Lattice Hadron Spectroscopy

Part I: The anomalous magnetic moment of the muon

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Helmholtz International Summer School
"Hadron Structure, Hadronic Matter, and Lattice QCD"

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A practical introduction

Disclaimer

The material covered in this lecture is not entirely original, it has been previously shown and has been compiled here by the author for pedagogical presentation. As such figures and slides may have been copied from other sources.

Exercises

A number of home and in class exercises are strewn across the lectures. These are purely voluntary. Solutions in general will not be given. However, we can discuss the solution to the exercises outside of class, or if you do them any time in the future after the school, by email.

Discussions and questions

Please, feel free to ask questions and contribute to the discussion points. There is so much accumulated knowledge in the room, if the author cannot answer a question, there will be someone who can. We are eager to share our experiences!

Outline

Intro to the anomalous magnetic moment

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-

Dispersive approach and connection to lattice QCD

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-

Lattice calculation in practice

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Intro to the anomalous magnetic moment

- Basic idea and status of experiment
- Theory prediction and tension

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- a_μ^{HLO} via dispersion relation and connection to lattice
- Derivation of the Time-Momentum representation method

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Lattice calculation in practice

- Explicit contractions for the vector meson current
- Analysing lattice results and extrapolations
- Systematic uncertainties:
  - lattice spacing
  - pion mass
  - finite volume
- Things yet unmentioned: Disconnected diagrams, more flavors, QED effects, signal-to-noise deterioration, ...

Introduction and the derivation of the lattice observable

The magnetic moment $\vec{\mu}$ determines the shift of a particle's energy in the presence of a magnetic field $\vec{B}$

$$V = -\vec{\mu} \cdot \vec{B}$$

where the spin $\vec{S}$ of the particle contributes

$$\vec{\mu} = g\left(\frac{e}{2m}\right)\vec{S}$$

with electric charge e, particle mass m and Landé factor g.

The anomalous magnetic moment $a = (g-2)/2$ accounts for radiative corrections to the result found by Dirac $g = 2$.

Since $a_l \propto m_{l=e,\mu,\tau}$ one is led to believe that precision studies of a_l are a good way to reveal new physics.

Particularly interesting is the case $l = \mu$

To access a_μ experimentally, note: The momentum vector of a muon moving in a circle in a static magnetic field rotates with the cyclotron angular frequency

$$\vec{\omega}_c = \frac{e\vec{B}}{m}$$

However, the Larmor spin precession frequency is the same as for the particle at rest

$$\vec{\omega}_L = g\frac{e\vec{B}}{2m}$$

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$$\vec{\omega}_L = g\frac{e\vec{B}}{2m}$$

insert a=(g-2)/2

$$\Rightarrow \vec{\omega}_L = (1 + a_\mu)\frac{e\vec{B}}{m}$$

This means: Measuring $\vec{\omega}_a = \vec{\omega}_L - \omega_c = a_\mu(e/m)\vec{B}$ one gets a direct handle on a_μ!

Experimental technique since CERN-II

$$a_\mu = \frac{g-2}{2} \propto \frac{\omega_a}{B}$$

Polarized muons — Make a pion beam, then select highest energy muons from parity violating $\pi \to \mu + \nu_\mu$ decay

↓

Precession in uniform B-field — Storage ring with ultra-precise dipole B-field. Allow muons to precess through as many g-2 cycles as possible.

↓

Measure muon spin direction vs time — In parity violating decay $\mu \to e + \nu_e + \nu_\mu$, the positron is preferentially emitted in the muon spin direction

Taken from a presentation at FCM2015

New experiment: Fermilab E989

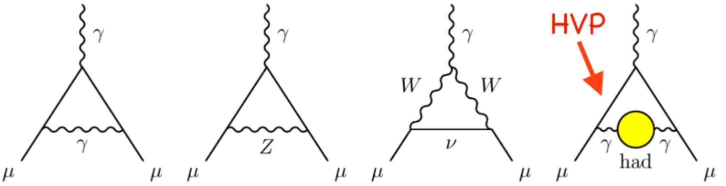
Aims at a 4× reduction in experimental uncertainty. Need better theory!

(Higher muon intensity at FNAL, same storage ring)

Alternative proposal using ultra-cold muons allowing for a 66cm storage device proposed at J-PARC (E34)

Taken from a presentation at LATTTICE2017

To access a_μ theoretically one needs to understand diagrams such as



$$a_\mu = a_\mu^{QED} + a_\mu^{EW} + a_\mu^{HAD}$$

Note: The three left diagrams (or two contributions to a_μ) can be handled well using perturbation theory.

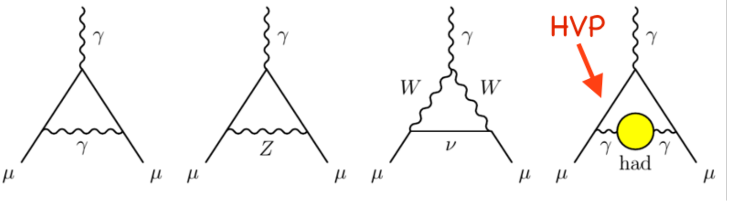
Theory status for a_μ – summary

Contribution	Value $\times 10^{10}$		Uncertainty $\times 10^{10}$
QED (5 loops)	11 658 471.895		0.008
EW	15.4		0.1
HVP LO	692.3		**4.2**
HVP NLO	-9.84		0.06
HVP NNLO	1.24		0.01
Hadronic light-by-light	10.5		**2.6**
Total SM prediction	11 659 181.5	!!	4.9
BNL E821 result	11 659 209.1		6.3
FNAL E989/J-PARC E34 goal			$\approx$ **1.6**

We currently observe a $\sim 3\sigma$ tension !

Taken from a presentation at LATTTICE2017

To access a_μ theoretically one needs to understand diagrams such as



The diagram on the right, connected to a_μ^{had}, the hadronic vacuum polarization, is the least well determined from theory!

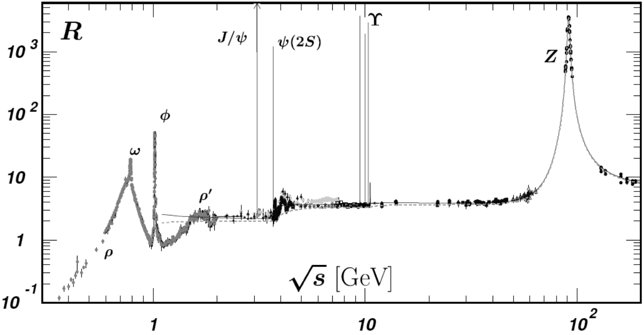
Can we accurately determine the hadronic contribution a_μ^{had} from first principles?

To find an approach from lattice QCD, let's take a closer look at how the hadronic diagram can be computed in phenomenology. From the PDG:

> one currently relies on a dispersion relation approach to evaluate the lowest-order (*i.e.*, $\mathcal{O}(\alpha^2)$) hadronic vacuum polarization contribution $a_\mu^{\text{Had}}[\text{LO}]$ from corresponding cross section measurements [15]
>
> $$a_\mu^{\text{Had}}[\text{LO}] = \frac{1}{3}\left(\frac{\alpha}{\pi}\right)^2 \int_{m_\pi^2}^{\infty} ds \, \frac{K(s)}{s} R^{(0)}(s) \, , \qquad (10)$$

Here, $K(s)$ is a known electromagnetic Kernel and $s = q^2 = \omega^2$ denotes a given momentum transfer, while the R-ratio is

$$R(s) = \frac{\sigma(e^+e^- \to \text{hadrons})}{4\pi\alpha(s)^2/3s}$$



via optical theorem the R-ratio is directly related to the spectral function of the e/m, or vector meson, current:

$$\rho(s) = \frac{R(s)}{12\pi^2}$$

One possibility to determine a_μ^{had} is therefore to numerically fold and integrate the R-ratio data with the Kernel.

The Kernel function $K(s)$ depends on the mass of the lepton, indeed it enters as $1/m_\mu^4$:

$$a_\mu^{\rm HLO} = \left(\frac{\alpha}{\pi}\right)^2 \int_0^\infty {\rm d}Q^2 K_E(Q^2)\widehat{\Pi}(Q^2),$$

$$\widehat{\Pi}(Q^2) = 4\pi^2\left[\Pi(Q^2) - \Pi(0)\right],$$

with the kernel given by[1] [3]

HVP!

$$K_E(s) = \frac{1}{m_\mu^2} \cdot \hat{s} \cdot Z(\hat{s})^3 \cdot \frac{1 - \hat{s}Z(\hat{s})}{1 + \hat{s}Z(\hat{s})^2},$$

$$Z(\hat{s}) = -\frac{\hat{s} - \sqrt{\hat{s}^2 + 4\hat{s}}}{2\hat{s}}, \qquad \hat{s} = \frac{s}{m_\mu^2}.$$

$$\widehat{\Pi}(Q^2) = \frac{Q^2}{3}\int_0^\infty ds \frac{R(s)}{s(s+Q^2)}.$$

Taken from arxiv:1107.4388.

Home exercise:

In arxiv:1107.4388 a parametrisation of $R(s)$ is given. Implement the above equations in a small program to perform the integration numerically and obtain your own value for a_μ^{had}. What regions of s is the result particularly sensitive to? What is the impact of the strong m_l dependence in the kernel? You can check this by entering e.g. the electron, muon and tau masses.

The key quantities needed to compute a_μ^{had} are the lepton mass and the spectral function in terms of the R-ratio, as it enters through the auxiliary definition $\hat{\Pi}(Q^2)$.

This gives two handles to approach a_μ^{had} from lattice QCD:

1. attempt to calculate $R(s)$
2. directly compute

$$\hat{\Pi}(Q^2) = \frac{Q^2}{3} \int_0^\infty ds \frac{R(s)}{s(s+Q^2)}$$

Which option should we choose?

Notice, compared to the lattice spectral functions introduced before (e.g. in Gert's or Olaf's lectures) we have the relation:

$$\frac{\rho_{lat}(\omega)}{\omega^2} = \rho(\sqrt{s})$$

This means there is a direct connection between the R-ratio and the lattice vector meson correlation function via:

$$G(t, T = 0, N_T = \infty) = \int_0^\infty \rho_{lat}(\omega) \cdot \exp[-\omega t]$$

(see e.g. Gert's or Olaf's lectures)

Procedure:

- Calculate the correlation function $G(t)$ on the lattice
- Reconstruct the spectral funciton $\rho_{lat}(\omega)$
- Insert it into the dispersion relation and integrate to get a_μ^{had}

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But: The reconstruction of the spectral function is highly non-trivial and subject to large systematic uncertainties!

Better: Have a closer look at $\hat{\Pi}(Q^2)$.

$\hat{\Pi}(Q^2) = \Pi(Q^2) - \Pi(0)$ is the subtracted vacuum polarization function that can be computed from the vacuum polarization tensor (see e.g. Peskin-Schroeder):

$$\Pi_{\mu\nu}(Q) = \Big(Q_\mu Q_\nu - \delta_{\mu\nu} Q^2\Big)\Pi(Q^2)$$

the latter is directly related to the electromagnetic (vector meson) current

$$J_\mu(x) = \frac{2}{3}\bar{u}(x)\gamma_\mu u(x) - \frac{1}{3}\bar{d}(x)\gamma_\mu d(x) - \frac{1}{3}\bar{s}(x)\gamma_\mu s(x) + ...$$

via

$$\Pi_{\mu\nu}(Q) = \int d^4x e^{iQ\cdot x}\langle J_\mu(x) J_\nu(0)\rangle$$

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via

$$\Pi_{\mu\nu}(Q) = \int d^4x e^{iQ\cdot x}\langle J_\mu(x) J_\nu(0)\rangle$$

Hold on! We've seen $\langle J_\mu(x) J_\nu(0)\rangle$ before! This also gives the vector meson correlator:

$$G(t) = \int d^3x \langle J_\mu(x) J_\nu(0)\rangle$$

Setting (w.l.o.g) $\mu = \nu = z$ and $Q = (\omega, \vec{k} = 0)$, we find $G(t)$ and $\Pi_{zz}(Q = (\omega, \vec{k} = 0))$ are connected via Fourier transform:

$$G(t) = -\int \frac{d\omega}{2\pi}\, e^{i\omega t}\, \Pi_{zz}(\omega, \vec{k} = 0)$$

exploiting the relation between the $\Pi(Q^2)$ and $\Pi_{\mu\nu}(Q)$ we have

$$\Pi_{zz}(\omega, \vec{k} = 0) = -\omega^2 \Pi(\omega^2)$$

Therefore:

$$\Pi(\omega^2) = \frac{1}{\omega^2} \int dt\, e^{-i\omega t}\, G(t)$$

The vacuum polarization function entering a_μ^{had} is directly related to the lattice correlator by integration!

Therefore:

$$\Pi(\omega^2) = \frac{1}{\omega^2} \int dt \, e^{-i\omega t} \, G(t)$$

The vacuum polarization function entering a_μ^{had} is directly related to the lattice correlator by integration!

Procedure:

- Calculate the correlation function $G(t)$ on the lattice
- Simplify it's relation to the HVP and integrate
- Insert the result into the dispersion relation and integrate to get a_μ^{had}

Advantage: $G(t)$ is a raw lattice observable, it's systematics can be tightly controlled and improved.

Alternative: Compute $\Pi_{\mu\nu}(Q)$ directly on the lattice and derive $\Pi(Q^2)$. Both approaches are equivalent. But we need to subtract for $\hat{\Pi}(Q^2)$ and $\Pi_{\mu\nu}(Q) \leftarrow \Pi(Q^2)$ diverges as $Q^2 \to 0$.

In class exercise

Let's more closely examine

$$\Pi(\omega^2) = \frac{1}{\omega^2} \int dt\, e^{-i\omega t}\, G(t)$$

1. expand this expression around $\omega = 0$ using

$$\exp(-ixt) \stackrel{x \to 0}{=} 1 - itx - \frac{t^2 x^2}{2} + \frac{1}{6} i t^3 x^3 + \frac{t^4 x^4}{24} + \ldots$$

2. simplify the expansion by using that $G(t)$ is even and real
3. why is it sufficient to expand two orders to obtain an accurate result for $\Pi(0)$? Argue!
4. use the expanded and the unexpanded equations to form $\hat{\Pi}(\omega^2)$
5. simplify once more using the even and real properties of $G(t)$
6. how does $\hat{\Pi}(\omega^2)$ behave for $\omega \to 0$ and $\omega \to \infty$?

With

$$\Pi(\omega^2) = \frac{1}{\omega^2}\int dt\, e^{-i\omega t}\, G(t)$$

we can expand around $\omega \sim$ small for a more practical relation:

$$\Pi(\omega^2) \overset{\omega\to 0}{=} \frac{1}{\omega^2}\int dt G(t) - \frac{1}{2}\int dt\, t^2\, G(t) + ...$$

where we used that G(t) even and real, so

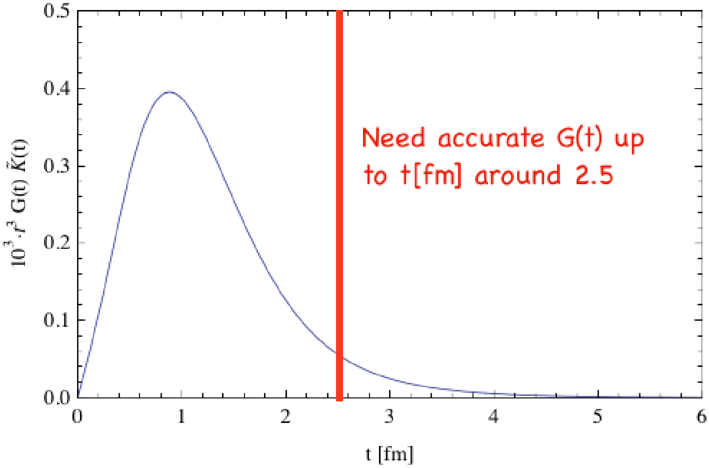
$$\begin{aligned}
\hat{\Pi}(\omega^2) &= \Pi(\omega^2) - \Pi(0) \\
&= \int dt\, G(t)\Big[\frac{e^{i\omega t} - 1}{\omega^2} + \frac{t^2}{2}\Big] \\
&= 2\int_0^\infty dt\, G(t)\Big[\frac{t^2}{2} - \frac{1-\cos(\omega t)}{\omega^2}\Big] \text{ , G(t) even and real} \\
&= \int_0^\infty dt\, G(t)\Big[t^2 - \frac{4\sin^2(\omega t/2)}{\omega^2}\Big]
\end{aligned}$$

$$\hat{\Pi}(\omega^2) = \int_0^\infty dt\, G(t) \Big[t^2 - \frac{4\sin^2(\omega t/2)}{\omega^2} \Big]$$

This final expression can be inserted into the dispersion relation to obtain a_μ^{had} from first principles after calculating the correlator G(t).
This approach is referred to as "TMR"(time momentum representation) method.

Remark:

- The lattice extent in t is finite, the integral to ∞ cannot be rigorously performed.
  $\Rightarrow$ since $G(t) \sim \exp[-m_V t]$ decays faster than t^2 the large t contribution is suppressed. However, enough of $G(t)$ must be accurately obtained to ensure a precise determination of the integral.

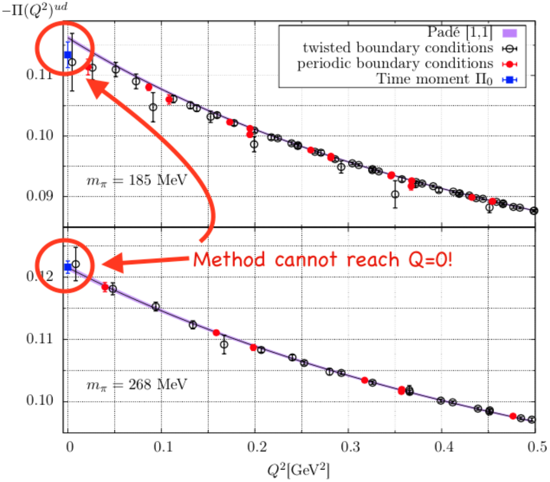

0.5
0.4
0.3
0.2
0.1
0.0
0
1
2
3
4
5
6
$10^3 \cdot t^3\, G(t)\, \tilde{K}(t)$
t [fm]
Need accurate G(t) up to t[fm] around 2.5

Home exercise:

Use the program you made from the first exercise and determine the Euclidean correlator from the parametrisation of $R(s)$. Can you reproduce the value for a_μ^{had} you had before from using the TMR instead?

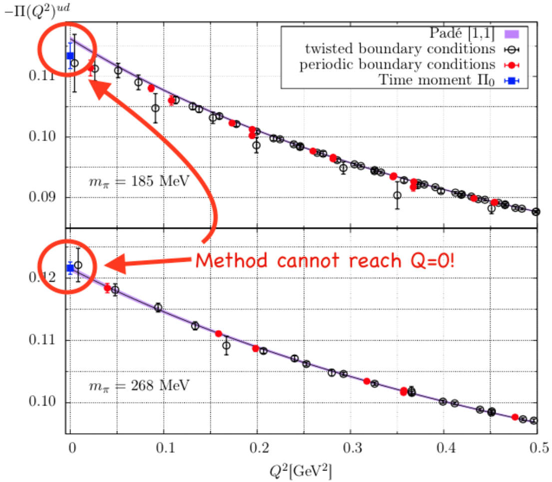
There are other approaches to determine the crucial $\hat{\Pi}(Q^2)$:

One is the aforementioned direct calculation from the HVP tensor. This has the problem that the $Q^2 = 0$ point is not directly calculable.



Calculated by CLS-Mainz

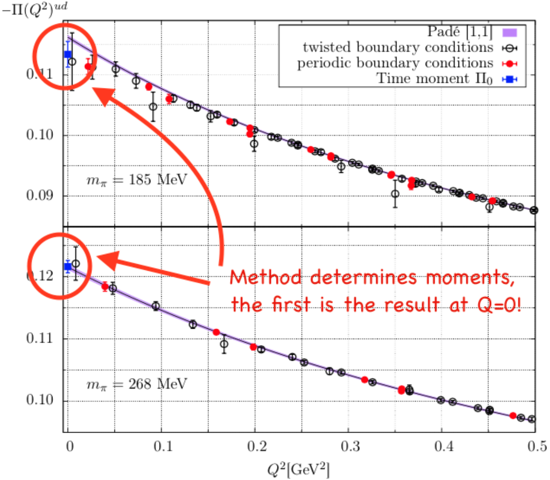
In this case the HVP $\Pi(Q^2)$ is fit to Padé-Ansätze to fix $\Pi(0)$. However, since the Kernel in a_μ^{had} is very sensitive to the low Q^2 region a significant systematic is thereby introduced!



Calculated by CLS-Mainz

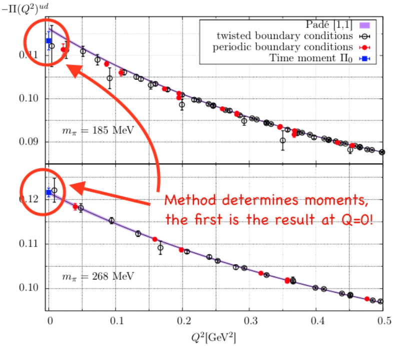
There are other approaches to determine the crucial $\hat{\Pi}(Q^2)$:

Another is the determination of moments $\Pi^{(0)}$, $\Pi^{(2)}, \ldots$ of the expansion of $\Pi(Q^2)$ around $Q^2 = 0$. The TMR and time moments are directly related, as the latter are calculated order by order in the same expansion.



Calculated by CLS-Mainz

Time moments are currently the most accurate method to determine a_μ^{had}. This is because only two to three moments in the expansion (same as the one for the TMR) are required to fix the overall accuracy of a_μ^{had} to $\sim 1\%$.



Calculated by CLS-Mainz

Discussion

Can we understand intuitively why this method might be the most precise?

In all approaches the main quantity to be calculated is the vector meson correlator. How can this be done on the lattice?

Practical lattice calculation

We need to calculate the e/m current correlation function: $\langle J_\mu(x) J_\mu(0) \rangle$ where

$$J_\mu(x) = \frac{2}{3}\bar{u}(x)\gamma_\mu u(x) - \frac{1}{3}\bar{d}(x)\gamma_\mu d(x) - \frac{1}{3}\bar{s}(x)\gamma_\mu s(x) + ...$$

In the following we will restrict ourselves to only u and d quarks. (Note: In a typical lattice calculation these will also be mass degenerate!)

Omitting also charge for the moment, we have:

$$\Big\langle J_\mu(x) J_\mu(0) \Big\rangle = \Big\langle \Big(\bar{u}(x)\gamma_\mu u(x) + \bar{d}(x)\gamma_\mu d(x)\Big) \overline{\Big(\bar{u}(0)\gamma_\mu u(0) + \bar{d}(0)\gamma_\mu d(0)\Big)} \Big\rangle$$

To replace and contract the individual quarks with quark propagators to form the correlation function we use Wick's theorem.

In other words, a string of creation and annihilation operators can be rewritten as the normal-ordered product of the string, plus the normal-ordered product after all single contractions among operator pairs, plus all double contractions, etc., plus all full contractions.

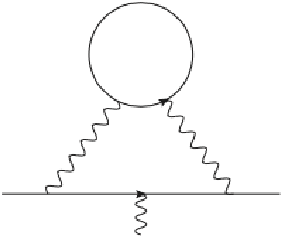
Applying the theorem to the above examples provides a much quicker method to arrive at the final expressions.

from wikipedia

First set of contractions: Connect the quarks with antiquarks and vice versa in the respectively opposite hadron

$$\left\langle J_\mu(x) J_\mu(0) \right\rangle = \left\langle \overbrace{\left(\bar{u}(x)\gamma_\mu u(x) + \bar{d}(x)\gamma_\mu d(x) \right)}^{\text{Meson sink}} \overbrace{\left(\bar{u}(0)\gamma_\mu u(0) + \bar{d}(0)\gamma_\mu d(0) \right)}^{\text{Meson source}} \right\rangle$$

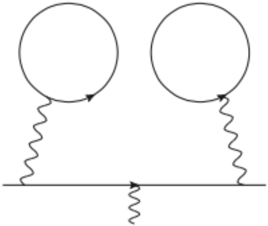
Quark lines connect only in opposite hadron = connected diagram



Second set of contractions: Connect the quarks with antiquarks and vice versa in their own hadron

$$\left\langle J_\mu(x) J_\mu(0) \right\rangle = \left\langle \overset{\text{Meson sink}}{\left(\bar{u}(x)\gamma_\mu u(x) + \bar{d}(x)\gamma_\mu d(x) \right)} \overbrace{\left(\bar{u}(0)\gamma_\mu u(0) + \bar{d}(0)\gamma_\mu d(0) \right)}^{\text{Meson source}} \right\rangle$$

Quark lines connect within the hadron = disconnected diagram



Computing disconnected diagrams is orders of magnitude more difficult (=costly) than computing connected ones.

Why?

To understand we look at how a quark propagator is calculated numerically:

- The quark propagator is related to the fermion operator.

$$[D(x,y)]^{spin}_{colour}[S_q(y,z)]^{spin}_{colour} = [\delta_{x,z}]^{spin}_{color}$$

- here: D is the fermion operator, S the quark propagator we want
- The quark source $[\delta_{x,z}]^{spin}_{color}$ needs to be specified
- To obtain $[S_q(y,x)]^{spin}_{colour}$ the fermion operator needs to be inverted. This amounts to inverting a $N_L^3 \times N_T \times N_{spin} \times N_{color}$, sparse, complex Matrix.

Computing disconnected diagrams is orders of magnitude more difficult (=costly) than computing connected ones.

$$[D(x,y)]^{spin}_{colour}[S_q(y,z)]^{spin}_{colour} = [\delta_{x,z}]^{spin}_{color}$$

- Obtaining $[S_q(y,x)]^{spin}_{colour}$ amounts to inverting a $N_L^3 \times N_T \times N_{spin}^2 \times N_{color}^2$, sparse, complex Matrix.
- This can be efficiently done for propagators originating from a single (or locally smeared) source point.
- This type of point-to-all propagator starts at a freely chosen origin and goes to all points on the lattice.
- To compute a disconnected diagram we need an all-to-all propagator, and therefore in principle need to solve for sources at **all** points. Ergo it is much more expensive.

Result: We, and most lattice groups, will restrict ourselves to calculating only the connected piece of the e/m correlator. This is a large systematic and progress has only recently (2015/16) been made.

Discussion

There are methods to handle all-to-all propagators, although we will not cover them here.
Still: Can you think of any efficient ways to tackle the problem of calculating an all-to-all propagator? Let's discuss!

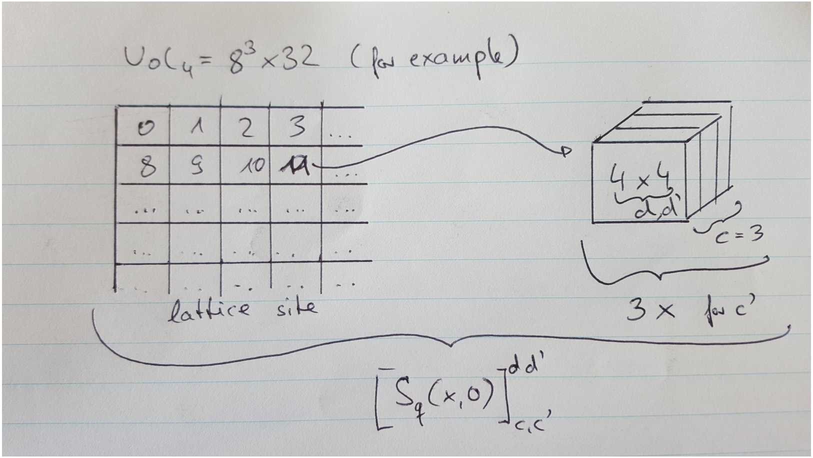
Another look at point-to-all propagators $[S_q(y,x)]^{spin}_{color}$

- let's set the source at x=0 w.l.o.g and x=all spacetime points
- in fact S has two sets of color/spin-indices: one for the source and one for the sink side

$$[S_q(x,0)]^{spin}_{color} = [S_q(x,0)]^{d,d'}_{c,c'}$$

- this means for every lattice site index the propagator is a 12×12 matrix of complex entries.

$\Rightarrow$ numerically, performing Wick-contractions really means correctly multiplying and summing the elements of $12 \times 12 \times \mathrm{Vol}_4 \times \mathrm{complex}$ arrays



$Vol_4 = 8^3 \times 32$ (for example)
0 1 2 3 ...
8 9 10 11 ...
lattice site
4 x 4
d,d'
c = 3
3 x for c'
$\left[S_q(x,0)\right]^{dd'}_{c,c'}$

The prescription how to multiply and sum is given by the Wick-contractions we found before, but we need to insert our point-to-all propagators $[S_q(y,x)]^{s,s'}_{c,c'}$:

$$\left\langle J_\mu(x) J_\mu(0) \right\rangle = \left\langle \left(\bar{u}(x)\gamma_\mu u(x) + \bar{d}(x)\gamma_\mu d(x) \right) \overline{\left(\bar{u}(0)\gamma_\mu u(0) + \bar{d}(0)\gamma_\mu d(0) \right)} \right\rangle$$

$$\| \text{ On the lattice typically: } q = u = s \ , \quad S_q = S_u = S_d$$

$$= \left\langle 2 \cdot \left(\bar{q}^d(x)\gamma_\mu^{d,f} q^f(x) \right) \overline{\left(\bar{q}^{d'}(0)\gamma_\mu^{d',f'} q^{f'}(0) \right)} \right\rangle$$

$$= \left\langle 2 \cdot [S_q(x,0)]^{d,f'}_{c,c'} [\Gamma_\mu]^{d,f} \ \overline{[S_q(x,0)]}^{f,d'}_{e,e'} \overline{[\Gamma_\mu]}^{d',f'} \right\rangle$$

Recall: $[\Gamma_\mu]^{d,d'} = \gamma_\mu$ has only spin indices, the overlined propagator denotes "backwards" propagation as opposed to the un-overlined "forwards".

- We need to "turn around" one of the point-to-all propagators that we have, if we do not want to invert from all-to-point. We use:

$$\overline{[S_q(x,0)]}^{d,d'}_{c,c'} \equiv \gamma_5 \, [S_q(x,0)]^{d,d'}_{c,c'} \, \gamma_5^\dagger$$

The e/m correlator becomes:

$$\Big\langle J_\mu(x) J_\mu(0) \Big\rangle = \Big\langle 2 \cdot [S_q(x,0)]^{d,f'}_{c,c'} [\Gamma_\mu]^{d,f} \; \overline{[S_q(x,0)]}^{f,d'}_{e,e'} \overline{[\Gamma_\mu]}^{d',f'}$$

$$= \Big\langle 2 \cdot \underbrace{[S_q(x,0)]^{d,f'}_{c,c'} \, \gamma_\mu}_{[\tilde{S}_q(x,0)]} \; \underbrace{\gamma_5 [S_q(x,0)]^{f,d'}_{e,e'} \, \gamma_5^\dagger \gamma_\mu^\dagger}_{[\tilde{S}_q(x,0)]} \Big\rangle$$

$$= 2 \cdot \sum_{N_{conf}} \Big(\sum_x \Big(\mathrm{Tr}_{\mathrm{spin}} \mathrm{Tr}_{\mathrm{color}} [\tilde{\mathrm{S}}_\mathrm{q}(\mathrm{x},0)] \cdot [\tilde{\mathrm{S}}_\mathrm{q}(\mathrm{x},0)] \Big) \Big)$$

whereby N_{conf} in principle denotes all possible gaugefield configurations. Naturally, in a numerical calculation only a finite number of these is available.

Numerically:

$$[\tilde{S}_q(x,0)] = [S_q(x,0)]\,\gamma_\mu\gamma_5$$

for example

$$[\tilde{S}_q(x,0)] = \begin{pmatrix} a11 & a12 & a13 & a14 \\ a21 & a22 & a23 & a24 \\ a31 & a32 & a33 & a34 \\ a41 & a42 & a43 & a44 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}$$

As noted: In lattice QCD the task is to properly invert, add, multiply and sum the elements of large matrices.

In class exercise

$$\left\langle J_\mu(x) J_\mu(0) \right\rangle = \left\langle 2 \cdot [S_q(x,0)]^{d,f'}_{c,c'} [\Gamma_\mu]^{d,f} \, \overline{[S_q(x,0)]}^{f,d'}_{e,e'} \overline{[\Gamma_\mu]}^{d',f'} \right.$$

$$= \left\langle 2 \cdot \underbrace{[S_q(x,0)]^{d,f'}_{c,c'} \gamma_\mu}_{[\tilde{S}_q(x,0)]} \underbrace{\gamma_5 [S_q(x,0)]^{f,d'}_{e,e'} \gamma_5^\dagger \gamma_\mu^\dagger}_{[\tilde{S}_q(x,0)]} \right\rangle$$

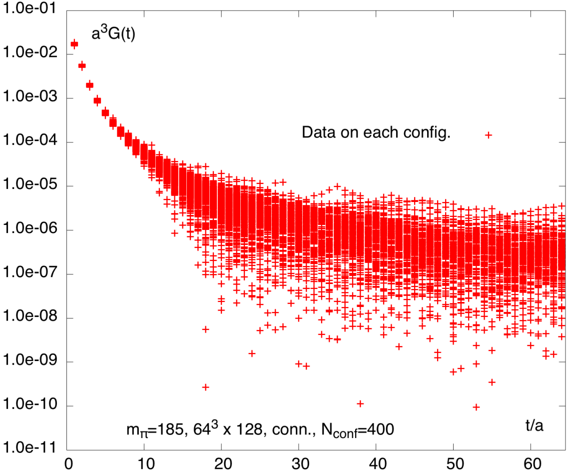
$$= 2 \cdot \sum_{N_{conf}} \left(\sum_x \left(\mathrm{Tr}_{\mathrm{spin}} \mathrm{Tr}_{\mathrm{color}} [\tilde{\mathrm{S}}_\mathrm{q}(\mathrm{x},0)] \cdot [\tilde{\mathrm{S}}_\mathrm{q}(\mathrm{x},0)] \right) \right)$$

In the interpolating operator, i.e. current, for a pion correlator the gamma matrix γ_μ is replaced with γ_5. What changes in the overall calculation? How can be most efficiently calculate a pion correlator on the lattice?

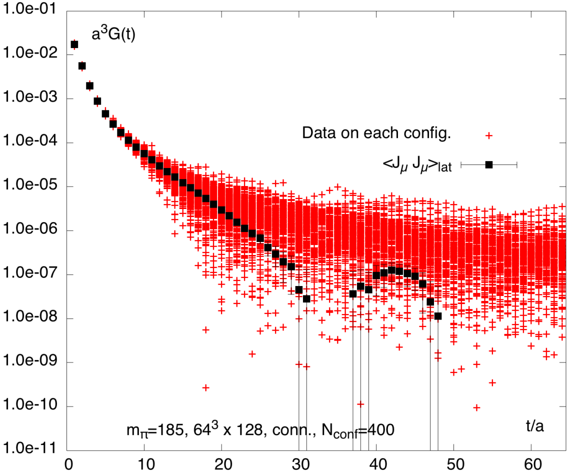
Home exercise

A small free propagator is provided in the contraction package located at `https://github.com/RJHudspith/Contractual_Obligations`. It is open source, so you may download and compile it. Contracting the propagator provided you will obtain a free lattice meson correlator. It is the numerical analog to the calculation in Gert's lecture.

$$\left\langle J_\mu(x) J_\mu(0) \right\rangle = 2 \cdot \sum_{N_{conf}} \underbrace{\left(\sum_x \left(\mathrm{Tr}_{\mathrm{spin}} \mathrm{Tr}_{\mathrm{color}} [\tilde{\mathrm{S}}_{\mathrm{q}}(\mathrm{x}, 0)] \cdot [\tilde{\mathrm{S}}_{\mathrm{q}}(\mathrm{x}, 0)] \right) \right)}_{\text{result obtained per propagator inversion}}$$



$$\left\langle J_\mu(x) J_\mu(0) \right\rangle$$



This is the data for $G(t)$ that we need to determine a_μ^{had}. Even after taking the gauge average the result is still quite noisy.

Back to calculating a_μ

- We have formulated a direct link between $G(t)$ and $\hat{\Pi}(Q^2)$

$$\hat{\Pi}(\omega^2) = \int_0^\infty dt\, G(t)\Big[t^2 - \frac{4\sin^2(\omega t/2)}{\omega^2}\Big]$$

- insert this into the dispersion relation to find a_μ^{HLO}

$$a_\mu^{HLO} = \left(\frac{\alpha}{\pi}\right)^2 \int_0^\infty dQ^2 K_E(Q^2, m_\mu^2) \int_0^\infty dt\, G(t) \left[t^2 - \frac{4\sin^2(Qt/2)}{Q^2}\right]$$

- Define:

$$\tilde{K}(t, m_\mu) = 4\pi^2 \int_0^\infty dQ^2 K_E(Q^2, m_\mu^2) \left[t^2 - \frac{4\sin^2(Qt/2)}{Q^2}\right]$$

- $$a_\mu^{HLO} = \left(\frac{\alpha}{\pi}\right)^2 \int_0^\infty dt\, G(t)\, \tilde{K}(t, m_\mu)$$

  simple, concise relation of known functions and the lattice $G(t)$ to get $a_\mu^{had}[LO]$

Note: To carry on with our procedure we need to handle the integration to ∞.

Two options:

- Truncate the integral
- Extrapolate the correlator

Here we will follow the approach to extrapolate the correlator.

In a system where $\sqrt{(2m_\pi)^2 + (2\vec{p}_{lat}^{min})^2} > m_\rho$ the vector meson is stable and the long time behavior of $G(t)$ is governed by the exponentially decaying ρ-particle ground state.

$$G(t \gg) = A_\rho \exp[-m_\rho t]$$

- Fit to this Ansatz and extend $G(t)$ in the integral relation for $\hat{\Pi}(Q^2)$.
- This is related, but not the same as, the idea of vector meson dominance (VMD).

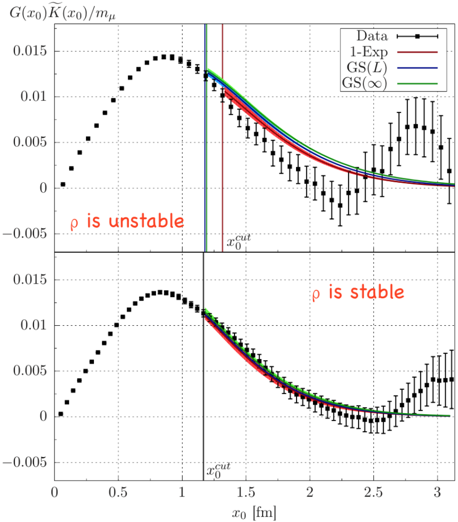
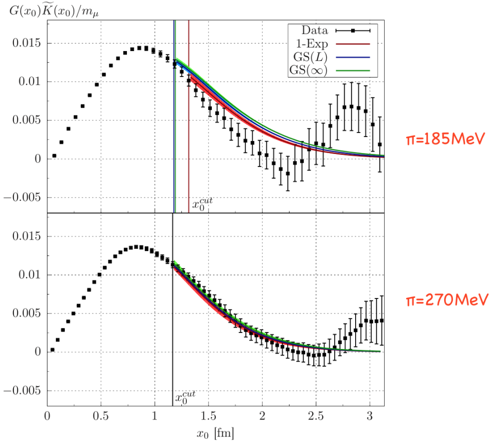


Figure 2: Data for the light quark contribution to the integrand $\widetilde{K}(x_0; m_\mu)\, G^{ud}(x_0)$, scaled in units of the muon mass for ensembles G8 (top) and O7 (below). The coloured bands, which show the various methods to constrain the long-distance behaviour, start at the respective value of x_0^{cut} as indicated by the vertical lines.

Calculated by CLS-Mainz

What if m_ρ decays as in physics?

- In this case we need to take into account $\pi\pi$-states
- For the calculation here that was only possible using a model.
  - Here the known Gounaris-Sakurai model was used, where the width and mass input parameters were fitted to the data
  - Advantage: In this model also finite volume effects due to the finiteness of the lattice box can be estimated
- Still: This is a significant systematic in the approach shown here!



Observations:

- Data much better behaved for larger m_π (very typical behaviour)
- The whole peak can be described by data for $m_\pi = 270\text{MeV}$
- For $m_\pi = 185\text{MeV}$ large fluctuation visible
- Large part of the peak needs to be fitted $\Rightarrow$ systematics?

Having looked at the integrand of

$$a_\mu^{HLO} = \left(\frac{\alpha}{\pi}\right)^2 \int_0^\infty dt\, G(t)\, \tilde{K}(t, m_\mu)$$

what is the final result for a_μ^{HLO}?

Note, it will depend on m_π, a[fm] and $m_\pi L$ of the lattice calculation.

To determine a_μ^{HLO} at the physical point several extrapolations have to be made, these are ...

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what is the final result for a_μ^{HLO}?

Note, it will depend on m_π, a[fm] and $m_\pi L$ of the lattice calculation.

To determine a_μ^{HLO} at the physical point several extrapolations have to be made, these are

- chiral limit $m_\pi^{lat} \to m_\pi$
- continuum limit $a \to 0$
- infinite volume limit $m_\pi L \to \infty$

m_π[MeV]	a[fm]	$m_\pi L$	$(a_\mu^{had}[LO])_{ud}^{conn}$
495	0.0755(9)(7)	6.0	278(04)
381	0.0755(9)(7)	4.7	342(06)
331	0.0755(9)(7)	4.0	355(14)
281	0.0755(9)(7)	5.0	407(13)
437	0.0658(7)(7)	4.7	314(04)
311	0.0658(7)(7)	5.0	395(11)
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π down to 185MeV

3 lattice spacings Multiple volumes!

Data for all three extrapolations available - in principle

m_π[MeV]	a[fm]	$m_\pi L$	$(a_\mu^{had}[LO])^{conn}_{ud}$
495	0.0755(9)(7)	6.0	278(04)
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π down to 185MeV

3 lattice spacings Multiple volumes!

In principle?

- Only few volumes, close together (must rely on GS model for good extrapolation!)
- $m_\pi(a)$ don't match up

- Have to perform a combined chiral-continuum(-volume) extrapolation
- But: A rigorous form for the extrapolation is not known!
- A number of plausible extrapolations have to be performed,.e.g [1705.01775]

$$\begin{aligned}
\text{Fit A:} \qquad & \alpha_1 + \alpha_2 m_\pi^2 + \alpha_3 m_\pi^2 \ln m_\pi^2 + \alpha_4 a, \\
\text{Fit B:} \qquad & \beta_1 + \beta_2 m_\pi^2 + \beta_3 m_\pi^4 + \beta_4 a, \\
\text{Fit C:} \qquad & \gamma_1 + \gamma_2 m_\pi^2 + \gamma_3 a, \\
\text{Fit D:} \qquad & \delta_1 + \delta_2 a,
\end{aligned}$$

Discussion

What are the individual terms, can you guess? Let's discuss!

$$
\begin{aligned}
&\text{Fit A:} \quad \alpha_1 + \alpha_2 m_\pi^2 + \alpha_3 m_\pi^2 \ln m_\pi^2 + \alpha_4 a, \\
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&\text{Fit C:} \quad \gamma_1 + \gamma_2 m_\pi^2 + \gamma_3 a, \\
&\text{Fit D:} \quad \delta_1 + \delta_2 a,
\end{aligned}
$$

- Fit A:
  - Assume $\mathcal{O}(a)$ lattice spacing effects
  - Assume leading chiral correction in m_π^2
  - Include possible chiral logarithm $m_\pi^2 \ln m_\pi^2$

$$\begin{aligned}
\text{Fit A:} &\quad \alpha_1 + \alpha_2 m_\pi^2 + \alpha_3 m_\pi^2 \ln m_\pi^2 + \alpha_4 a, \\
\text{Fit B:} &\quad \beta_1 + \beta_2 m_\pi^2 + \beta_3 m_\pi^4 + \beta_4 a, \\
\text{Fit C:} &\quad \gamma_1 + \gamma_2 m_\pi^2 + \gamma_3 a, \\
\text{Fit D:} &\quad \delta_1 + \delta_2 a,
\end{aligned}$$

- Fit B:
  - Assume $\mathcal{O}(a)$ lattice spacing effects
  - Assume leading chiral correction in m_π^2
  - Assume correction in m_π^4

$$
\begin{aligned}
\text{Fit A:} &\quad \alpha_1 + \alpha_2 m_\pi^2 + \alpha_3 m_\pi^2 \ln m_\pi^2 + \alpha_4 a, \\
\text{Fit B:} &\quad \beta_1 + \beta_2 m_\pi^2 + \beta_3 m_\pi^4 + \beta_4 a, \\
\text{Fit C:} &\quad \gamma_1 + \gamma_2 m_\pi^2 + \gamma_3 a, \\
\text{Fit D:} &\quad \delta_1 + \delta_2 a,
\end{aligned}
$$

- Fit C:
  - Assume $\mathcal{O}(a)$ lattice spacing effects
  - Include only leading chiral correction in m_π^2
- Fit D:
  - Assume only $\mathcal{O}(a)$ lattice spacing effects

TMR	light	strange	charm
Fit ansatz	A, B	A, B, C	C, D
Cuts in m_π and a	no cuts cut 1 cut 2 cuts 1 and 2	cut 2 cuts 1 and 2	cut 2 cuts 1 and 2
IR regime	single exponential‡ Gounaris-Sakurai	single exponential	single exponential
Current renormalization	$Z_V^{(m_{ud})}$	$Z_V^{(m_s)}$	$Z_V^{(m_c)}$ $Z_V(1+b_V a m_c)$

*cut 1: $m_\pi < 400\,\mathrm{MeV}$

†cut 2: $a < 0.07\,\mathrm{fm}$

‡ single exponential is not used as a variation with the GS model including the FV correction

Work of CLS-Mainz,[1705.01775]

Discussion

Do you see the reasoning behind choosing the different fits? Let's discuss the procedure.

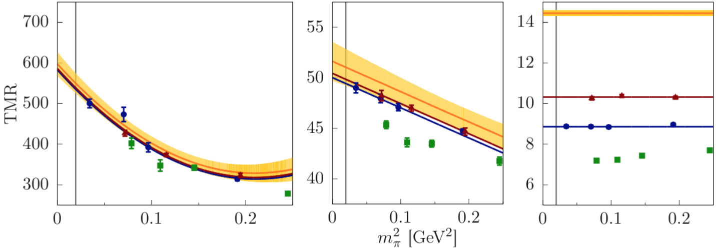
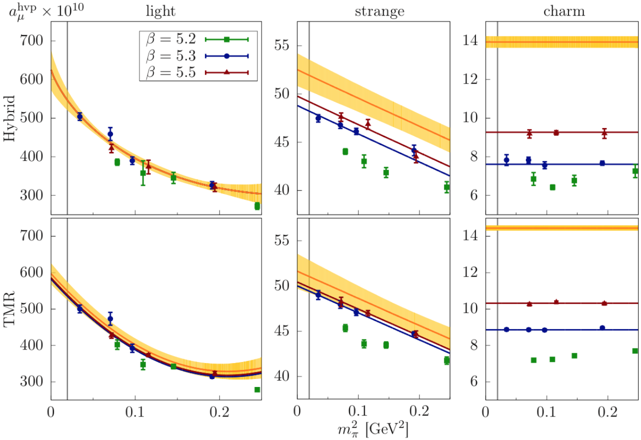


Figure 3: Examples of chiral and continuum extrapolations of the light, strange and charm quark contributions to a_μ^{hvp} for the hybrid (above) and TMR (below) methods. Yellow bands correspond to the chiral behaviour in the continuum limit, while the dark red and blue curves represent the pion mass dependence at $\beta = 5.5$ and 5.3. The physical value of the pion mass is indicated by the vertical lines.

- Procedure works well for strange and charm quarks
- Light quarks very difficult, yet still very constrained and significant
- Typical feature: The extrapolated error is much larger than that of the individual points!
- Cross check: Repeat analysis using one or more of the other approaches other than TMR.



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- Hybrid of time-moments for $\Pi(0)$ and the HVP for $\Pi(Q^2 > 0)$
- Results are consistent

Final results:

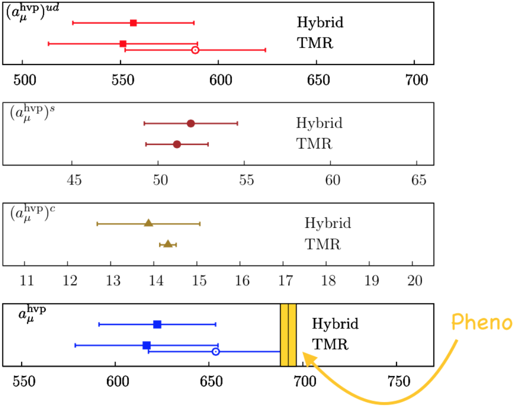
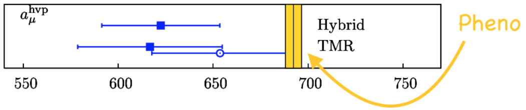


Figure 4: Comparison of results for the different flavour contributions to $a_\mu^{\rm hvp}$ in units of 10^{-10}. Open circles denote the results based on the finite-volume corrected estimates of the light quark contribution. The yellow vertical band denotes the result obtained from dispersion theory [3].

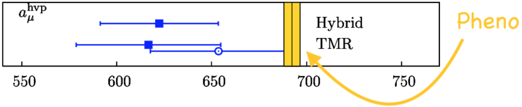
Adding the contributions from the light, strange and charm quarks we arrive at

$$a_\mu^{\rm hvp} = (654 \pm 32_{\rm stat} \pm 17_{\rm syst} \pm 10_{\rm scale} \pm 7_{\rm FV} {}^{+0}_{-10\,{\rm disc}}) \cdot 10^{-10}. \tag{37}$$

Final summary



What have we learned? Conclusions?



What have we learned? Conclusions?

- Getting $G(t)$ accurately is difficult
- Controlling the extrapolations is not trivial
- Typically not all the values m_π, a[fm], $m_\pi L$ one would like are available
- Typically not all effects can be included

Effects on final result?

- $a_\mu^{had}[LO]$ is "truncated". Here it is $(a_\mu^{had}[LO])^{conn,disc=guessed,L\ll\infty,m_\pi^{lat}>m_\pi}_{ud,s=quench,c=quench}$.
- In other calculations different "truncations".
- The error budget after extrapolation is large.

- Calculating $a_\mu^{had}[LO]$ from ab initio is a pressing issue.
- Obtaining precision that is competitive with respect to experiment is the stated goal, but difficult to achieve.

- In detail we have gone through a typical lattice calculation of $a_\mu^{had}[LO]$ and saw the problems or issues arise.
- Many effects are still not properly accounted for, e.g. disconnected diagrams and QED effects.
- Innovations are necessary and many new ideas are being tested. But new approaches also have new systematics, hidden and visible.

- The TMR is not the best approach, indeed it is equivalent to the others. Still it shows the problems faced clearly. It reminds us:

Free lunch theorem: There is no free lunch.