Instead of seeing the separation of scales as a drawback, we can embrace it and use it to our advantage ⇒ Effective Field Theory.

We give up the requirement that the theory is valid at all scales, as long as it produces accurate quantitative results at the relevant scales of the problem. Corrections need to be known. (see e.g. hep-ph/0410047)

General setup: Physics at a low energy scale described by effective d.o.f treated explicitly while physics @ higher scales enters via ① contact interactions ② low-energy constants (Wilson prefactors in L)

Construct a Lagrangian in low energy d.o.f. that reproduces physical observables.

For heavy quarks: $\frac{\Lambda_{QCD}}{M} \ll 1$ no spontaneous $Q\bar{Q}$ pair production
$\frac{T}{M} \ll 1$ no thermal $Q\bar{Q}$ pair production
⟩ Non-relativistic quarks.

How to construct the most general Lagrangian for non-rel. quarks compatible with the symmetries of QCD? (No general formulas available)

Clever Trick: Foldy-Tani-Wouthuysen transformation + Expand the exponentials

$$\bar{Q}(x)\,(D_\mu \gamma^\mu + m)\,Q \qquad Q \to Q' = e^{S'} e^{S} Q \qquad \bar{Q} \to \bar{Q}' = \bar{Q}\, e^{S} e^{S'}$$

$$S = \frac{1}{2M} D_i \gamma_i = -S^\dagger \qquad S' = \frac{1}{4M^2} \gamma_0 \gamma_k F_{k0} \qquad Q = \begin{pmatrix} \psi \\ \chi \end{pmatrix}$$

From the appearing terms construct Lagrangian $L = \sum_n \frac{c_n(\alpha_s(M), \mu)}{M^n} O_n(\mu, m\sigma, m\sigma^2)$

$$L^{NRQCD} = \psi^\dagger \underbrace{(iD_0 + H_\psi)}_{L_\psi} \psi + \chi^\dagger \underbrace{(-iD_0 + H_\chi)}_{L_\chi = L_\psi^*} \chi + \text{contact terms} \underbrace{- \frac{1}{4} F^{\mu\nu} F_{\mu\nu} + \bar{q}(\dots) q}_{\text{light d.o.f @ scale } \mu}$$

$$iD^0 = i\partial^0 - gA^0 \qquad i\vec{D} = i\vec{\partial} + g\vec{A} \qquad E^i = F^{i0} \qquad B^i = -\varepsilon_{ijk} F^{jk}/2$$

$$L_\psi = \psi^\dagger \left(iD_0 + \frac{c_1}{2m_1} D_i^2 + \underbrace{\frac{c_4}{8M^3} D^4}_{*} + \frac{c_F}{2M} \vec{\sigma} \cdot \vec{B} \cdot g + \frac{c_D}{8M} (\vec{D} \cdot g\vec{E} - g\vec{E} \cdot \vec{D}) + \dots \right) \psi$$

* relativistic correction $\sqrt{p^2 + M^2} \approx M + \frac{p^2}{2M} - \frac{p^4}{8M^3}$ $\quad c_k = c_4 = 1$ $\quad$ Constraints on c_i's as remnant of underlying Lorentz invariance.

In general c's can receive non-trivial contributions from UV physics @ the scale $\mu \sim M$, can even be complex: Need <u>matching</u>

Compute correlation functions both in QCD and NRQCD with the same physics content and set equal at some scale. That gives the c_i's. Carry out QCD computations either perturbatively or using lattice QCD. (In the following neglected)

$$j^\mu_k(x) = b_1(\psi^\dagger \sigma_k \chi) + O\left(\frac{1}{M^2}\right)$$

Note: Explicit mass term 2M from FTW removed from L_ψ. All energies are shifted and absolute energy scale needs to be calibrated.

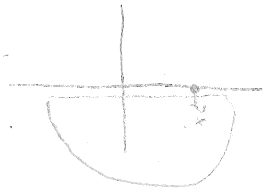
Let us have a look at the non-interacting theory ($A^\mu=0$)

$$\int D\psi D\psi^\dagger\, \psi^\dagger\psi\, e^{iS^{NRQCD}} = \int D\psi D\psi^\dagger\, \psi^\dagger\psi\, e^{i\psi^\dagger \underbrace{\left(+iD_0+\frac{p^2}{2M}\right)}_{K_\psi^{-1}}\psi}$$

$$= K\det(K^{-1}) \approx K \qquad K_\psi^{-1}K_\psi = \text{"}\mathbb{1}\text{"} \Rightarrow \left(i\partial_0+\frac{p^2}{2M}\right)K_\psi(x,x') = \delta(x-x')$$

Solve for the propagator explicitly: $\left(i\partial_0+\frac{p^2}{2M}\right)K_\psi=\delta$ $\quad \underbrace{\left(-i\partial_0+\frac{p^2}{2M}\right)K_\chi=\delta}_{\text{exercise}}$

$$K(t,\vec x) = \frac{1}{(2\pi)^4}\int d\omega\int d^3p\, e^{-i\omega t}\, e^{i\vec p\vec x}\,\frac{-1}{\omega-\frac{p^2}{2M}+i\varepsilon}$$



$$= \frac{i\Theta(t)}{(2\pi)^3}\int d^3p\, e^{-i\frac{p^2}{2M}t}\, e^{i\vec p\vec x}$$

$$= \frac{i\Theta(t)}{(2\pi)^2}\int_0^\infty dp\int_{-1}^{1} dy\, e^{-i\frac{p^2}{2M}t}\, e^{ipry}\, p^2 \qquad -\frac{it}{2M}\left(p^2-pr\frac{2M}{t}+\frac{M^2}{t^2}r^2-\frac{M^2}{t^2}r^2\right)$$

$$= \frac{i\Theta(t)}{(2\pi)^2}\int_0^\infty dp\,\frac{p}{ir}\left(e^{ipr}-e^{-ipr}\right)e^{+ip^2\frac{t}{2M}} \qquad = -\frac{it}{2M}\left[\left(p-\frac{Mr}{t}\right)^2-\frac{M^2}{t^2}r^2\right]$$

$$= \frac{\Theta(t)}{(2\pi)^2}\int_{-\infty}^{\infty} dp\,\frac{p}{ir}\,e^{ipr}\,e^{+ip^2\frac{t}{2M}} = \frac{\Theta(t)}{(2\pi)^2}\,e^{i\frac{M}{2t}r^2}\int_{-\infty}^{\infty}dp'\,\frac{p'+\frac{M}{t}r}{r}\,e^{-i\frac{t}{2M}p'^2}$$

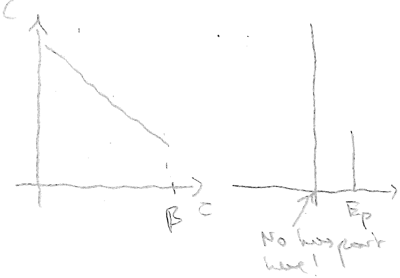
$$= \frac{\Theta(t)}{(2\pi)^2}\,\frac{\sqrt{\pi}}{2}\left(\frac{2M}{t}\right)^{3/2} e^{i\frac{M}{t}r^2} \qquad \frac{1}{2}\left(\frac{2M}{t}\right)\left(\frac{2\pi}{t}\right)^{1/2}\sqrt{\pi}$$

Quick check for $\tilde A\neq 0$ but $M\to\infty$ $\quad (i\partial_0-gA^0)K=\delta \Rightarrow K=\Theta(t)\,e^{ig\int_t^1 A^0(s)\,ds}\,\delta^3(x)$

$\Rightarrow$ There is only a single forward propagating mode.

Time-slice correlator (single heavy quark) $\quad C(\tau)=K(\tau,\vec p)=e^{-\frac{p^2}{2M}\tau}\,\Theta(\tau)$

$\rho(\omega)=\Theta(\omega)\,\delta\!\left(\omega-\frac{p^2}{2M}\right)$

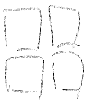


Now put NRQCD on the lattice: $\quad t\to i\tau \quad D_t\to iD_\tau$

$a\Delta^+\psi(x)=U_{x,\mu}\psi(x+a\hat\mu)-\psi(x) \quad \Delta^\pm=\frac{1}{2}(\Delta^+ + \Delta^-) \quad \Delta^2=\sum_i\Delta_i^-\Delta_i^+$ $\quad$ (see e.g. PRD 43 (1991) 196; hep-lat/9205007)

discretized Lagrangian becomes series in powers of $\frac{\Lambda}{Ma}$

$$S^{NRQCD}_{lat,\psi}=\psi^\dagger K^{-1}\psi=\sum_{x,\tau}\psi^\dagger\left(\Delta_4^+-\sum_{j=1}^{3}\frac{\Delta_j^+\Delta_j}{2M}+\frac{\vec\sigma\cdot\vec B}{2M}+\dots\right)\psi_{x,\tau} \qquad B=$$



clover leaf approximation

The free propagator from $\int_{-\pi/a}^{\pi/a} d^4p\,\psi^\dagger(p)\left[i\tilde p_4 a-\frac{\tilde p_4^2a^2}{2}+\frac{1}{2M}\sum_j(\hat p_j)^2\right]\psi(p)$

since $e^{i\tilde p_4 a}-1=i\tilde p_4 a-\frac{a^2\tilde p_4^2}{2}$

This expression only vanishes once in the Brillouin zone $\to$ no doublers.

How to compute heavy quark propagation on the lattice? (VI)

$$K^{-1}K = \mathbb{1} \qquad \Big(\Delta_4^+ - \sum_{j=1}^{3} \frac{\Delta_j^+ \Delta_j^-}{2M} \Big) K_{x,\tau} = \delta_{x,\tau}$$

This is a discretized stochastic diffusion equation (U_μ fluctuate)

① Initial condition kick: $K_{x,0} = \eta$ with $\langle \eta \rangle = 0$ $\langle \eta \eta' \rangle = \delta_{xx'}$
use different noise for each configuration.

② $K_{x,\tau+1} = \underbrace{U^+_{x,\tau} \,(1 - aH)}_{\mathrm{i}\neq\, e^{-aH_{\mathrm{eff}}}} K_{x,\tau}$ for better stability usually define lepage parameter n.

$$= \Big(1 - \frac{a}{2n} H\Big)^n U^+_{x,\tau} \Big(1 - \frac{a}{2n} H\Big)^n K_{x,\tau}$$

=> Much simpler than relativistic theory, no inversion of Dirac operator.

Now we are ready to compute heavy quarkonium current-current correlators:

Channel	Relativistic	NRQCD
$^3S_1$	$\bar{Q}\gamma^\mu Q$	$\chi^+ \sigma_i \psi$
$^3P_1$	$\bar{Q}\gamma_5\gamma^\mu Q$	$\chi^+(\overleftrightarrow{\Delta}_i \sigma_j - \overleftrightarrow{\Delta}_j \sigma_i)\psi$

$$D(\tau,\vec{x}) = \int \mathcal{D}[\psi,\psi^+] \int \mathcal{D}[\chi,\chi^+] \int \mathcal{D}U \; (\chi^+\sigma^i\psi)(\chi^+\sigma_i\psi)^+ \, e^{-S[U] - S^{\mathrm{NRQCD}}[\psi,\psi^+,\chi,\chi^+,U]}$$

Integrating out the fermions gives $D(\tau,\vec{x}) = \int \mathcal{D}U \, \mathrm{Tr}\big[K^+_\chi(\tau,x)\,\sigma_i\, K_\psi(\tau,x)\,\sigma_i \big] e^{-S[U]}$

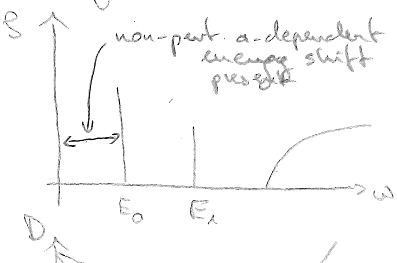
Quarkonium at rest $D(\tau) = \int d^3x \, D(\tau,\vec{x})$

Strategy: compute the evolution of single heavy quark then combine into quarkonium correlator. (In Euclidean time $K_\psi = K_\chi$)

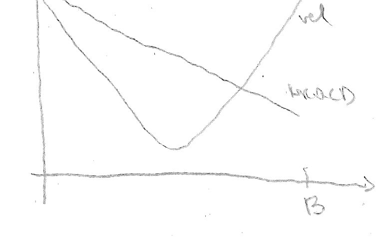
<u>At $T=0$</u> $D = \sum_x \langle 0| J(x,\tau) J^+(x,0) |0\rangle \overset{\text{NRQCD time evol}}{=} \sum_n \sum_x \langle 0| J(x,0)\, e^{-H\tau} \overset{\text{state with } Q\bar{Q}}{|n\rangle} \langle n| J^+(x,0) |0\rangle$

$$= \sum_n \underbrace{\sum_x |\langle 0| J(0,x) |n\rangle|^2}_{A_n} e^{-E_n \tau}$$

In general even at $T>0$ $\quad D(\tau) = \int d\omega \, e^{-\omega\tau} \rho(\omega)$



- Because E_0 still contains an Energy shift, since we subtracted $2M$ in L^{NRQCD}. Need to calibrate for each lattice spacing by fixing e.g. $^3S_1$ mass to PDG value.



- Positive - Absence of backward travelling mode -> no symmetry in τ -> all data points usable.
  - Due to energy shift, exponential decay weaker, better S/N ratio.

What is the meaning of A_n?

$\psi_n(x) = \langle 0|\chi^\dagger(x/2)\,\psi(-x/2)|n\rangle$ (VII)
Bethe-Salpeter-Wavefunction

Related to the decay of heavy quarkonium, e.g. into dileptons.

$$\Gamma(Q\bar{Q}\to \ell^+\ell^-) = \sum_n \frac{2\,\mathrm{Im}(f_n(\Lambda))}{M^{d_n-4}}\langle Q\bar{Q}|O_n(\Lambda)|Q\bar{Q}\rangle$$

(see e.g. hep-ph/9407339)

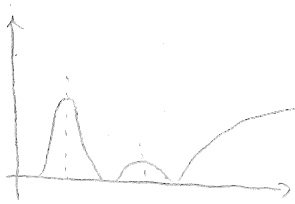
which in NRQCD is mediated via contact interactions, since it describes physics at the scale M which was integrated out. (not explicit in the EFT)

$$\Gamma(\Upsilon/\psi\to e^+e^-) = \frac{2\,\mathrm{Im}\, f_{ee}(^3S_1)}{M^2}\left|\langle 0|\chi^\dagger\sigma\psi|\Upsilon/\psi\rangle\right|^2 + \frac{2\,\mathrm{Im}\, g_{ee}(^3S_1)}{M^4}\,\mathrm{Re}\left[\langle \Upsilon/\psi|\psi^\dagger\sigma\chi|0\rangle\langle 0|\chi^\dagger\sigma(-\tfrac{i}{2}\overleftrightarrow{D})^2\psi|\Upsilon/\psi\rangle\right]$$

other channels can also contain color octet contributions.

Decay rates can be systematically related to non-relativistic wavefunctions at the origin $A_n(^1S_0) = |\psi_n(0)|^2$ via Bethe-Salpeter wavefunction.
(current research interest: use BSW to derive nuclear force from lattice QCD)

What is the situation at finite temperature?



Spectrum gives information about dilepton emission of thermal $Q\bar{Q}$ in a medium: $R_{e\bar{e}} \propto \int dp_0 \int d^3p\, \frac{\rho(P)}{P^2}\, n_B(p_0)$

reduces to the $T=0$ formula if one assumes $\rho(P) = \rho(\omega=\sqrt{p^2+M^2})$ and $\rho(\omega) = \delta(\omega - E_0)$

Lattice NRQCD at finite temperature:

The derivation of the lattice action proceeds equally if $T\neq 0$, i.e. for a finite Euclidean time extent. As long as $T \ll M$ the Wilson coefficients only receive small radiative corrections, which is neglected in current $T>0$ NRQCD studies. Solve diffusion equation along finite τ-axis.

Gain some intuition via the free theory:

remember: $K_\psi(\tau) = \frac{\Theta(\tau)}{(2\pi)^2}\frac{\sqrt{\pi}}{2}\left(\frac{2M}{\tau}\right)^{3/2} e^{-\frac{M}{\tau}r^2}$ $\quad K_\chi^\dagger = \frac{\Theta(\tau)}{(2\pi)^2}\frac{\sqrt{\pi}}{2}\left(\frac{2M}{\tau}\right)^{3/2} e^{-\frac{M}{\tau}r}$

$$D(\tau) = \int d^3x\, K_\psi K_\chi^\dagger \propto \left(\frac{2M}{\tau}\right)^3 \int d^3x\, e^{-\frac{2M}{\tau}r^2} \propto \left(\frac{2M}{\tau}\right)^3\left(\frac{2M}{\tau}\right)^{-3/2} \propto \left(\frac{2M}{\tau}\right)^{3/2}$$

$\underline{D(\tau) \propto \int d^3p\, e^{-2E_p\tau}}$ which with $\underline{\rho(\omega) = \int d\omega\, e^{-\omega\tau}\rho(\omega)}$

leads to $\rho(\omega) = \int d^3p\, \delta(\omega - 2E_p) = \frac{N_c}{\pi} M^{3/2}\omega^{1/2}\,\Theta(\omega)$

Comparison to relativistic free spectrum (see. hep-lat/0507004)

$$\rho(\omega) = \Theta(\omega^2-4M^2)\,\frac{N_c}{8\pi\omega}\sqrt{\omega^2-4M^2}\,\big(1-2n_F(\omega/2)\big)\,\big(\omega^2 K + 4M^2 K\big) + 2\pi\omega\,\delta(\omega)\,N_c\,\chi$$

$$\underset{\omega\to\infty}{\approx}\ \omega^2$$

Now the lattice NRQCD version: We get the dispersion relation from our time evolution operator $K_{x,\tau+1} = \left(1 - a\frac{p^2}{2M}\right)K_{x,\tau} \Rightarrow aE_p = -\log\left(1-\frac{p^2}{2Ma}\right)$

C++ exercise: Compute the NRQCD free spectral function for $Ma = 5$

Let us have a look at some actual real-world NRQCD data:

HotQCD $N_f = 2+1$ HISQ ensembles $48^3\times 12$ $(T>0)$ $48^3\times 32, 48, 64$ $(T\approx 0)$

$M_b a \in [2.8 \ldots 0.9]$ $n=4$ $\boxed{\text{Fig 8}}$ $\boxed{\text{Fig 9}}$ (see also: 1409.3630; 1402.6210; 1109.4496)

⇒ Notice mass difference between different lattice spacings → NRQCD energy shift

⇒ First hint at in-medium modification from correlator ratios $\boxed{\text{Fig 10}}$ $\boxed{\text{Fig 11}}$

Correlators only give us information about global in-medium changes but we are interested in changes of individual states. Need spectral information (mass & width @ $T>0$). Once peaks acquire thermal width effective mass analysis fails (no plateau!)

Spectral function reconstruction using Bayesian inference

In general we have the relation $D(\tau) = \int d\omega\, K(\tau,\omega)\,\rho(\omega)$ which needs to be inverted. Since ρ can contain many features needs to be well resolved along ω. $D_i = \sum_{l=1}^{N_\omega} \Delta\omega_l\, K_{il}\, \rho_l$

Relativistic $Q\bar{Q}$: $K = \frac{\cosh(\omega(\tau-\beta/2))}{\sinh(\omega\beta/2)}$

NRQCD $Q\bar{Q}$: $K = e^{-\omega\tau}$

Inversion is $\underbrace{\text{ill-posed}}_{N_\tau \ll N_\omega}$ and $\underbrace{\text{ill-conditioned}}_{D_i = D_i^{ideal} + \eta}$: Naive inversion in the case $N_\tau = N_\omega$ leads to exponential increase of noise.

∞ solutions to χ^2 fit

Question: How to systematically regularize the problem?

Personal answer: Bayesian inference that allows to incorporate additional prior information on the spectrum. Subjective view of probability, assign $P[X]$ even if X is not random variable.

Multiplication law for prob.

$$P(\rho,D,I) = P[\rho|D,I]\,P[D|I] = P[D|\rho,I]\,P[\rho|I] \;>\; \underset{\text{test function}}{\overset{\text{Posterior}}{P[\rho|D,I]}} = \frac{\overset{\text{Likelihood}}{P[D|\rho,I]}\;\overset{\text{Prior}}{P[\rho|I]}}{\underset{\text{Evidence}}{P[D|I]}}$$

see also: Phys. Rep. 269 (1996) 133