Two-Component Landau Liquid Model for Two-Color QCD

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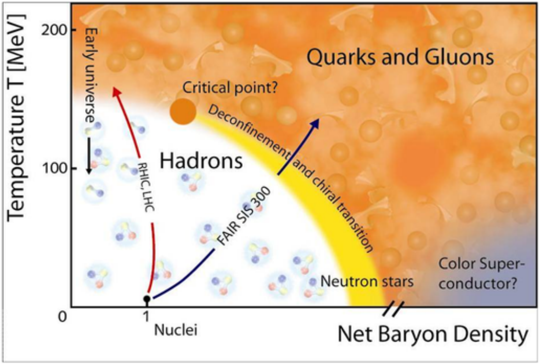
Introduction

In conditions of high temperature and density, it is expected that QCD goes through a phase transition between confined hadrons and a *Quark-Gluon Plasma* (QGP).



Speculated matter phase diagram, from theoretical modeling and some experimental data

There is experimental experimental evidence for QGP from RHIC and LHC

Introduction

Today challenge in the field: describe the phase-diagram of QCD

What we know

- Equation of State similar of an ideal gas
- Behaves as a liquid of low viscosity
- Perturbative QCD is unable to describe QGP

Chernodub *et al.* suggested to model the QGP by a two-component model, usually used for superfluids (Theor. Math. Phys., v. 170, 2012)

About Hydrodynamics

Hydrodynamics

Effective theory applied in situations when conservation laws alone are able to describe a system

Relativistic Hydrodynamic Equations for Frictionless Fluid

$$\partial_\mu T^{\mu\nu} = 0$$
$$\partial_\mu (n u^\mu) = 0$$

- n is a number density, e.g. barionic number
- u^μ is the four-velocity of the fluid

From Lorentz invariance[1]

$$T^{\mu\nu} = (\varepsilon + P)\, u^\mu u^\nu - g^{\mu\nu} P$$
$$u^\mu = \frac{dx^\mu}{d\tau}$$

[1] We adopt $g^{\mu\nu} = \mathrm{diag}(1, -1, -1, -1)$.

The Two-Fluid Model

Proposed by Tisza (1938, 1940, 1947) and Landau (1941) independently to describe ^{4}He. Its characteristics

- Two-compenents: normal and superfluid.
- The superfluid component is a Bose-Einstein condensate.
- Each component interpenetrates each other.
- They have different four-velocity and densities.

The conserved quantities become

$$J^\mu = \rho_n u_n^\mu + \rho_s u_s^\mu$$
$$T^{\mu\nu} = T_n^{\mu\nu} + T_s^{\mu\nu} = (\varepsilon_n + P_n)\, u_n^\mu u_n^\nu + \mu \rho_s u_s^\mu u_s^\nu - g^{\mu\nu} P$$

Note, we used the thermodynamic relation $\varepsilon_s + P_s = sT + \mu\rho_s$ and that the condensate has zero entropy ($s = 0$).

Linear Response Theory

Small fluctuations around equilibrium can be computed by assuming that the system responds linearly to a small perturbation

Pick a perturbation on H

$$\delta H(t) = \int_V d^3x \, U_j(\mathbf{x}, t)\theta(-t)T^{0j}(\mathbf{x}, t)$$

- From Heisenberg picture $\dfrac{\partial \langle T^{0i}(\mathbf{x}, t)\rangle}{\partial t} = i \left\langle [\delta H(t), T^{0i}(\mathbf{x}, t)] \right\rangle$
- Supposing δH small

$$\langle T^{0i}(\mathbf{x}, t)\rangle \cong \langle T^{0i}(\mathbf{x}, 0)\rangle + i \int_0^t dt' \left\langle [\delta H(t'), T^{0i}(\mathbf{x}, 0)] \right\rangle + \mathcal{O}(\delta H^2)$$

or

$$\delta\langle T^{0i}(\mathbf{x}, t)\rangle = i \int_0^t dt' \int_V d^3x' \, U_j(\mathbf{x}', t') \, \theta(-t') \left\langle [T^{0j}(\mathbf{x}', t'), T^{0i}(\mathbf{x}, 0)] \right\rangle$$

Linear Response Theory

Retarded Green Function

$$G^{ij,R}(\mathbf{x}' - \mathbf{x}, t') \equiv i\theta(-t') \left\langle [T^{0j}(\mathbf{x}', t'), T^{0i}(\mathbf{x}, 0)] \right\rangle$$

It is useful to go to momentum space in the $\mathbf{x} = 0$ coordinate

$$\int d^4x' e^{-ik^\mu x_\mu} G^{ij,R}(\mathbf{x}' - 0, t') = \tilde{G}^{ij,R}(\mathbf{k}, \omega)$$

And from rotational invariance, we may write

$$\tilde{G}^{ij,R}(\mathbf{k}, 0) = \frac{k^i k^j}{\mathbf{k}^2} G^{\parallel}(\mathbf{k}) + \left(\delta^{ij} - \frac{k^i k^j}{\mathbf{k}^2} \right) G^{\perp}(\mathbf{k})$$

Linear Response Theory

Known property of superfluid: $\nabla \times \mathbf{v} = 0$

- If perturbation parallel to u_s^i, U^i will be added to both velocity components.
- If perturbation perpendicular to u_s^i, U^i will be added only to normal component.

Using $\varepsilon_n + P_n = sT + \mu\rho_n$ and taking $\mathbf{k} \to 0$

$$\lim_{\mathbf{k}\to 0} G^{\parallel}(\mathbf{k}) = sT + \mu\rho \qquad \lim_{\mathbf{k}\to 0} G^{\perp}(\mathbf{k}) = sT + \mu\rho_n$$

Or

$$\lim_{\mathbf{k}\to 0} \tilde{G}^{ij,\,R}(\mathbf{k},0) = \delta^{ij}[sT + \mu(2\rho_n + \rho_s)] + \frac{k^i k^j}{\mathbf{k}^2}\mu\rho_s$$

Test Proposal

Chernodub, Verschelde e Zhakarov, **Theor. Math. Phys.**, v. 170, p. 211–216, 2012 proposed to compute in a LQCD simulation

$$G_R^{0i,0j}(k) = -i \int d^4x \exp(ik^\mu x_\mu)\theta(x_0) \left\langle \left[T^{0i}(x), T^{0j}(0)\right] \right\rangle .$$

Confrontation between lattice data for $G_R^{0i,0j}(k)$ and the general shape may show if the model is compatible (or not!) with QGP.

Regularization via Lattice Discretization

QCD version:

$$\langle \mathcal{O}(\bar{\psi}, \psi, A_\mu) \rangle = \frac{\int \mathcal{D}[\bar{\psi}(x)\psi(x)A_\mu(x)]\, \mathcal{O}(\bar{\psi}, \psi, A_\mu) e^{iS_{QCD}(\bar{\psi}, \psi, A_\mu)}}{\int \mathcal{D}[\bar{\psi}(x)\psi(x)A_\mu(x)] e^{iS_{QCD}(\bar{\psi}, \psi, A_\mu)}}$$

$$\mathcal{D}[\bar{\psi}(x)\psi(x)A_\mu(x)] = \mathcal{D}\bar{\psi}(x)\mathcal{D}\psi(x)\mathcal{D}A_\mu(x)$$

- QCD has UV divergences
- Introduction of 4D lattice of spacing a does the job of a regulator
- Lattice in spatial directions is infinite, but in time direction contains only $N_\tau = \beta/a$ sites.

Notice: This implies that taking the limit $a \to 0$ implies adjusting the bare parameters as to keep an observable constant, e.g. equal its physical value.

QFT at High Temperature and Finite Density

Expected value of an operator $\mathcal{O}$

$$\langle \mathcal{O} \rangle = \mathrm{Tr}\left[\mathcal{O} e^{-\beta(\hat{H}-\mu\hat{N})}\right] = \int d\phi \, \langle\phi| \, e^{-\beta(\hat{H}-\mu\hat{N})} \, |\phi\rangle$$

Some transformations

- Split the exponential in N slices: $e^{-\beta(\hat{H}-\mu\hat{N})} = \prod_{i=1}^{N} e^{-a(\hat{H}_i-\mu\hat{N}_i)}$
- Insert completeness relations $\mathbb{1} = \int d\phi_i \, |\phi_i\rangle \langle\phi_i|$ and $\mathbb{1} = \int \frac{d\pi_i}{2\pi} \, |\pi_i\rangle \langle\pi_i|$ between slices.
- Perform same procedure for spatial directions and take continuum limit $(a \to 0)$

$$\langle \mathcal{O} \rangle = \frac{1}{Z} \int \mathcal{D}\phi \mathcal{D}\pi \exp\left\{ - \int_0^\beta d\tau \int d^3x \; i\pi\partial_\tau\phi - \mathcal{H} + \mu\mathcal{N} \right\} .$$

Some approximations

- Quenched simulations: We do not include dynamical fermions
  - Main reason: dynamical fermions simulations implies the need to invert high-dimension matrices after each Monte Carlo step (hybrid MC simulation), i.e. its slow.
  - From data from early LQCD simulations, there is the observation that the qualitative behavior are not affected
  - Source of systematical error
- Use of two-color QCD, i.e. SU(2) gauge symmetry
  - Less memory use, one of the main bottle necks during simulations [SU(N) groups have $N(N-1)$ parameters]
  - One may use it as basis for a full SU(3) simulations
  - Not recommended to study phase transition (has different phase transition order)

Energy-momentum tensor on a Gauge Theory

$$T^R_{\mu\nu} = Z_T \left\{ T^{(1)}_{\mu\nu} + z_T T^{(3)}_{\mu\nu} + z_S \left[T^{(2)}_{\mu\nu} - \langle T^{(2)}_{\mu\nu} \rangle_0 \right] \right\}$$

$$T^{(1)}_{\mu\nu} = (1 - \delta_{\mu\nu}) \frac{1}{g_0^2} F^a_{\mu\nu} F^a_{\mu\nu}$$

$$T^{(2)}_{\mu\nu} = \delta_{\mu\nu} \frac{1}{4g_0^2} F^a_{\alpha\beta} F^a_{\alpha\beta}$$

$$T^{(3)}_{\mu\nu} = \delta_{\mu\nu} \frac{1}{g_0^2} \left(F^a_{\mu\alpha} F^a_{\mu\alpha} - \frac{1}{4} F^a_{\alpha\beta} F^a_{\alpha\beta} \right)$$

Note que

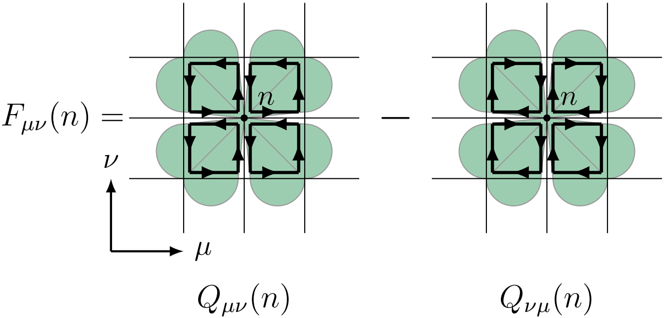
- We need only off-diagonal elements, thus $T^R_{0i} = Z_T T^{(1)}_{0i}$
- Renormalization is done by settiing the scale a of the lattice through changes in g_0.

Energy-momentum tensor on a Gauge Theory

For $F_{\mu\nu}$, we use the clover discretization

$$F^a_{\mu\nu}(n) = -\frac{i}{4a^2}\,\mathrm{Tr}\left\{\left[Q_{\mu\nu}(n) - Q_{\nu\mu}(n)\right]\lambda^a\right\}$$

$$Q_{\mu\nu}(n) = U_{\mu\nu}(n) + U_{\mu\nu}(n - a\hat{\mu}) + U_{\mu\nu}(n - a\hat{\nu}) + U_{\mu\nu}(n - a\hat{\mu} - a\hat{\nu})$$

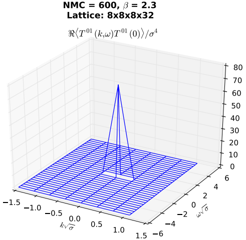


$\langle T^{0i}T^{0j}\rangle(k,\omega)$ at Lattice $8\times 8\times 8\times 32$

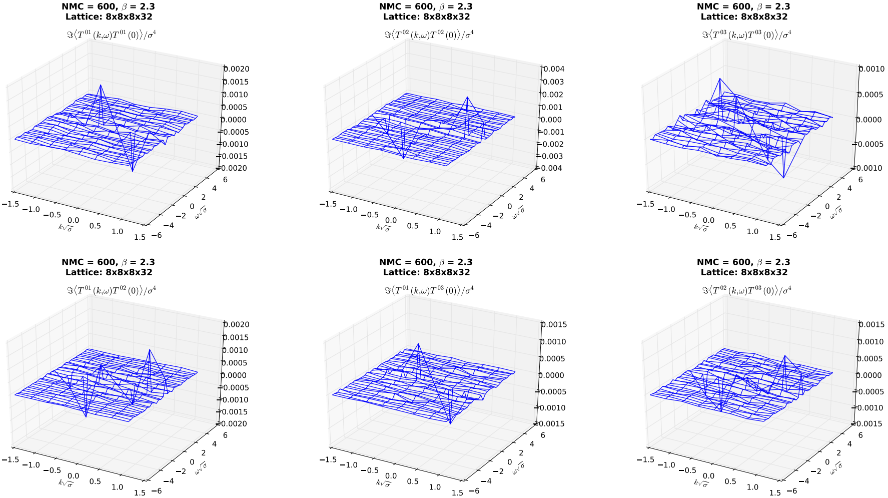
$$\beta = 2.3 \quad \sigma a^2 = 0.136$$

Polyakov Loop Compatible with zero, i.e. confined phase yet.

- $\Re[\langle T^{0i}T^{0j}\rangle(k,\omega)]$ ressembles $\delta(k)\delta(\omega)$



- Imaginary part is more rich. In particular for each $\{i,j\}$, we obtain a slightly different behavior



- Dependence on ω is $\delta(\omega)$
- For $k \to 0$, $\Im[\langle T^{0i}T^{0j}\rangle(k,\omega)] \to 0$

Considerations and Next Steps

- So far, we did not see any evidence of a two-component liquid.
  - But we were unable to cross to the deconfined phase yet.
  - It is a small lattice, so we do not have precision on $\mathbf{k} \to 0$.
- Increase lattice side, to tipically $N_s = 64$ at least.
  - With current size, reducing N_τ to 4 kept $P \sim 0$.
- Change color group to SU(3).

Thank you

Problema de Linde

A.D.Linde, Phys. Lett. **B 96**, 289 (1980)

$$\Omega_N(T) \sim g^6 T^4 \left[\frac{g^2 T}{m(T)} \right]$$

- $m(T)$ é a massa gluônica gerada dinamicamente.
- $\Omega_N(T)$ diverge para $m(T) < g^2 T$.
- Entretanto única informação possível de extrair perturbativamente: $m(T) \lesssim g^2 T$.
- Teoria de perturbação falha a temperatura finita.

Discretização Fermiônica

$$S_F = a^4 \sum_{m,n} \bar{\psi}_\alpha(m) K_{\alpha\beta}(m,n) \psi_\beta(n)$$

$$K^{\mathsf{naive}}_{\alpha\beta}(m,n) = \sum_{\mu=1}^{4} \frac{(\gamma_\mu)_{\alpha\beta}}{2a} \times \left[U_\mu(m) \delta_{n,\, m+a\hat{e}_\mu} - U^\dagger(m - a\hat{e}_\mu) \delta_{n,\, m-a\hat{e}_\mu} \right]$$

$$+ \, m \, \delta_{\alpha\beta} \delta_{mn}$$

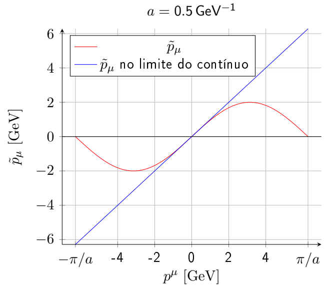
Discretrização Fermiônica

Uso da derivada simétrica na discretização da ação fermiônica é ingênuo

- Para permitir cálculo analítico: $U_\mu(x) = \mathbb{1}$

$$\langle \psi_\alpha(m)\bar{\psi}_\beta(n) \rangle = \lim_{a\to 0} \int_{-\pi/a}^{\pi/a} \frac{d^4p}{(2\pi)^4} \frac{\left[-i\sum_\mu \gamma_\mu \tilde{p}_\mu + M\right]_{\alpha\beta}}{\sum_\mu \tilde{p}_\mu^2 + M^2} e^{ip(m-n)}$$

$$\tilde{p}_\mu = \frac{1}{a}\sin(p_\mu a)$$



Discretrização Fermiônica

Solução: Adicionar termo extra

$$K^W_{\alpha\beta}(m,n) = K_{\alpha\beta}(m,n) - a\sum_{\mu=1}^{4}\delta_{\alpha\beta}\frac{U_\mu(n)\delta_{n,\,n+a\hat{e}_\mu} - 2\delta_{nm} + U^\dagger_\mu(m-a\hat{e}_\mu)\delta_{n,\,m-a\hat{e}_\mu}}{2a^2}$$

Com $U_\mu(x) = \mathbb{1}$, temos

$$\langle \psi_\alpha(m)\bar{\psi}_\beta(n)\rangle = \lim_{a\to 0}\int_{-\pi/a}^{\pi/a}\frac{d^4p}{(2\pi)^4}\frac{\left[-i\sum_\mu \gamma_\mu \tilde{p}_\mu + M(p)\right]_{\alpha\beta}}{\sum_\mu \tilde{p}^2_\mu + M(p)^2}e^{ip(m-n)},$$

$$M(p) = M + \frac{2}{a}\sum_{\mu=1}^{4}\sin^2(p_\mu a/2)$$

M é infinito no limite $a \to 0$, a menos que $p_\mu = 0$.

Motivando identificação de $\partial^\mu\varphi = v^\mu/\mu$

$$\phi(x) = \rho(x)e^{i\varphi(x)}$$

$$n = \sqrt{j^\mu j_\mu} = \rho^2\sqrt{\partial_\mu\varphi\partial^\mu\varphi}$$

$$j^\mu = \rho^2\partial^\mu\varphi = nv^\mu$$

$$v^\mu = \frac{\partial^\mu\varphi}{\sqrt{\partial_\nu\varphi\partial^\nu\varphi}}$$

Hunting the QGP

Sinais do QGP que os times do RHIC procuraram (2005):

1. Observação de propriedades do sistema compatível com a formação do QGP: Grandes densidades de energia, crescimento entrópico, platô na evolução de varíaveis termodinâmicas, expansão e tempo de vida anormais do sistema
2. Modificação de propriedades específicas de partículas: modificação da massa e da largura de ressonâncias e modificação da taxa de produção destas partículas

Em particular, medidas do "Fluxo Elíptico"[2] forneceram a evidência mais contundente da existência do QGP. Também foram observadas densidades de energia excepcionais

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[2] A medida da assimetria do fluxo de energia-momento da colisão

Acoplamento na QED e na QCD

QCD can be built from QED by changing the gauge symmetry group from U(1) $\rightarrow$ SU(3)

$$\mathcal{L}_{QED} = \sum_l \bar{\psi}_l (i\not{D} - m_l)\psi_l - \frac{1}{4} F^{\mu\nu} F_{\mu\nu}$$

$$\mathcal{L}_{QCD} = \sum_f \bar{\psi}_f^i (i\not{D}^{ij} - \delta^{ij} m_f)\psi_f^j - \frac{1}{4} G^{\mu\nu,\, a} G^a_{\mu\nu}$$

$$F^a_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

$$G^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g_0 f^{abc} A^b_\mu A^c_\nu$$

The extra term adds gluon-gluon interactions, with no analogous in QED and changes the behavior of the running coupling of QCD.

Commutators expectation value

- Path-integral approach automatically time-orders the operators.
- E.g. Free real scalar field of mass m

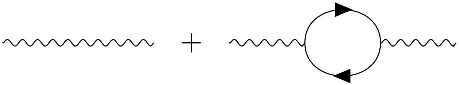
$$\langle\hat{\phi}(x)\hat{\phi}(y)\rangle = \frac{1}{Z}\int \mathcal{D}\phi e^{iS}\phi(x)\phi(y)$$
$$= -\frac{1}{2}\int \frac{d^4p}{(2\pi)^4}\frac{e^{-ip\cdot(x-y)}}{p^2-m^2} = \langle T\hat{\phi}(x)\hat{\phi}(y)\rangle$$

- Thus, if we compute $\langle T^{0i}(x)T^{0j}(y)\rangle$ for all x, y, we can easily obtain the commutator
$\langle[T^{0i}(x), T^{0j}(0)]\rangle = \langle T^{0i}(x)T^{0j}(0)\rangle - \langle T^{0j}(0)T^{0i}(x)\rangle$
- On the lattice

$$\langle T^{0i}(m)T^{0j}(0)\rangle = \frac{1}{N}\frac{1}{L/a}\sum_{k=1}^{L/a}\sum_{n=1}^{N} T^{0i}_n(m+k)T^{0j}_n(k)$$

Acoplamento na QED e na QCD

Cálculo ingênuo destes diagramas da QED resulta em infinitos



Solução: Modifica-se a Lagrangiana, introduzindo constantes Z_i, $i = 0,\ 1,\ 2,\ 3$ e um *regulador* na integral do loop

$$\mathcal{L} = -\frac{1}{4} Z_3 F_{\mu\nu} F^{\mu\nu} + Z_2 \bar{\psi} \left[i\partial\!\!\!/ + e \frac{Z_1}{Z_2} A\!\!\!/ \right] \psi - Z_0 m \bar{\psi} \psi + \text{Termo de Fixação de Gauge}$$

$$= -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \bar{\psi} \left[i\partial\!\!\!/ + e A\!\!\!/ \right] \psi - m \bar{\psi} \psi - \frac{1}{4} (Z_3 - 1) F_{\mu\nu} F^{\mu\nu}$$

$$+ i(Z_2 - 1) \bar{\psi} \partial\!\!\!/ \psi + (Z_1 - 1) e \bar{\psi} A\!\!\!/ \psi - (Z_0 - 1) m \bar{\psi} \psi + \text{Termo de Fixação de Gauge}$$

Nota: Nenhuma simetria é violada pela introdução de Z_i

Comparação entre QED e QCD

$$\begin{aligned}
G^{\mu\nu}(q^2) &= -i\frac{g^{\mu\nu}}{q^2+i\varepsilon}[1+\Pi(q^2)] \\
\Pi(q^2) &= \frac{ie^2}{(D-1)q^2}\times \\
&\int \frac{d^Dp}{(2\pi)^D}\int_0^1 dx \frac{4(2-D)(p^2-p\cdot q)+4Dm^2}{\{x(p^2-m^2)+(1-x)[(p-q)^2-m^2]\}^2} + (Z_3-1)^{(1)} \\
&= -\frac{4e^2}{(4\pi)^{-D/2}}\Gamma\left(\frac{4-D}{2}\right)
\end{aligned}$$

In D dimensions

Acoplamento na QED e na QCD

- Z_i arbitrários. Depende da escolha do esquema de renormalização e das condições de renormalização impostas
- Exemplo: Renormalização da carga $e_R^2(q^2) = Z_3(\mu)e^2$. Esquema $\overline{\text{MS}}$ com a condição de renormalização ser tal que o polo de $G^{\mu\nu}(q^2)$ seja $q^2 = \mu^2$
- Das equações do grupo de renormalização (supondo $-q^2 \gg m_R^2$)

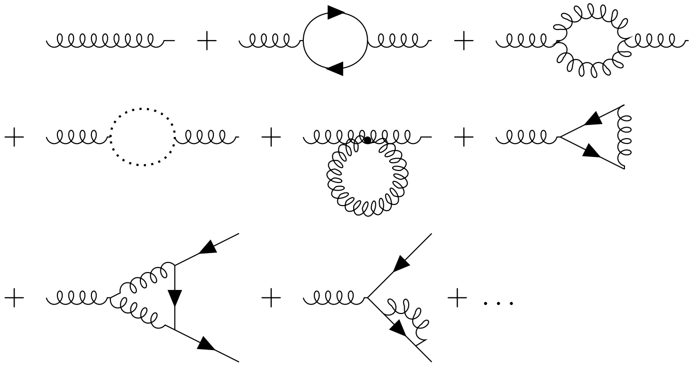
$$e_R^2(q^2) = \frac{e_R^2(\mu^2)}{1 - C e_R^2(\mu^2)\ln(|q^2|/\mu^2)} \qquad C = C(N_f) > 0$$

- Se para $|q^2| = \Lambda^2$ tivermos $1 - C e_R^2(\mu^2)\ln(\Lambda^2/\mu^2) = 0$ então

$$e_R^2(q^2) = \frac{1}{C\ln\left(\Lambda^2/|q^2|\right)}$$

Acoplamento na QED e na QCD

Termo extra na lagrangiana da QCD introduz mais diagramas contribuindo para a renormalização do acoplamento



Nota: O tracejado indica uma partícula “ghost”, que é somente um artifício matemático para fixar gauge sem a introdução de graus físicos adicionais de liberdade

Acoplamento na QED e na QCD

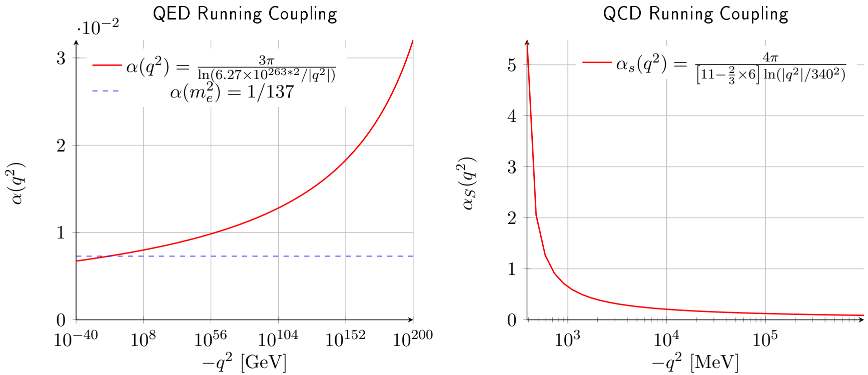
Novamente, via equações do grupo de renormalização

$$\alpha_s(q^2) = \frac{\alpha_s(\mu^2)}{1 + \frac{11 - \frac{2}{3}N_f(|q^2|)}{4\pi}\alpha_s(\mu^2)\ln\left(\frac{|q^2|}{\mu^2}\right)}$$

Impondo $1 + \frac{11 - \frac{2}{3}N_f(\Lambda^2_{QCD})}{4\pi}\alpha_s(\mu^2)\ln\left(\frac{\Lambda^2_{QCD}}{\mu^2}\right) = 0$

$$\alpha_s(q^2) = \frac{4\pi}{\left[11 - \frac{2}{3}N_f(q^2)\right]\ln(|q^2|/\Lambda^2_{QCD})}$$

Acoplamento na QED e na QCD



Conclusion: QCD it is not perturbative at low energies regime (below $\sim 1\,\text{GeV}$)

Obs.: Para o acoplamento da QED, utilizamos $C = 1/3\pi$ e $\alpha^{-1}(M_Z) = 127.940$. Para o acoplamento da QCD, utilizamos $N_f = 3$ e $\Lambda_{QCD} = 340$. Valores retirados da edição de 2014 do PDG

Renormalização na Rede

Simples retirada do regulador obviamente reintroduzirá as divergências.
Alternativas:

- Renormalização perturbativa: $g_0 \to g(a) = Z(a) g_0$
  - Com uma dada condição de normalização fixada, e.g. $M_{\text{Hádron}} = \text{Massa Física}$
- Estudo na "região de escala"
  - Medir duas grandezas[3], e.g. $\langle \psi_\alpha(m) \psi_\beta(n) \rangle / M^3$ e $|m - n| / \hat{M}$
  - Se ao variarmos os parâmetros da rede, mantendo $|m - n| / \hat{M}$ fixo, atingirmos uma região em que $\langle \psi_\alpha(m) \psi_\beta(n) \rangle / M^3$ independe de M, então estamos em um regime em que podemos extrair física da simulação

[3] Grandezas com um circunflexo são medidas em unidades da rede, e.g. $\hat{M} = M/a$

Exemplo: Hidrodinâmica do Campo Escalar com Quebra Espontânea de Simetria

$$\mathcal{L} = \partial_\mu \phi^* \partial^\mu \phi - \lambda(\phi^* \phi - v^2)^2$$

Leis de conservação

- Invariância translacional: $\partial_\mu T^{\mu\nu} = 0$
- Fase global ($\phi' = \phi e^{i\alpha}$): $\partial_\mu j^\mu = 0$

Equações de movimento clássicas via parênteses de Poisson:
$\frac{\partial A}{\partial t} = \{\mathcal{H}, A\}$

Método de cálculo: $\{A, B\} \rightarrow \frac{1}{i\hbar}[\hat{A}, \hat{B}] \rightarrow$ Resultado desejado

Exemplo:

$$\{p_i, x_j\} = -\delta_{ij}$$

$$[\hat{p}_i, \hat{x}_j] = -i\hbar\delta_{ij} \rightarrow \{p_i, x_j\} = \frac{1}{i\hbar}[\hat{p}_i, \hat{x}_j] = -\delta_{ij}$$

Exemplo: Hidrodinâmica do Campo Escalar com Quebra Espontânea de Simetria

Passo a passo:

1. Resolver Eqs. de Movimento, tipicamente EDPs de segunda ordem
   - Como temos 2 campos, teremos duas equações e assim 4 constantes de integração: A, B, C e D
2. Calcular Função de Partição: $Z = \int dA\, dB\, dC\, dD\, e^{-\beta \int_V d^3x\, T^{00}(x)}$
3. Calcular Energia Livre de Helmholtz: $F = -\frac{1}{\beta} \ln Z$
4. Com isto, é possível identificar componentes microscópicas com as macroscópicas, e.g. $p = \frac{\partial F}{\partial V}\Big|_{T,n}$

Exemplo: Hidrodinâmica do Campo Escalar com Quebra Espontânea de Simetria

Resultado Final

$$T^{\mu\nu} = (\varepsilon + P)u^\mu u^\nu - Pg^{\mu\nu} + f^2 \partial^\mu \varphi \partial^\nu \varphi$$

- f é a variável termodinâmica conjugada a $\partial_\mu \varphi$
- Interpreta-se $\partial^\nu \varphi = \mu v^\nu$ e por conseguinte $f^2 = \rho_s/\mu$. Logo

$$T^{\mu\nu} = (\varepsilon + P)u^\mu u^\nu - Pg^{\mu\nu} + \mu \rho_s v^\mu v^\nu$$

Note:

- Componente superfluida não contribui com a entropia e pressão: $\varepsilon = -P + sT + \mu\rho_n$.
- Este é o caso para um líquido ideal. $T^{\mu\nu}$ ainda admite correções (não conservativas), proporcionais a coeficientes de transporte, como viscosidade

General shape of $T^{\mu\nu}$ for perfect fluids

- Force acting on area element perpendicular to k direction: $f_i = T_{ik} d\sigma_k$
- Four momentum $P^\mu = \int d^4x T^{\mu 0}$
- In a referential frame comoving with the fluid:
  - Isotropic force, and thus $T_{ik} = \delta_{ik} P$
  - $P^\mu = (E, 0, 0, 0)$
  - $u^\mu = (1, 0, 0, 0)$
- Pressure and energy measured in the comoving referencial are Lorentz invariant (same reason as proper time)

Thus, from Lorentz invariance[4]

$$T^{\mu\nu} = (\varepsilon + P)\, u^\mu u^\nu - g^{\mu\nu} P$$

$$u^\mu = \frac{dx^\mu}{d\tau}$$

[4] We adopt $g^{\mu\nu} = \mathrm{diag}(1, -1, -1, -1)$.

Perturbation around steady flow

What is the expected value of $\langle T^{0i}(\mathbf{x},t)\rangle$ when we perturb the fluid?

- Liquid described by Hamiltonian $\hat{H}$ and eigenstates $|\psi\rangle$.
- Perturbation given by $\delta H(t) = \int d^3x\, U_j e^{-ik^\mu x_\mu}\,\theta(-t) T^{0j}(\mathbf{x},t)$
- Using Heisenberg picture, one gets
$$\frac{\partial\langle T^{0i}(\mathbf{x},t)\rangle}{\partial t} = i\left\langle[\delta H(t), T^{0i}(\mathbf{x},t)]\right\rangle$$
- Supposing δH small, we get

$$\begin{aligned}\langle T^{0i}(\mathbf{x},t)\rangle &= \langle T^{0i}(\mathbf{x},0)\rangle + i\int_0^t dt'\left\langle[\delta H(t'), T^{0i}(\mathbf{x},t')]\right\rangle \\ &\cong \langle T^{0i}(\mathbf{x},0)\rangle + i\int_0^t dt'\left\langle[\delta H(t'), T^{0i}(\mathbf{x},0)]\right\rangle + \mathcal{O}(\delta H^2)\end{aligned}$$

Perturbation around steady flow

Bringing everything together

$$\delta\langle T^{0i}(\mathbf{x},t)\rangle = i\int d^4x' U_j e^{-ik^\mu x_\mu}\,\theta(-t')\left\langle[T^{0j}(\mathbf{x}',t'),T^{0i}(\mathbf{x},0)]\right\rangle$$

- Definition $G^{ij,\,R}(\mathbf{x}'-\mathbf{x},t') \equiv i\theta(-t')\left\langle[T^{0j}(\mathbf{x}',t'),T^{0i}(\mathbf{x},0)]\right\rangle$.

$$\delta\langle T^{0i}(0,t)\rangle = \int d^4x' U_j e^{-ik^\mu x_\mu} G^{ij,\,R}(\mathbf{x}',t') = U_j\tilde{G}^{ij,\,R}(\mathbf{k},\omega)$$

Decompose Green's function into transverse and longitudinal components

$$\tilde{G}^{ij,\,R}(\mathbf{k},0) = \frac{k^i k^j}{\mathbf{k}^2}G^{\parallel}(\mathbf{k}) + \left(\delta^{ij} - \frac{k^i k^j}{\mathbf{k}^2}\right)G^{\perp}(\mathbf{k})$$

Perturbation around steady flow

Known property of superfluid: $\nabla \times \mathbf{v} = 0$

- If perturbation parallel to u_s^i:
  $u_n^i \to u_n^i + U^i e^{-ik^\mu x_\mu} \quad u_s^i \to u_s^i + U^i e^{-ik^\mu x_\mu}$

  $$\delta\langle T^{0i}(0,0)\rangle = [\varepsilon_n + P_n + \mu\rho_s]U^i = G^{\parallel}(\mathbf{k})k^i k^j U_j/\mathbf{k}^2$$

- If perturbation perpendicular to u_s^i: $u_n^i \to u_n^i + U^i \quad u_s^i \to u_s^i$

  $$\delta\langle T^{0i}(0,0)\rangle = [\varepsilon_n + P_n]U^i = G^{\perp}(\mathbf{k})(\delta^{ij} - k^i k^j/\mathbf{k}^2)U_j$$

Using $\varepsilon_n + P_n = sT + \mu\rho_n$ and taking $\mathbf{k} \to 0$

$$\lim_{\mathbf{k}\to 0} G^{\parallel}(\mathbf{k}) = sT + \mu\rho \qquad \lim_{\mathbf{k}\to 0} G^{\perp}(\mathbf{k}) = sT + \mu\rho_n$$

Or

$$\lim_{\mathbf{k}\to 0} \tilde{G}^{ij,\,R}(\mathbf{k},0) = \delta^{ij}[sT + \mu(2\rho_n + \rho_s)] + \frac{k^i k^j}{\mathbf{k}^2}\mu\rho_s$$

Renormalization

- Actual measurements from the lattice are dimensionless: $a^2 F_{\mu\nu}(m)$, $a^4 T_{\mu\nu}(m)$, $a^8 T_{0i}(0) T_{0j}(m)$
- We do not know a priori the value of a (only g_0)
- Sommer parameter: enables determination via

$$F(r_0) r_0^2 = 1.65 \qquad r_0 \simeq 0.5\,\text{fm} \simeq 2.5\,\text{GeV}^{-1} \qquad F(r) = \frac{dV(r)}{dr}$$

  - These values comes from modeling $c\bar{c}$ and $b\bar{b}$ spectra with an effective potential $V(r)$ in non-relativistic approximation.
- It is possible to compute $aV(m)$ for two infinitely heavy quarks on the lattice

$$m_0^2 \left. \frac{d[aV(m)]}{dm} \right|_{m=m_0} = a^2 m_0^2 \left. \frac{dV(r)}{dr} \right|_{r=r_0} = r_0^2 F(r_0) = 1.65$$

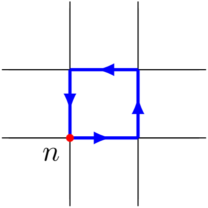
- Once m_0 is found, then one finds $a = (0.5\,\text{fm})/m_0$

Wilson action

$$S_{QCD} = S_G + S_F$$

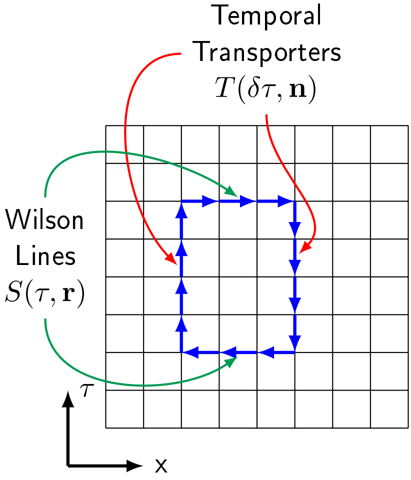
$$S_G = \frac{2N}{g_0^2} \sum_n \sum_{C(\mu,\nu)} \left[1 - \frac{1}{N} \Re \, \mathrm{Tr} \, U_{\mu\nu}(n) \right]$$

$$U_{\mu\nu}(n) \equiv U_\mu(n) U_\nu(n + a\hat{e}_\mu) U_\mu^\dagger(n + a\hat{e}_\nu) U_\nu^\dagger(n)$$



We will ignore fermions in our work

Wilson Loop



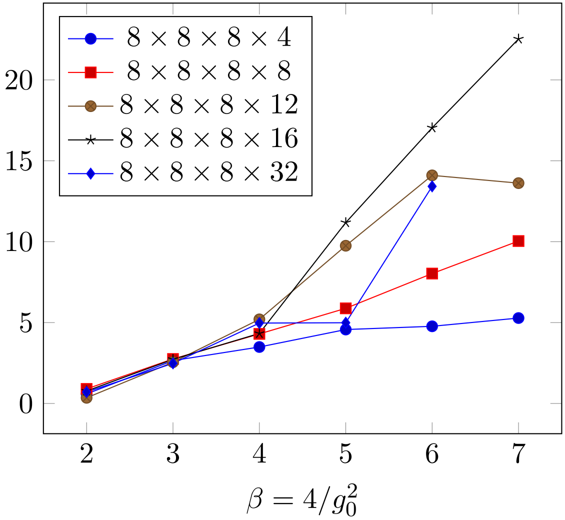
- Basically a plaquette of non-unitary side
- One may choose a gauge where all temporal links along a time-line is unit, except by one link. ($T = \mathbb{1}$)
- $W(\delta\tau, \mathbf{r}) = \langle \mathrm{Tr}[S(\tau_1, \mathbf{r})S^\dagger(\tau_2, \mathbf{r})] \rangle$
- It is possible to show that

$$\lim_{m_q \to \infty} \langle \psi_q(m)\bar{\psi}_q((n)) \rangle = \prod_{k=m}^{n} U_{\mu_k}(k)$$

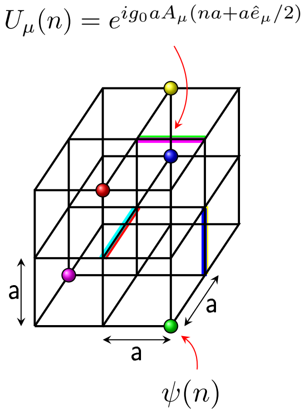
- We interpret $W(\delta\tau, \mathbf{r})$ as an infinitely heavy pair of quark-antiquark interacting, i.e.

$$W(\delta\tau, \mathbf{r}) \propto e^{-a\delta\tau V(\mathbf{m})}$$

Lattice potential results



Regularization via Lattice Discretization



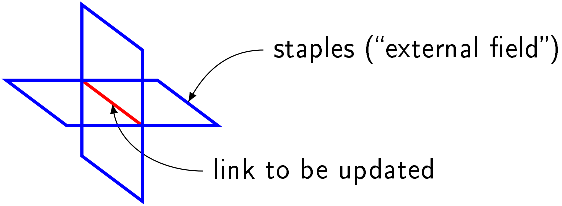
- Discretization acts as cutoff momentum $p^\mu > \pi/a$
- Integral becomes mathematically identical to a classical partition function

$$\langle \mathcal{O} \rangle = \frac{1}{Z} \prod_n \left[\prod_\mu \left(\sum_{U_\mu(n)} \right) \sum_{\bar{\psi}(n)} \sum_{\psi(n)} \right] \mathcal{O} \; e^{-S^E_{QCD}}$$

- We may use Monte Carlo simulations to estimate $\langle \mathcal{O} \rangle$
- WARNING: The use of simulations implies finite-size lattice in spatial directions, i.e. adds an infrared cutoff which may be a source of systematical error.

Heat-Bath for SU(2) simulation

- The 6 staples are seen as an external field



Note: 3D slice of the 4D space

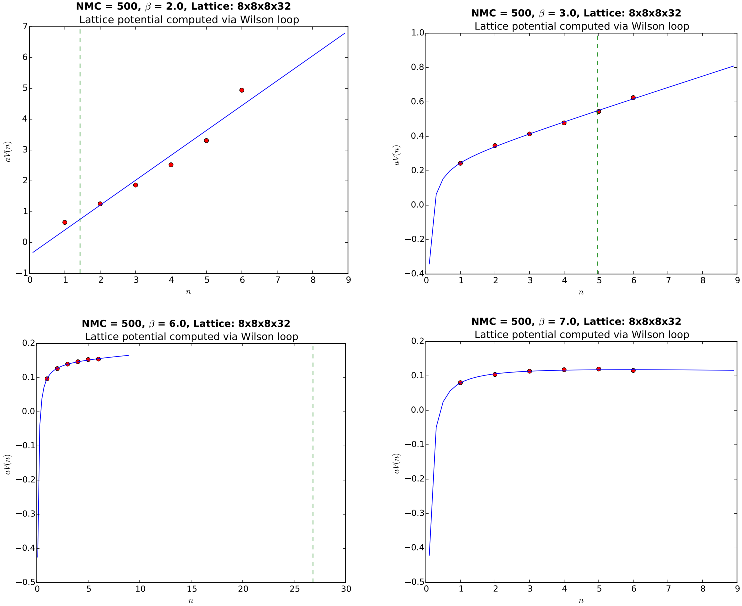
- New SU(2) element sorted follwing distribution
  $dP \cong dU \exp\left[\frac{2}{g_0^2}\,\mathrm{Tr}\left(U \sum_{\alpha=1}^{6} \tilde{U}_\alpha\right)\right]$
- One MC step is defined as this process repeated over the entire lattice
- Expectation value: $\langle \mathcal{O} \rangle = \frac{1}{N}\sum_{n=1}^{N} \mathcal{O}_n$
- As usual in MC methods: Correlation between successive MC steps introduces systematic errors

Order parameter: The Polyakov Loop

We need as well an order parameter to determine in which phase we are.

- Polyakov loop: temporal transport of the size of the entire lattice and closed due to periodic boundary condition.
- Interpreted as infinitely massive quark at rest. $L(x) \propto e^{-F_q}$
- Confined phase: $F_q \to \infty \implies L(x) = 0$
- Deconfined phase: $F_q \to 0 \implies L(x) = \text{constant}$

What happens at $\beta > 4$



A grain of salt

- Kalaydzhyan; Nucl.Phys. A913 (2013): Argues that "*The "normal" component of the fluid is the thermalized matter in common sense, while the "superfluid" part consists of long wavelength (chiral) fermionic states moving independently.*"
- Chernodub, Doorsselaere, Kalaydzhyan, Verschelde; Phys.Lett. B730 (2014): Gluon condensation allows quarks propagates over large distances along flux tubes.

We may be neglecting the normal component when we neglect fermions.