Spectral quantities in thermal QCD

I: introduction

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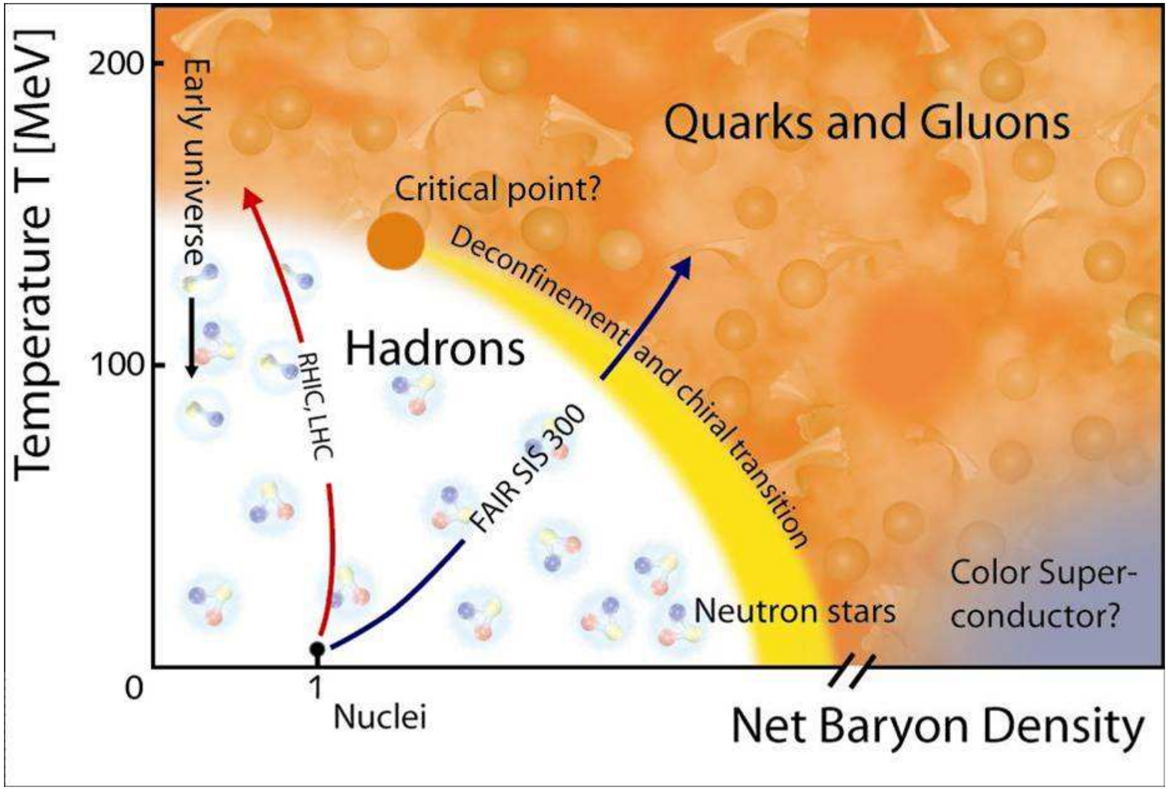
Strongly interacting matter: QCD

hadronic and quark-gluon plasma: rich topic

spectral quantities

close to equilibrium

out of equilibrium



Strongly interacting matter: QCD

spectral quantities

- (de)confinement: light hadron spectrum ($\pi, \rho, N, ...$)
  - in-medium modification as $T \sim 0 \to T_c$
  - $T > T_c$: transition to quark degrees of freedom

Strongly interacting matter: QCD

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  - $T \to T_c$: emergent degeneracy $(\rho \leftrightarrow a_1, N \leftrightarrow N^*, ...)$

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- chiral symmetry
  - $T \to T_c$: emergent degeneracy $(\rho \leftrightarrow a_1, N \leftrightarrow N^*, ...)$
- heavy quarks/quarkonium
  - survival in QGP, channel dependent ($\bar{c}c, \bar{b}b$)
  - sequential melting, effective thermometer
- ...

Strongly interacting matter: QCD

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close to equilibrium

- linear response, external perturbations
  - plasma oscillations, correlation times

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- transport: evolution of nearly conserved quantities
  - energy-momentum: viscosities η, ξ
  - charges: conductivity σ, light/heavy quark diffusion D

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  - plasma oscillations, correlation times
- transport: evolution of nearly conserved quantities
  - energy-momentum: viscosities η, ξ
  - charges: conductivity σ, light/heavy quark diffusion D
- non-strongly interacting signatures
  - thermal radiation: photon emission rate
  - dilepton production rate
- $\ldots$

Strongly interacting matter: QCD

<u>spectral quantities</u>

<u>close to equilibrium</u>

<u>out of equilibrium</u>

- “arbitrary” initial state: evolution in real time
  - heavy ion collision
  - ...

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requires use of effective (field) theories

- hydrodynamics – kinetic theory, (classical) particle dynamics – classical field dynamics, ...

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relation

- transport coefficients as low-energy constants
- spectral understanding, (quasi)particles

Strongly interacting matter: QCD

spectral quantities

close to equilibrium

out of equilibrium (not discussed further, except as above)

addresses seemingly very different questions:

- yet information is contained in thermal correlators

all thermal correlation functions contain *all* the information
(Euclidean, Feynman, Wightman, retarded, advanced, statistical, . . .)

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all thermal correlation functions contain *all* the information
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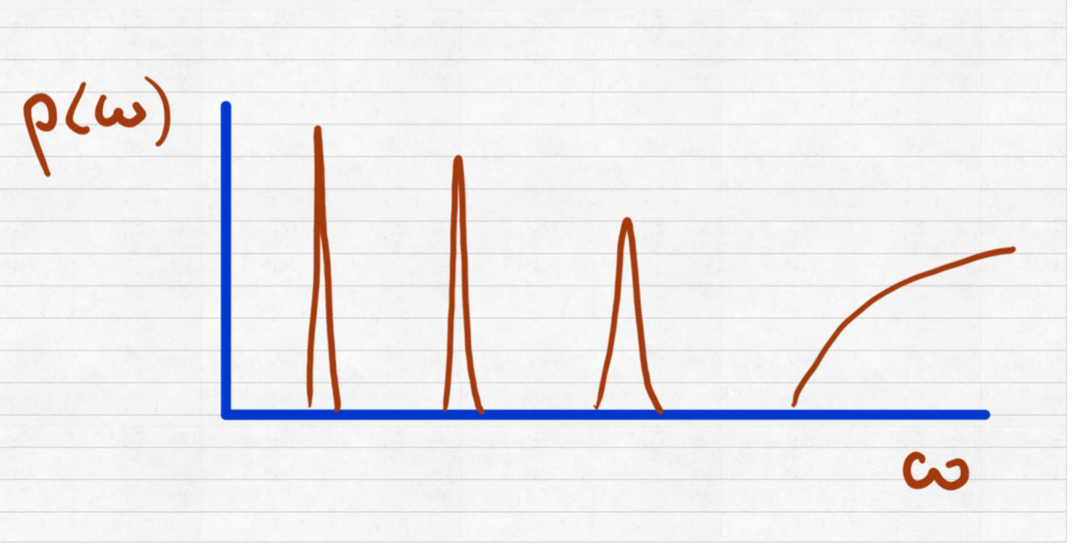
- but info might be more accessible in specific form
- or easier defined in certain representations

$\Rightarrow$ spectral function $\rho(\omega, \mathbf{p})$

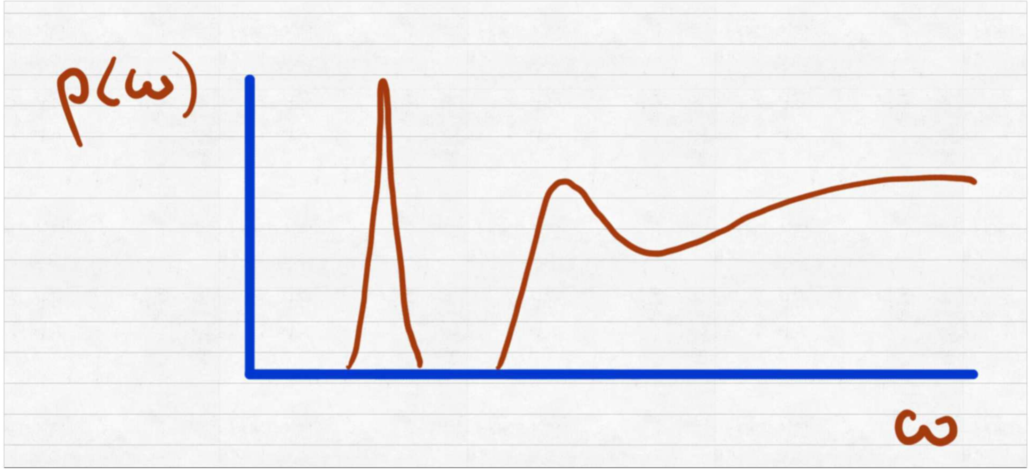
Spectral functions

- spectral quantities at low temperatures



Spectral functions

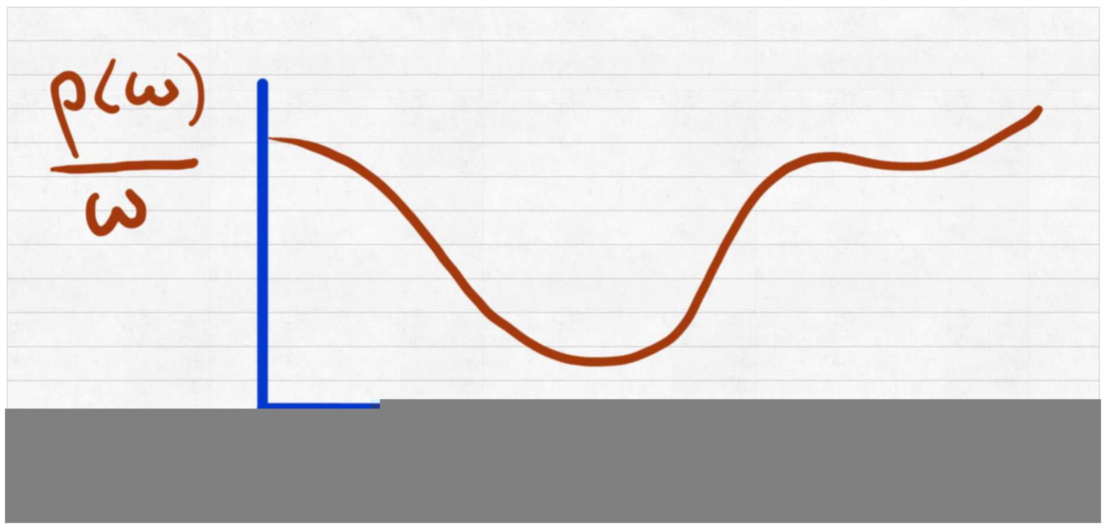
- spectral quantities at low temperatures
- spectral quantities at higher temperatures



thermal broadening, dissociation, quarkonium

Spectral functions

- spectral quantities at low temperatures
- spectral quantities at higher temperatures
- transport at small frequencies



Spectral functions

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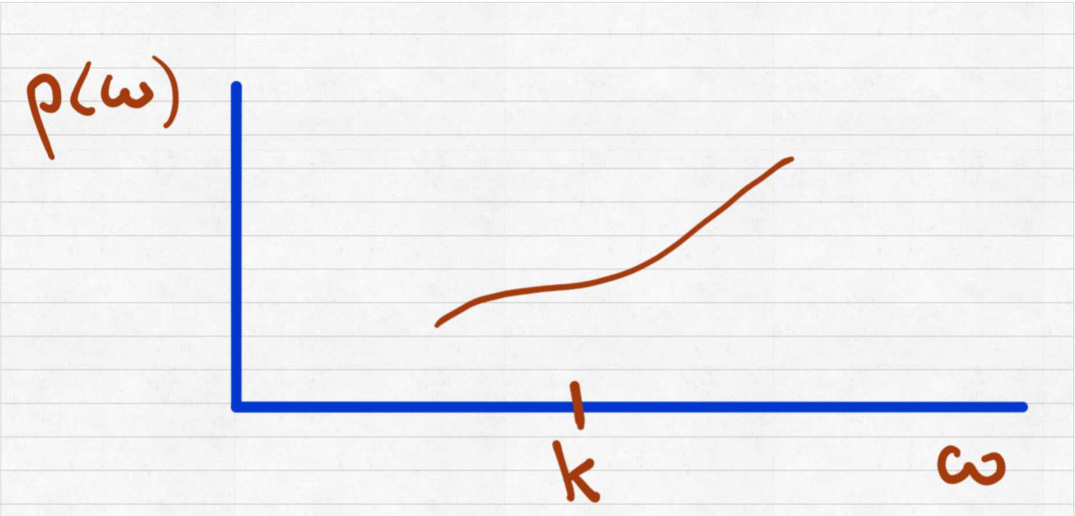
- photon/dilepton production in kinematic range



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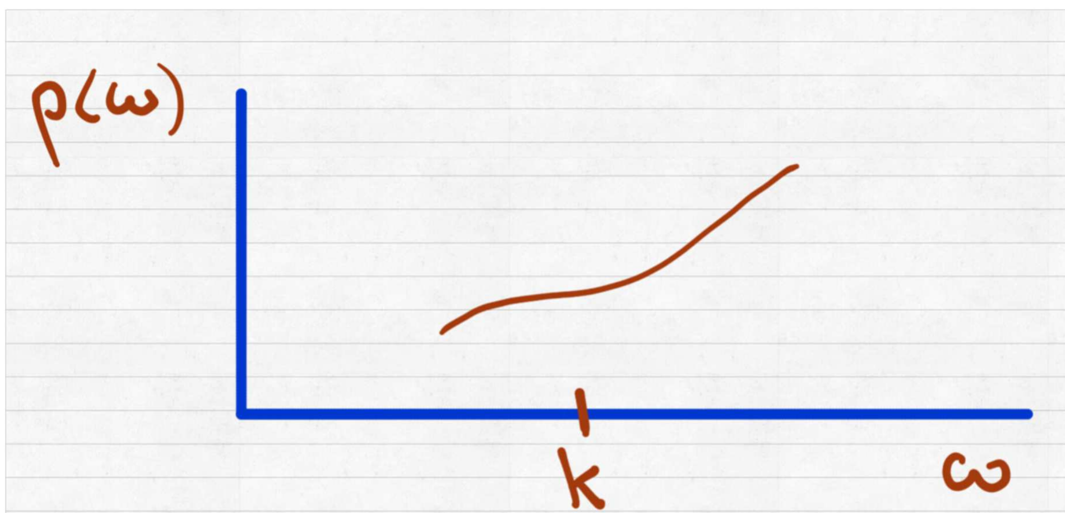

- lots of diverse information in various regimes

Outline

plethora of Green functions

start with basic thermal field theory discussion

- free scalar field
- express Green functions in terms of spectral function
- extend nonperturbatively using KMS condition
- relate euclidean and spectral functions

make connection with lattice QCD

Le Bellac *Thermal Field Theory* Cambridge University Press

Laine & Vuorinen *Basics of Thermal Field Theory* arXiv:1701.01554

Green functions in thermal equilibrium

many Green functions, special role for spectral function

let's start with free fields

- consider one mode of free scalar bosonic field $\phi(x)$

$$\phi_{\mathbf{k}}(t) = \frac{1}{\sqrt{2\omega_{\mathbf{k}}}}\left(a_{\mathbf{k}}e^{-i\omega_{\mathbf{k}}t} + a^\dagger_{\mathbf{k}}e^{i\omega_{\mathbf{k}}t}\right) \qquad \omega_{\mathbf{k}} = \sqrt{\mathbf{k}^2 + m^2}$$

with $\qquad [\phi_{\mathbf{k}}(t), \pi_{\mathbf{k}'}(t)] = i\delta_{\mathbf{k}\mathbf{k}'} \qquad [a_{\mathbf{k}}, a^\dagger_{\mathbf{k}'}] = \delta_{\mathbf{k}\mathbf{k}'}$

- statistics $\qquad \langle a^\dagger_{\mathbf{k}} a_{\mathbf{k}'}\rangle = n_B(\omega_{\mathbf{k}})\delta_{\mathbf{k}\mathbf{k}'}$
- Bose distribution $\qquad n_B(\omega) = 1/(e^{\omega/T} - 1)$

all expectations values are $\propto \delta_{\mathbf{k}\mathbf{k}'}$: consider only one mode

Green functions in thermal equilibrium

- Wightman functions (not time-ordered)

$$G_{\mathbf{k}}^{>}(t-t') = \langle \phi_{\mathbf{k}}(t)\phi_{\mathbf{k}}(t')\rangle \qquad G_{\mathbf{k}}^{<}(t-t') = \langle \phi_{\mathbf{k}}(t')\phi_{\mathbf{k}}(t)\rangle$$

- insert field expansion

$$\begin{aligned} G_{\mathbf{k}}^{>}(t-t') &= \frac{1}{2\omega_{\mathbf{k}}}\left(e^{-i\omega_{\mathbf{k}}(t-t')} + 2n_B(\omega_{\mathbf{k}})\cos[\omega_{\mathbf{k}}(t-t')]\right) \\ &= G_{\mathbf{k}}^{<}(t'-t) \end{aligned}$$

- Feynman propagator (time-ordered)

$$G_{\mathbf{k}}^{F}(t-t') = \theta(t-t')G_{\mathbf{k}}^{>}(t-t') + \theta(t'-t)G_{\mathbf{k}}^{<}(t-t')$$

- consider real and imaginary components

Green functions in thermal equilibrium

- real part, anticommutator, statistical two-point function

$$F_{\mathbf{k}}(t-t') = \frac{1}{2}\left[G^{>}_{\mathbf{k}}(t-t') + G^{<}_{\mathbf{k}}(t-t')\right] = \frac{1}{2}\left\langle\{\phi_{\mathbf{k}}(t), \phi_{\mathbf{k}}(t')\}\right\rangle$$
$$= \frac{1}{2\omega_{\mathbf{k}}}\left[1 + 2n_B(\omega_{\mathbf{k}})\right]\cos[\omega_{\mathbf{k}}(t-t')]$$

- imaginary part, commutator, spectral function

$$\rho_{\mathbf{k}}(t-t') = G^{>}_{\mathbf{k}}(t-t') - G^{<}_{\mathbf{k}}(t-t') = \left\langle[\phi_{\mathbf{k}}(t), \phi_{\mathbf{k}}(t')]\right\rangle$$
$$= \frac{1}{i\omega_{\mathbf{k}}}\sin[\omega_{\mathbf{k}}(t-t')]$$

note:

- ρ does not know about thermal occupation numbers
- $\frac{\partial}{\partial t}\rho_{\mathbf{k}}(t-t')\big|_{t=t'} = -i = \langle[\pi_{\mathbf{k}}(t), \phi_{\mathbf{k}}(t)]\rangle$ commutation relation

Green functions in thermal equilibrium

- retarded and advanced Green functions

$$G^R_{\mathbf{k}}(t-t') = i\theta(t-t')\left\langle[\phi_{\mathbf{k}}(t), \phi_{\mathbf{k}}(t')]\right\rangle$$
$$= i\theta(t-t')\rho_{\mathbf{k}}(t-t') = G^A_{\mathbf{k}}(t'-t)$$

- causality:

$G^R_{\mathbf{k}}(t-t') = 0$ when $t < t'$ $\qquad$ $G^A_{\mathbf{k}}(t-t') = 0$ when $t > t'$

go to frequency space:

$$f(\omega) = \int_{-\infty}^{\infty} dt\, e^{i\omega t} f(t) \qquad f(t) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi}\, e^{-i\omega t} f(\omega)$$

Green functions in thermal equilibrium

spectral function

- in real-time $\rho_{\mathbf{k}}(t-t') = -i\sin[\omega_{\mathbf{k}}(t-t')]/\omega_{\mathbf{k}}$

$$\rho_{\mathbf{k}}(\omega) = \frac{2\pi}{2\omega_{\mathbf{k}}}\left[\delta(\omega-\omega_{\mathbf{k}}) - \delta(\omega+\omega_{\mathbf{k}})\right] = 2\pi\epsilon(\omega)\delta\left(\omega^2-\omega_{\mathbf{k}}^2\right)$$

- single-particle peak at $\omega = \pm\omega_{\mathbf{k}}$
- odd function: $\epsilon(\omega) \equiv \theta(\omega) - \theta(-\omega)$ $\rho_{\mathbf{k}}(-\omega) = -\rho_{\mathbf{k}}(\omega)$
- sum rule

$$\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\,\omega\rho_{\mathbf{k}}(\omega) = \int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\,\frac{2\pi\omega}{2\omega_{\mathbf{k}}}\left[\delta(\omega-\omega_{\mathbf{k}}) - \delta(\omega+\omega_{\mathbf{k}})\right] = 1$$

- consequence of commutation relation $\frac{\partial}{\partial t}\rho_{\mathbf{k}}(t-t')\big|_{t=t'} = -i$

Green functions in thermal equilibrium

Wightman/statistical two-point functions in frequency-space

- all proportional to spectral function

$$G^>_{\mathbf{k}}(\omega) = [n_B(\omega) + 1]\,\rho_{\mathbf{k}}(\omega)$$
$$G^<_{\mathbf{k}}(\omega) = n_B(\omega)\rho_{\mathbf{k}}(\omega)$$
$$F^>_{\mathbf{k}}(\omega) = \left[n_B(\omega) + \frac{1}{2}\right]\rho_{\mathbf{k}}(\omega)$$

- with statistical factors, such that e.g.

$$\rho = G^> - G^< \qquad\qquad F = \frac{1}{2}\left(G^> + G^<\right)$$

Green functions in thermal equilibrium

retarded/advanced Green functions: causality

- analytical in upper/lower half place

$$G_{\mathbf{k}}^{R/A}(\omega) = \frac{1}{\omega_{\mathbf{k}}^2 - (\omega \pm i\epsilon)^2}$$

- retarded: poles at $\omega = \pm\omega_{\mathbf{k}} - i\epsilon$, below real axis
- $G_{\mathbf{k}}^{A}(\omega) = G_{\mathbf{k}}^{R*}(\omega) = G_{\mathbf{k}}^{R}(-\omega)$
- relate to spectral function via dispersion relation

$$G_{\mathbf{k}}^{R}(\omega) = \int_{-\infty}^{\infty} \frac{d\omega'}{2\pi} \frac{\rho_{\mathbf{k}}(\omega')}{\omega' - \omega - i\epsilon}$$

verify using

$$\int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{e^{-i\omega t}}{\omega + i\epsilon} = -i\theta(t)$$

Green functions in thermal equilibrium

- combine dispersion relation $G^R_{\mathbf{k}}(\omega) = \int_{-\infty}^{\infty} \frac{d\omega'}{2\pi} \frac{\rho_{\mathbf{k}}(\omega')}{\omega' - \omega - i\epsilon}$
- with identity $$\frac{1}{x + i\epsilon} = \mathcal{P}\frac{1}{x} - i\pi\delta(x)$$

  or
  $$\frac{1}{x + i\epsilon} - \frac{1}{x - i\epsilon} = \frac{-2i\epsilon}{x^2 + \epsilon^2} = -2i\pi\delta(x)$$
- this yields
  $$\rho_{\mathbf{k}}(\omega) = -i\left[G^R_{\mathbf{k}}(\omega) - G^A_{\mathbf{k}}(\omega)\right] = 2\mathrm{Im}\, G^R_{\mathbf{k}}(\omega)$$
- spectral density is
  - imaginary part of retarded Green function
  - discontinuity across the real axis

Green functions in thermal equilibrium

in summary:

in thermal equilibrium, all Green functions can be expressed in terms of the corresponding spectral function, combined with thermal occupation numbers

- so far this was demonstrated for free fields
- but also correct nonperturbatively, due to Kubo-Martin-Schwinger (KMS) condition

KMS condition

consider two Wightman functions for operators $A(t)$, $B(t)$

$$G^>(t_1 - t_2) = \langle A(t_1)B(t_2)\rangle \qquad G^<(t_1 - t_2) = \langle B(t_2)A(t_1)\rangle$$

- thermal expectation value: $\langle O\rangle = \frac{1}{Z}\mathrm{Tr}\, e^{-\beta H} O \quad Z = \mathrm{Tr}\, e^{-\beta H}$
- time evolution: $A(t) = e^{iHt}A(0)e^{-iHt}$
- note: in equilibrium $\Rightarrow$ time translation invariance $t_1 - t_2$

cyclicity of the trace

$$\begin{aligned} G^>(t_1 - t_2) &= \frac{1}{Z}\mathrm{Tr}\, e^{-\beta H} e^{iHt_1}A(0)e^{-iHt_1}e^{iHt_2}B(0)e^{-iHt_2} \\ &= \frac{1}{Z}\mathrm{Tr}\, e^{-\beta H} e^{\beta H} e^{iHt_2}B(0)e^{-iHt_2}e^{-\beta H}e^{iHt_1}A(0)e^{-iHt_1} \\ &= \frac{1}{Z}\mathrm{Tr}\, e^{-\beta H} B(t_2 - i\beta)A(t_1) = G^<(t_1 - t_2 + i\beta) \end{aligned}$$

KMS condition

identity $$G^>(t_1 - t_2) = G^<(t_1 - t_2 + i\beta)$$

- crucial: same hamiltonian for time evolution e^{iHt} and density matrix $e^{-\beta H}$
- hence only holds in thermal equilibrium
- since H includes interactions, holds nonperturbatively

in frequency space $$G^>(\omega) = e^{\beta\omega} G^<(\omega)$$

- relate all other Green functions
- spectral function $\rho(\omega) = G^>(\omega) - G^<(\omega) = (e^{\beta\omega} - 1)G^<(\omega)$
- hence $G^>(\omega) = [n_B(\omega) + 1]\,\rho(\omega)$ $\quad G^<(\omega) = n_B(\omega)\rho(\omega)$
- Bose distribution emerges *exactly*
- all *interesting* physics is hidden in spectral function

Fermions

few changes due to anticommuting fields

- Wightman functions

$$S^{>}(t_1-t_2) = \left\langle \psi(t_1)\bar{\psi}(t_2)\right\rangle \qquad S^{<}(t_1-t_2) = -\left\langle \bar{\psi}(t_2)\psi(t_1)\right\rangle$$

- KMS condition

$$S^{>}(t_1-t_2) = -S^{<}(t_1-t_2+i\beta) \qquad S^{>}(\omega) = -e^{\beta\omega}S^{<}(\omega)$$

- spectral function, anticommutator

$$\rho(t_1-t_2) = \left\langle \{\psi(t_1), \bar{\psi}(t_2)\}\right\rangle = S^{>}(t_1-t_2) - S^{<}(t_1-t_2)$$

- Fermi distribution $n_F(\omega) = 1/(e^{\omega/T}+1)$ emerges

$$S^{>}(\omega) = [1-n_F(\omega)]\,\rho(\omega) \qquad S^{<}(\omega) = -n_F(\omega)\rho(\omega)$$

Fermions vs bosons

- bosons

$$G^{>}(\omega) = [1 + n_B(\omega)]\, \rho(\omega) \qquad G^{<}(\omega) = n_B(\omega)\rho(\omega)$$

- Bose enhancement $1 \to 1 + n_B$
- fermions

$$S^{>}(\omega) = [1 - n_F(\omega)]\, \rho(\omega) \qquad S^{<}(\omega) = -n_F(\omega)\rho(\omega)$$

- Pauli blocking $1 \to 1 - n_F$
- general rule: from bosons to fermions $n_B \to -n_F$

connection with lattice QCD: euclidean time

Euclidean correlators

- interpret $e^{-\beta H}$ as evolution operator
- euclidean time $\tau = it$: $0 < \tau < \beta = 1/T$
- correlator $G^E_{\mathbf{k}}(\tau) = \langle \phi_{\mathbf{k}}(\tau)\phi_{\mathbf{k}}(0)\rangle$

$$G^E_{\mathbf{k}}(\tau) = T\sum_n e^{-i\omega_n\tau} G^E_{\mathbf{k}}(\omega_n) \qquad G^E_{\mathbf{k}}(\omega_n) = \int_0^{1/T} d\tau\, e^{i\omega_n\tau} G^E_{\mathbf{k}}(\tau)$$

- (anti)periodic boundary conditions for bosons (fermions)
- Matsubara frequencies $\omega_n = 2\pi nT$ $[(2n+1)\pi T]$
- free correlator for scalar field

$$G^E_{\mathbf{k}}(\tau) = \frac{\cosh[\omega_{\mathbf{k}}(\tau - 1/2T)]}{2\omega_{\mathbf{k}}\sinh(\omega_{\mathbf{k}}/2T)} \qquad G^E_{\mathbf{k}}(\omega_n) = \frac{1}{\omega_n^2 + \omega_{\mathbf{k}}^2}$$

Euclidean correlators

dispersion relation and analytical continuation

$$G^E_{\mathbf{k}}(\omega_n) = \frac{1}{\omega_n^2 + \omega_{\mathbf{k}}^2} \qquad G^R_{\mathbf{k}}(\omega) = \frac{1}{-(\omega + i\epsilon)^2 + \omega_{\mathbf{k}}^2}$$

- relation between euclidean and retarded Green functions

$$G^R_{\mathbf{k}}(\omega) = G^E_{\mathbf{k}}(i\omega_n \to w + i\epsilon)$$

- general dispersion relation

$$G^E_{\mathbf{k}}(\omega_n) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{\rho_{\mathbf{k}}(\omega)}{\omega - i\omega_n}$$

- all functions related by analyticity and KMS conditions

Thermal QCD

all functions related by analyticity and KMS conditions

- QCD: strongly interacting system
- do not rely on perturbation theory
- lattice simulations
- yield numerically determined euclidean correlators

make relation with spectral functions more explicit

Lattice correlators and spectral functions

- spectral relation between $G^E_{\mathbf{k}}(\tau)$ and $\rho_{\mathbf{k}}(\omega)$

$$G^E_{\mathbf{k}}(\omega_n) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{\rho_{\mathbf{k}}(\omega)}{\omega - i\omega_n}$$

- go to euclidean time

$$G^E_{\mathbf{k}}(\tau) = T \sum_n e^{-i\omega_n \tau} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{\rho_{\mathbf{k}}(\omega)}{\omega - i\omega_n}$$
$$= \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} K(\tau,\omega)\rho_{\mathbf{k}}(\omega)$$

with kernel

$$K(\tau,\omega) = T \sum_n \frac{e^{-i\omega_n \tau}}{\omega - i\omega_n}$$

Lattice correlators and spectral functions

- spectral relation $\quad G^E_{\mathbf{k}}(\tau) = \int \frac{d\omega}{2\pi} K(\tau,\omega)\rho_{\mathbf{k}}(\omega)$
- kernel

$$K(\tau,\omega) = T\sum_n \frac{e^{-i\omega_n\tau}}{\omega - i\omega_n} = \begin{cases} e^{-\omega\tau}\left[1+n_B(\omega)\right] & \text{bosons} \\ e^{-\omega\tau}\left[1-n_F(\omega)\right] & \text{fermions} \end{cases}$$

- bosons:
  $\rho_{\mathbf{k}}(-\omega) = -\rho_{\mathbf{k}}(\omega)$ odd $\Rightarrow$ odd part of kernel survives

$$K_B(\tau,\omega) = \ldots = \frac{\cosh[\omega(\tau - 1/2T)]}{\sinh(\omega/2T)}$$

note: $\quad 1+n_B(\omega) = \dfrac{e^{\omega/2T}}{2\sinh(\omega/2T)} \qquad n_B(\omega)+n_B(-\omega)+1 = 0$

- fermions: slightly more involved, see later

Lattice correlators and spectral functions

spectral relation $\qquad G^E(\tau) = \int \frac{d\omega}{2\pi} K(\tau,\omega)\rho(\omega)$

- if $G^E(\omega_n)$ is known analytically: simple route

$$G^E(\tau) \quad \to \quad G^E(\omega_n) \quad \to \quad G^R(\omega) \quad \to \quad \rho(\omega)$$

- but correlators only known numerically (i.e. with errors) at finite number of points
- lattice discretisation: $0 < \tau < 1/T$ with $1/T = a_\tau N_\tau$
  $\Rightarrow \quad$ at most N_τ points
- for bosons: kernel symmetric $K(\tau,\omega) = K(1/T-\tau,\omega)$
  $\Rightarrow \quad$ at most $N_\tau/2$ points
- $\rho(\omega)$ in principle continuous (and possibly nonanalytical) function with $-\infty < \omega < \infty$

Lattice correlators and spectral functions

spectral relation $G^E_{\mathbf{k}}(\tau) = \int \frac{d\omega}{2\pi} K(\tau,\omega)\rho_{\mathbf{k}}(\omega)$

- ill-posed problem: direct inversion not defined
- extract continuous function $\rho_{\mathbf{k}}(\omega)$ from finite $\mathcal{O}(N_\tau)$ number of numerically determined data points
- longstanding problem across science: image reconstruction
- provide some additional input to regulate the inversion:
  - Bayesian methods
  - Maximum Entropy Method (MEM)
  - Bayesian Reconstruction (BR)
  - Backus-Gilbert
  - Ansätze
  - ...

possibly see other talks (Rothkopf, Francis, Kaczmarek)

Thermal spectral functions

next time: what to expect?

- already encountered single-particle spectral function

$$\rho_{\mathbf{k}}(\omega) = 2\pi\epsilon(\omega)\delta\left(\omega^2 - \omega_{\mathbf{k}}^2\right)$$

- single-particle peak at $\omega = \pm\omega_{\mathbf{k}}$
- relevant for QCD at $T = 0$: spectrum of hadrons

increase the temperature:

- in-medium effects
- thermal broadening
- deconfinement
- symmetry restoration
- transport
- ...

discuss this for mesons and baryons