	9.528(79)	9.488(71)	9.443(76)
Configurations	400	800	195
Measurements	800	800	3078

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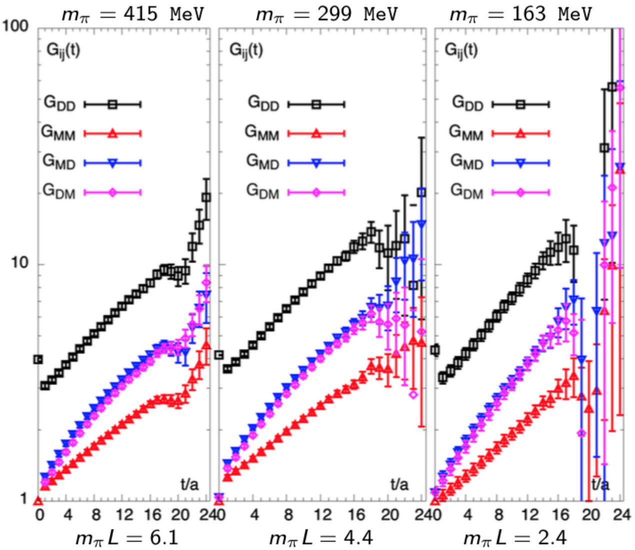
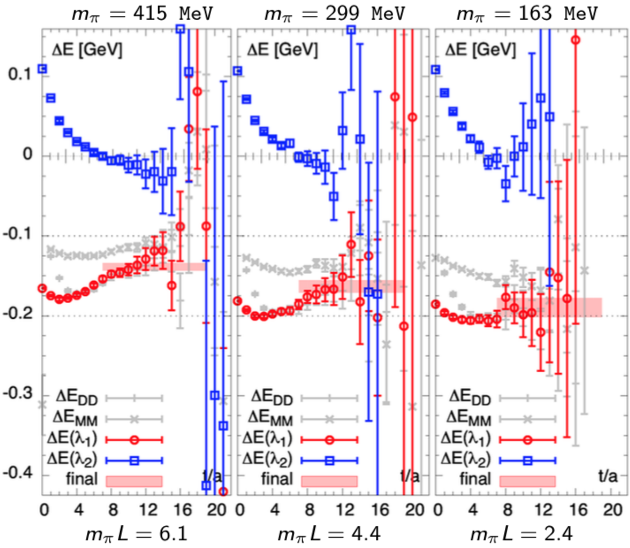
π = 415MeV π = 299MeV π = 163MeV

PhysRevLett.118.142001 using PACS-CS, $32^3 \times 64$, $N_f = 2+1$ Wilson-Clover ensembles



$\boxed{ud\bar{b}\bar{b}}$

PhysRevLett.118.142001 using PACS-CS, $32^3 \times 64$, $N_f = 2+1$ Wilson-Clover ensembles



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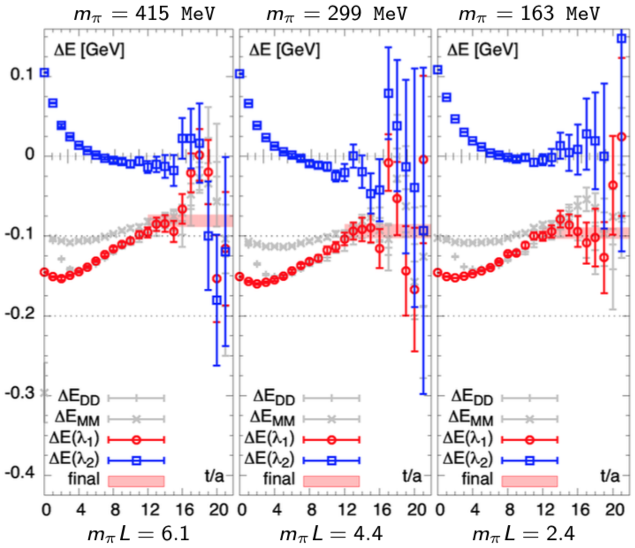
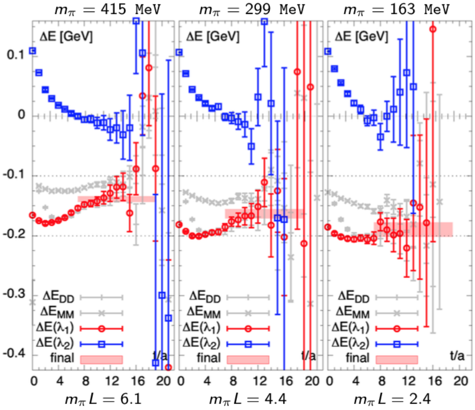


$\boxed{ls\bar{b}\bar{b}}$

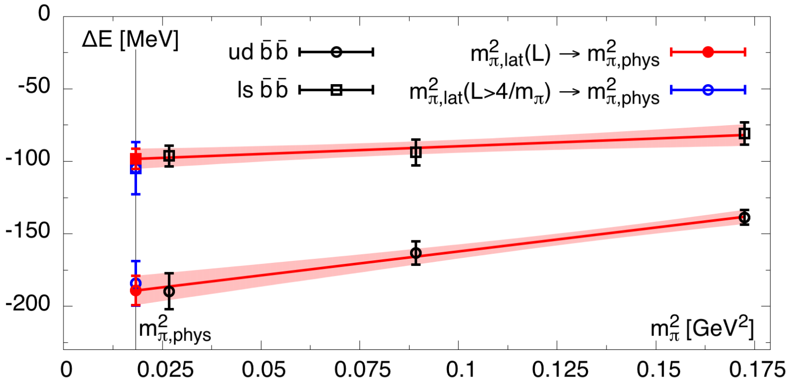
$ud\bar{b}\bar{b}$



Discussion

Time to think about systematics and extrapolations. What were the main ones again? How do you think could we handle them?

PhysRevLett.118.142001 using PACS-CS, $32^3 \times 64$, $N_f = 2+1$ Wilson-Clover ensembles



- We perform two combined chiral-volume extrapolations
- Both agree within errors
- The found binding is an order of magnitude deeper than the estimated maximum volume effects

Ensemble	$\Delta E_{ud\bar{b}\bar{b}}$[MeV]	$\Delta E_{\ell s\bar{b}\bar{b}}$[MeV]
E_H	-139(5)	-81(8)
E_M	-163(8)	-94(9)
E_L	-190(12)	-96(7)
Phys	-189(10)(3)	-98(7)(3)
	$M_{ud\bar{b}\bar{b}}$ [GeV]	$M_{\ell s\bar{b}\bar{b}}$ [GeV]
Predicted Mass	10.415(10)	10.594(8)

Ensemble and extrapolated physical-point (Phys) $ud\bar{b}\bar{b}$ and $\ell s\bar{b}\bar{b}$ binding energies from fitting all ensembles. Errors for the individual ensembles are statistical. For the extrapolated physical point entries, the first error is statistical and the second systematic. We provide a **prediction** for the physical masses of these states, errors have been added in quadrature.

Going further: Heavy quark mass dependence and post-diction

One of the predictions of this binding mechanism is that the binding energy increases with increasing heavy quark mass.

- On the lattice we have the unique capability to input unphysically heavy quark masses and check the predicted $\sim 1/m_Q$ behavior. (A vice just turned into a virtue)
- Excluding our previous results this means we can post-dict them via extrapolation/interpolation.
- With unphysically heavy quark masses also the $ud\bar{b}'\bar{b}$ flavor combination becomes available.

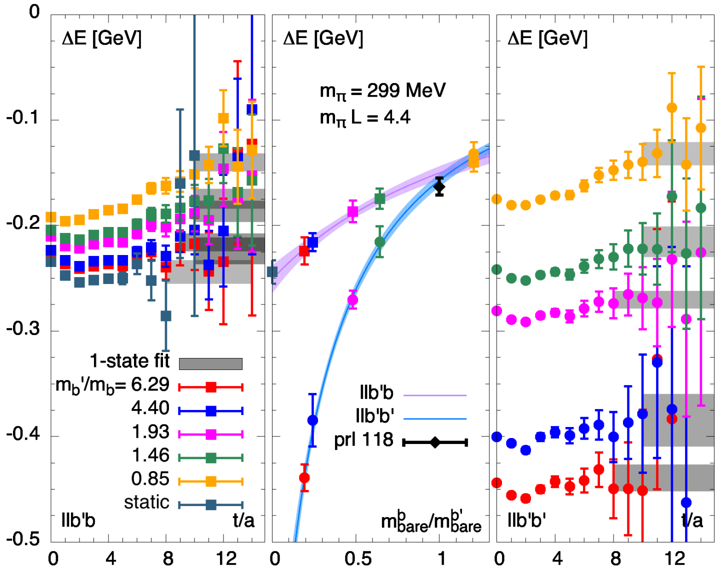
For tetraquarks with $\bar{Q}\bar{Q}'$ the 2×2 GEVP can be extended to 3×3 by exploiting the second possible meson-meson threshold:

New flavor combination enables formulation of 3×3 GEVP:

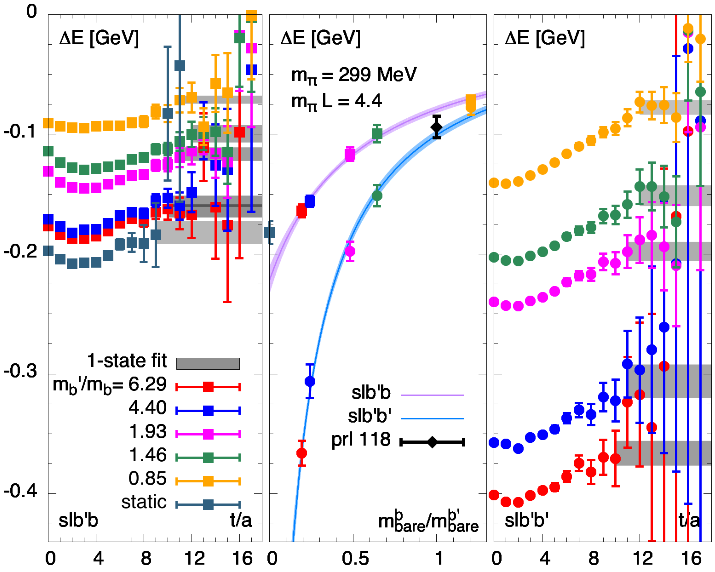
$$F(t) = \begin{pmatrix} G_{DD} & G_{DM} \\ G_{MD} & G_{MM} \end{pmatrix} \Rightarrow F(t) = \begin{pmatrix} G_{DD} & G_{DM_{12}} & G_{DM_{21}} \\ G_{M_{12}D} & G_{M_{12}M_{12}} & G_{M_{12}M_{21}} \\ G_{M_{21}D} & G_{M_{21}M_{12}} & G_{M_{21}M_{21}} \end{pmatrix}$$

Clean(er) extraction of ground state.

- All calculations at $m_\pi = 299$ MeV and $m_\pi L = 4.4$
- $m_{b'}/m_b \approx 6.29, 4.40, 1.93, 1.46, 0.85$ - tuned via dispersion relation of spin-averaged mass mesons
- Physical point is interpolated, Ansatz: $A/(m_{Q'} + M)$



- $1/m_Q$ confirmed. ✓
- Good intercept with previous results. ✓



- $1/m_Q$ confirmed. ✓
- Reasonable intercept with previous results. ✓

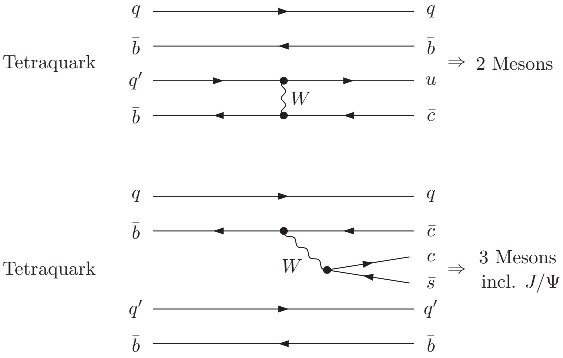
Conclusion

- Two tetraquark candidates at $\Delta E_{ud\bar{b}\bar{b}} = 189(10)(3)$ MeV and $\Delta E_{ls\bar{b}\bar{b}} = 98(7)(3)$ MeV

- Deeper binding with heavier quarks, $\sim 1/m_Q$ ✓
- Deepest binding for pairs $\bar{Q}\bar{Q}$ ✓
- Binding set by the lighter of Q, Q' for $\bar{Q}\bar{Q}'$ ✓
- Deeper binding for lighter quarks in the qq' component ✓

What about experimental detection?

With such deep ΔE, both tetraquarks decay only weakly **(wish-list ✓)**,



$ud\bar{b}\bar{b}$	$\rightarrow$	B^+D^0
	$\rightarrow$	$J/\psi B^+K^0$
$us\bar{b}\bar{b}$	$\rightarrow$	$B^+D_s^0$
	$\rightarrow$	$B_s\bar{D}^+$
	$\rightarrow$	$J/\psi B^+\phi$
	$\rightarrow$	$J/\psi B_sK^+$
$ds\bar{b}\bar{b}$	$\rightarrow$	$B^+D_s^-$
	$\rightarrow$	$B_s\bar{D}^0$
	$\rightarrow$	$J/\psi B^0\phi$
	$\rightarrow$	$J/\psi B_sK^0$

- Challenging for experiment, but favorable tags exist!

Final summary

What have we learned?

- In detail we went through the methods to extract state energies from lattice correlators
- We introduced the idea of using a basis of operators to form a GEVP
- Correctly identifying a bound state in a finite volume is a non-trivial task and we looked at the different L-dependencies for bound and scattering states
- Phenomenological intuition gave us a paradigm to build a tetraquark operator
- Implementing and calculating a set of operators we found strong indication that bound $ud\bar{b}\bar{b}$ and $ls\bar{b}\bar{b}$ tetraquark candidates exist.