Lattice QCD at finite chemical potential

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Plan

- Introduction
- Methods to solve sign problem
- Results of simulations

Bibliography

- **C. Gattringer, C.B. Lang,**
Quantum Chromodynamics on the Lattice - An introductory presentation, Springer (2010) (Chapter 12).

- **G. Aarts**
Introductory lectures on lattice QCD at nonzero baryon number. J.Phys.Conf.Ser. 706 (2016) no.2, 022004

- **Ph. de Forcrand**
Simulating QCD at finite density. PoS LAT2009 (2009) 010

- **Heng-Tong Ding**
Lattice QCD at nonzero temperature and density. PoS LATTICE2016 (2017) 022

QCD at nonzero T and μ

$$S_G[A] = \int_0^{1/T} dx_4 \int d^3x \, \frac{1}{2} \mathrm{Tr}\, F_{\mu\nu}(x) F_{\mu\nu}(x)$$

$$S_F[\bar{\psi}, \psi, A]$$

$$= \int_0^{1/T} dx_4 \int d^3x \sum_f \bar{\psi}_f(x) \left(\gamma_\mu D_\mu + m_f - \mu_f \gamma_0\right) \psi_f(x)$$

$T = 1/L_4$, L_4 - length in 4th direction

$L_4 = aN_4$, a – lattice spacing

Lattice action

$$S_W^G = \beta \sum_P \left(1 - \frac{1}{3} \mathrm{Re}\ \mathrm{Tr} U_P \right)$$

$$U_{\mu\nu}(s) = U_\mu(s) U_\nu(s + \hat{e}_\mu) U_\mu^\dagger(s + \hat{e}_\nu) U_\nu^\dagger(s) \,.$$

$$S_W^F = \bar{\psi}\, M(U)\, \psi$$

LQCD action with μ:

$$S_F(\psi, \bar{\psi}, U) = \bar{\psi}\, M(U)\psi$$

where $U_4 \rightarrow U_4 e^{a\mu}, \quad U_4^+ \rightarrow U_4^+ e^{-a\mu}$

$$< \mathcal{O} >= \frac{1}{\mathcal{Z}} \int \mathcal{D}U \, \mathcal{O}(U) \, e^{-S_{\rm eff}(U)}$$

$$\mathcal{Z} = \int \mathcal{D}U \, e^{-S_{\rm eff}(U)}$$

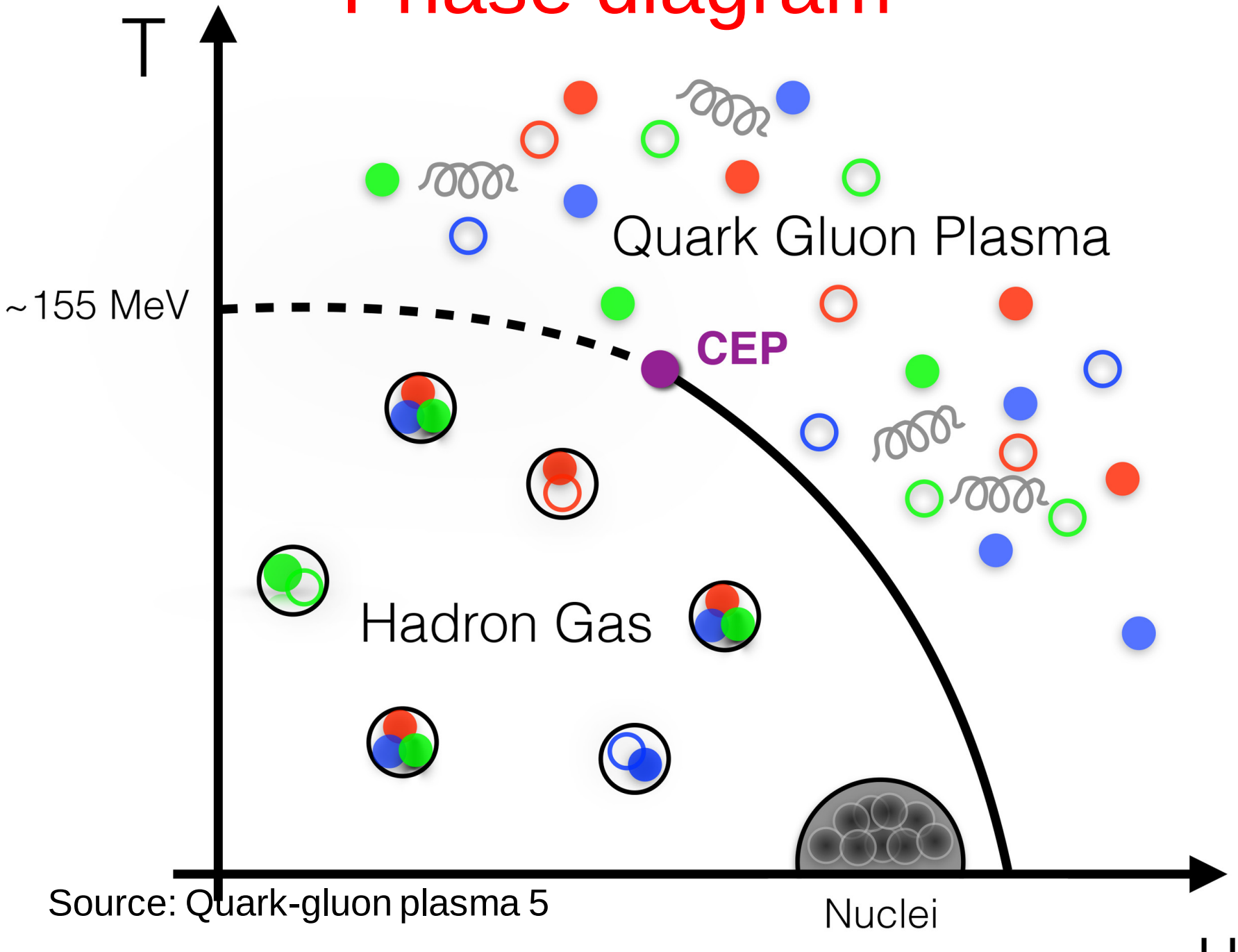
$$S_{\rm eff}(U) = S_W^G(U) - \sum_f \ln \det M_f(U)$$

$$\mathcal{D}U = \prod_{s,\mu} dU_\mu(s)$$

Configurations are generated with probability

$$P \sim e^{-S_{eff}}$$

Phase diagram



Source: Quark-gluon plasma 5

Sign problem

$\det(\not{D} + m + \mu\gamma_0)$ -- In the integral

We use $\gamma_5 \not{D} \gamma_5 = \not{D}^\dagger$

$$\gamma_5(\not{D} + m + \mu\gamma_0)\gamma_5 = \not{D}^\dagger + m - \mu\gamma_0$$

$$= (\not{D} + m - \mu^*\gamma_0)^\dagger$$

$$\det(\not{D} + m + \mu\gamma_0) = \det{}^*(\not{D} + m - \mu^*\gamma_0)$$

$$\det(\not{D} + m + \mu\gamma_0) = \det{}^*(\not{D} + m - \mu^*\gamma_0)$$

Determinant is real only for μ =0 and $\mu = i\mu_I$

This makes impossible to apply usual MCMC algorithm In case of real μ

Note, that for imaginary μ this problem is absent

Methods to solve sign problem

- Multi-Parameter Reweighting

Fodor, Katz, 2002

- Taylor expansion

Gottlieb et al. Phys.Rev.Lett. 59, 2247 (1987) (up to μ^2)
Allton et al., Phys.Rev. D71, 054508 (2005) (up to μ^6)

- Imaginary Chemical Potential

D'Elia, Lombardo, 2002

- Canonical ensemble approach

de Forcrand, Philipsen, 2002

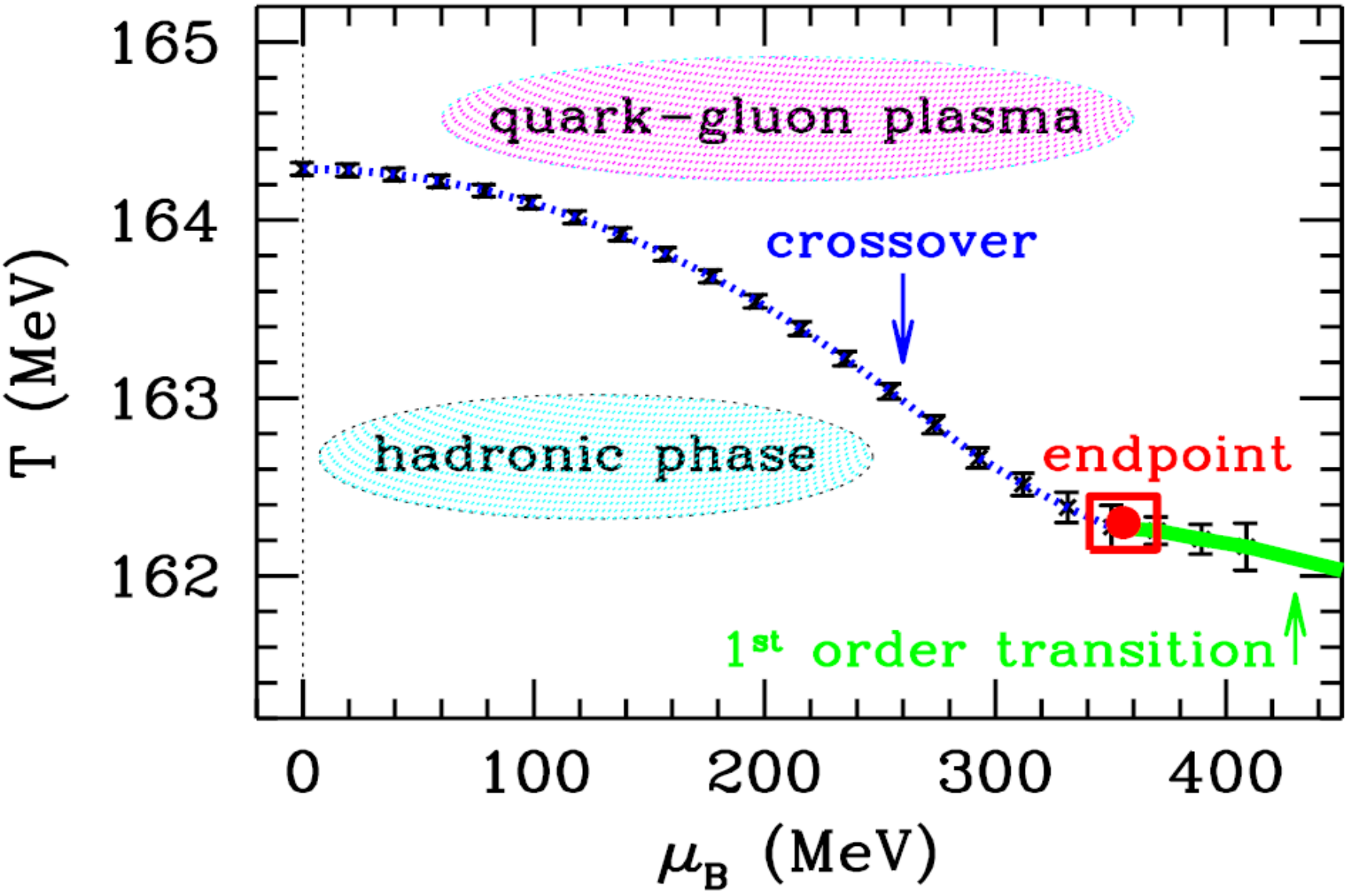
Multi-parameter reweighting

Fodor and Katz, Phys. Lett. B **534**, 87 (2002),
JHEP 0404 (2004) 050, JHEP 0203 (2002) 014

$$Z = \left\langle \frac{\mathrm{e}^{-S_g(\beta)} \det(M(\mu))}{\mathrm{e}^{-S_g(\beta_0)} \det(M(\mu = 0))} \right\rangle_{\mu=0,\beta_0}$$

-Simulations on $N_t = 4, N_s = 6, 8, 10, 12$;
at parameters $\mu = 0, \beta_0$, corresponding to $T = T_c$

- Parameters β and μ along the $\beta_c(\mu)$ line



Lee-Yang zeros of Z in complex β plane were used to determine the crossover line and CEP

Notations

pressure

$$\frac{p}{T^4} = \frac{1}{VT^3} \log Z(V, T, \mu)$$

Quark number density

$$n_f/T^3 = \frac{\partial p/T^4}{\partial \mu_f/T}$$

Susceptibility

$$\chi_{ff}/T^2 = \frac{\partial n_f/T^3}{\partial \mu_f/T}$$

Taylor expansion

S. Gottlieb et al., The Quark Number Susceptibility of High Temperature QCD, Phys.Rev.Lett. 59 (1987) 2247
(up to O(μ^2))

$$\frac{p}{T^4} = \sum_{n=0}^{\infty} c_n(T) \left(\frac{\mu_q}{T}\right)^n$$

$$\mu_u = \mu_d = \mu_q$$

The QCD Equation of State to O(μ^6) from Lattice QCD

Phys.Rev. D95 (2017) no.5, 054504

HotQCD collaboration

Simulation settings:

N_f=2+1 , staggered fermions (HISQ)

$\frac{m_s}{m_l}$ = 27 (m_π=140 MeV)

$\frac{m_s}{m_l}$ = 20 (m_π=160 MeV)

N_t=8, 10, 12, 16; $\frac{N_s}{N_t}$ = 4

High and low temperature limits

For free quark-gluon gas (Stefan-Boltzmann limit):

$$\frac{p_{SB}}{T^4} = \frac{8\pi^2}{45} + \sum_{f=u,d,..} \left[\frac{7\pi^2}{60} + \frac{1}{2}\left(\frac{\mu_f}{T}\right)^2 + \frac{1}{4\pi^2}\left(\frac{\mu_f}{T}\right)^4 \right]$$

This is valid for very high T

For low T – Hadron resonance gas (HRG) model

$$\frac{p}{T^4} = G(T) + F(T)\cosh\left(\frac{3\mu_q}{T}\right)$$

Three conserved charges: B, Q, S

respectively, μ_B, μ_Q, μ_S

$$\begin{aligned}
\mu_u &= \frac{1}{3}\mu_B + \frac{2}{3}\mu_Q \ , \\
\mu_d &= \frac{1}{3}\mu_B - \frac{1}{3}\mu_Q \ , \\
\mu_s &= \frac{1}{3}\mu_B - \frac{1}{3}\mu_Q - \mu_S
\end{aligned}$$

Taylor expansion

$$\frac{P}{T^4} = \frac{1}{VT^3} \ln \mathcal{Z}(T, V, \hat{\mu}_u, \hat{\mu}_d, \hat{\mu}_s)$$

$$= \sum_{i,j,k=0}^{\infty} \frac{\chi_{ijk}^{BQS}}{i!j!\,k!} \hat{\mu}_B^i \hat{\mu}_Q^j \hat{\mu}_S^k \qquad \hat{\mu} = \mu/T$$

Generalized susceptibilities:

$$\chi_{ijk}^{BQS} = \left. \frac{\partial P(T, \hat{\mu})/T^4}{\partial \hat{\mu}_B^i \partial \hat{\mu}_Q^j \partial \hat{\mu}_S^k} \right|_{\hat{\mu}=0}$$

Constraints on μ_Q and μ_S :

1) $\mu_Q = 0, \quad \mu_S = 0$

2) $n_Q = 0.4\, n_B \;, \quad n_S = 0$

Then

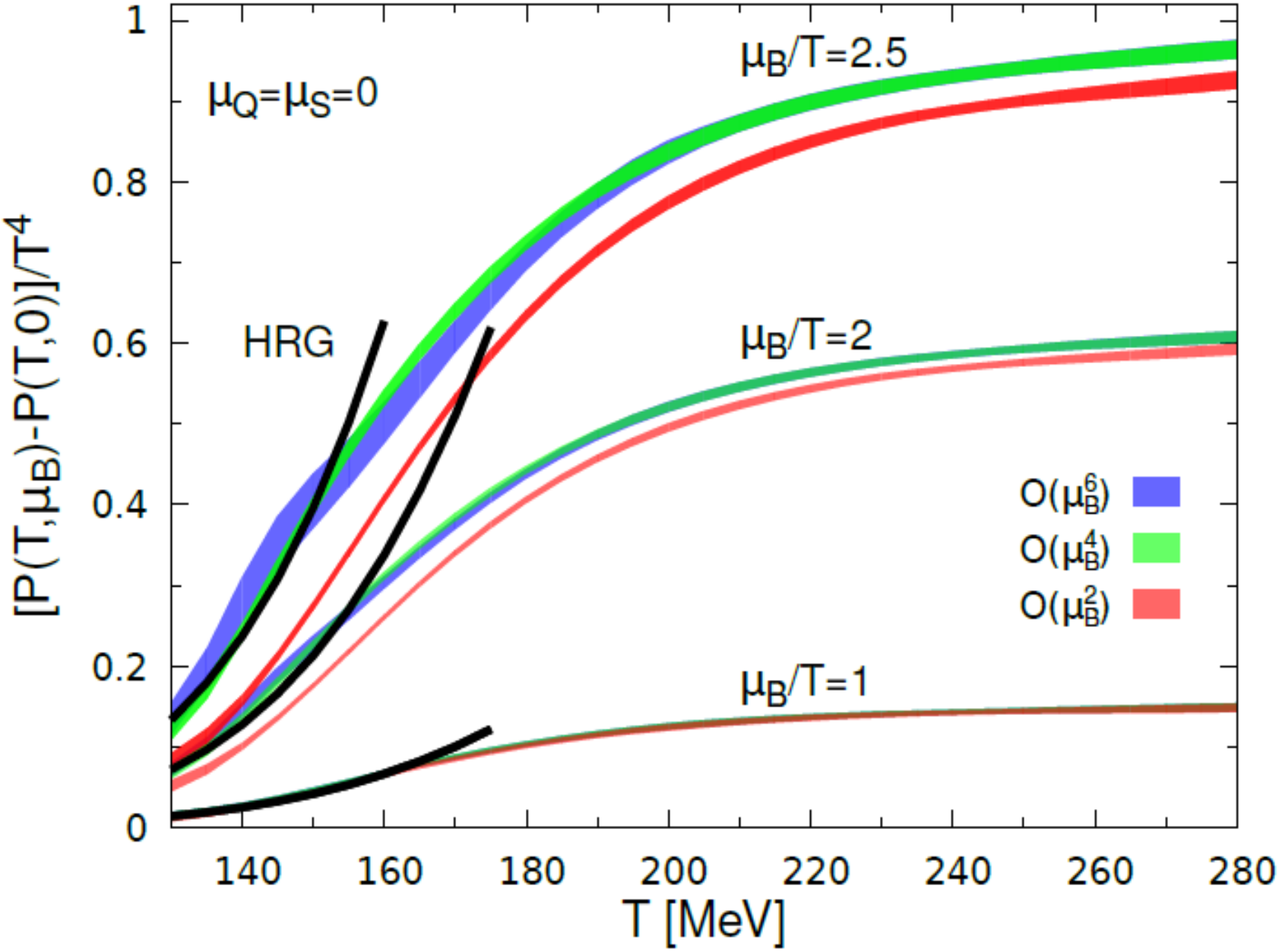
$$\frac{P}{T^4} = \sum_{k=1}^{\infty} P_{2k}(T)\hat{\mu}_B^{2k}$$

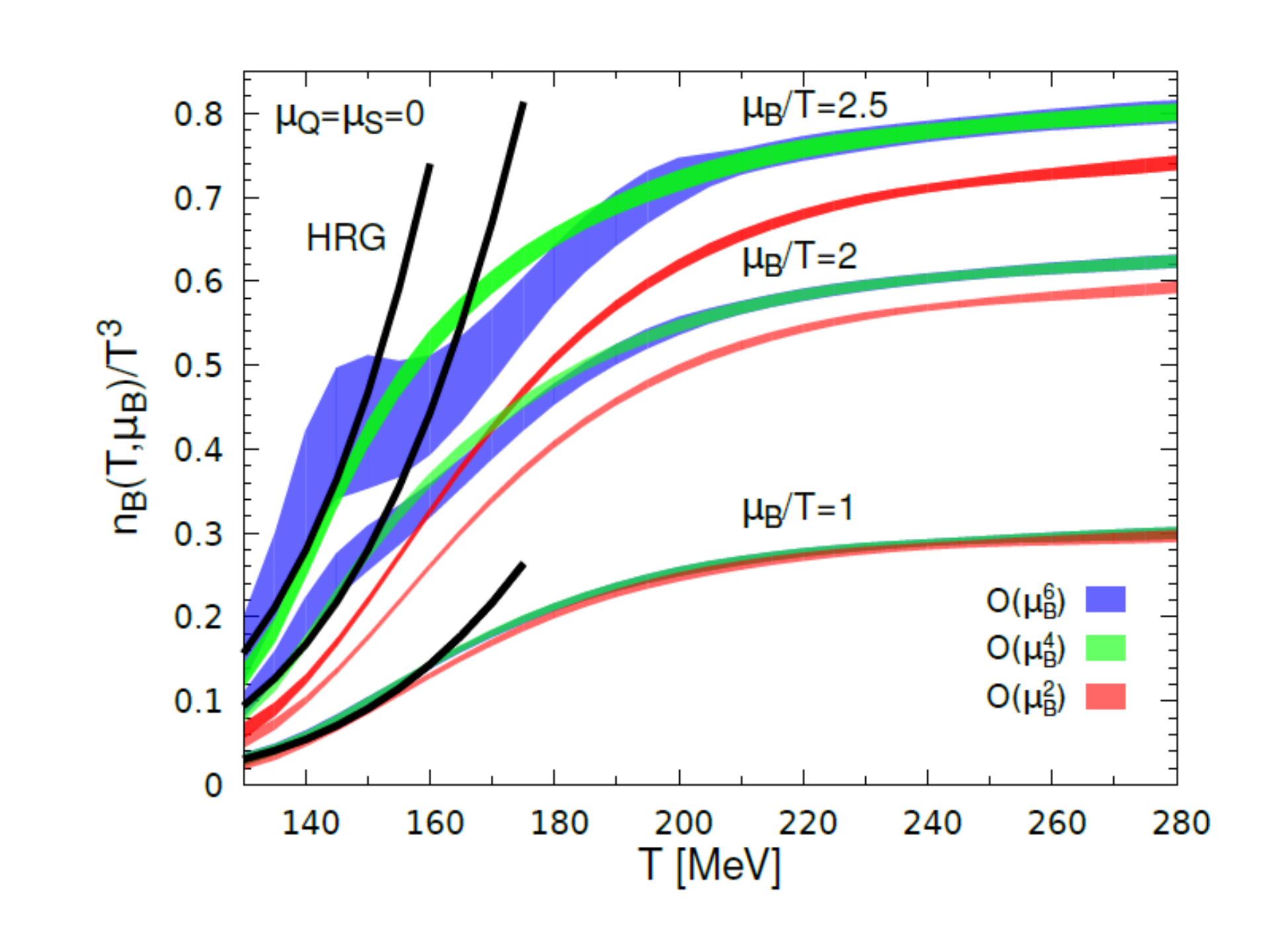
Terms of the form

$$\mathrm{Tr} M_f^{-1} M_f' M_f^{-1} M_f' .... M_f^{-1} M_f'$$

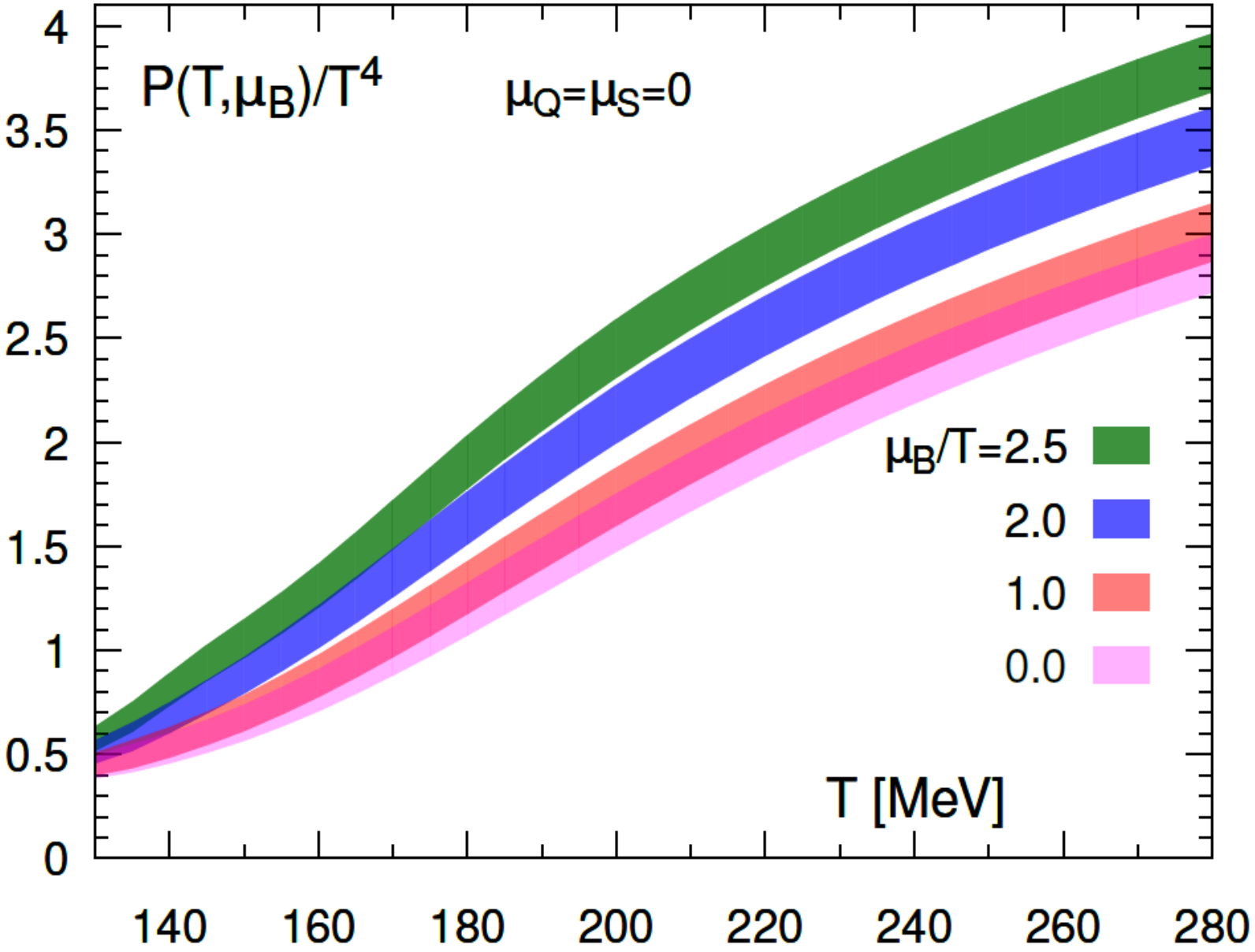
are to be computed using stochastic estimator method

For low temperatures number of trajectories: O(10^6)


$\mu_Q=\mu_S=0$
$\mu_B/T=2.5$
$\mu_B/T=2$
$\mu_B/T=1$
HRG
$O(\mu_B^6)$
$O(\mu_B^4)$
$O(\mu_B^2)$
$[P(T,\mu_B)-P(T,0)]/T^4$
T [MeV]
0
0.2
0.4
0.6
0.8
1
140
160
180
200
220
240
260
280


$\mu_Q=\mu_S=0$
HRG
$\mu_B/T=2.5$
$\mu_B/T=2$
$\mu_B/T=1$
$O(\mu_B^6)$
$O(\mu_B^4)$
$O(\mu_B^2)$
$n_B(T,\mu_B)/T^3$
T [MeV]
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
140
160
180
200
220
240
260
280

Full pressure



Convergence radius is $n \to \infty$ limit of

$$r_{2n}^{P} = \left| \frac{(2n+2)(2n+1)\chi_{2n}^{B}}{\chi_{2n+2}^{B}} \right|^{1/2}$$

For finite radius $|\chi_{n+2}^{B}/\chi_{n}^{B}| \sim n^2$

Note that for HRG model

$$|\chi_{n+2}^{B}/\chi_{n}^{B}|^{HRG} = 1$$

Assuming that the current results obtained with expansion coefficients up to 6th order are indicative for the behavior of higher order expansion coefficients and taking into account the current errors on 6th order expansion coefficients we concluded that at temperatures T > 135 MeV the presence of a critical point in the QCD phase diagram for $\mu_B < 2T$ is unlikely.

- Analytical continuation from imaginary μ

M. D'Elia, M.-P. Lombardo, Phys.Rev. D67 (2003) 014505

Ph. De Forcrand, O. Philipsen, Nucl.Phys. B642 (2002) 290-306

- Canonical ensemble approach

A. Hasenfratz, D. Toussaint, Nucl. Phys. B371 (1992) 539

Ph. De Forcrand, S. Kratochvila, Nucl.Phys.Proc.Suppl. 140 (2005) 514-516

Imaginary μ_q

At imaginary chemical potential $\mu_q = i\mu_{qI}$ the sign problem is absent and standard Monte Carlo algorithms can be applied to simulate Lattice QCD. Can we use this?

Study of QCD at nonzero μ_{qI} can provide us with information about physical range of μ_q

- extrapolation to $\mu_q = 0$ or analytical continuation to nonzero real μ_q

The QCD partition function Z is a periodic function of $\theta = \mu_{qI}/T$:

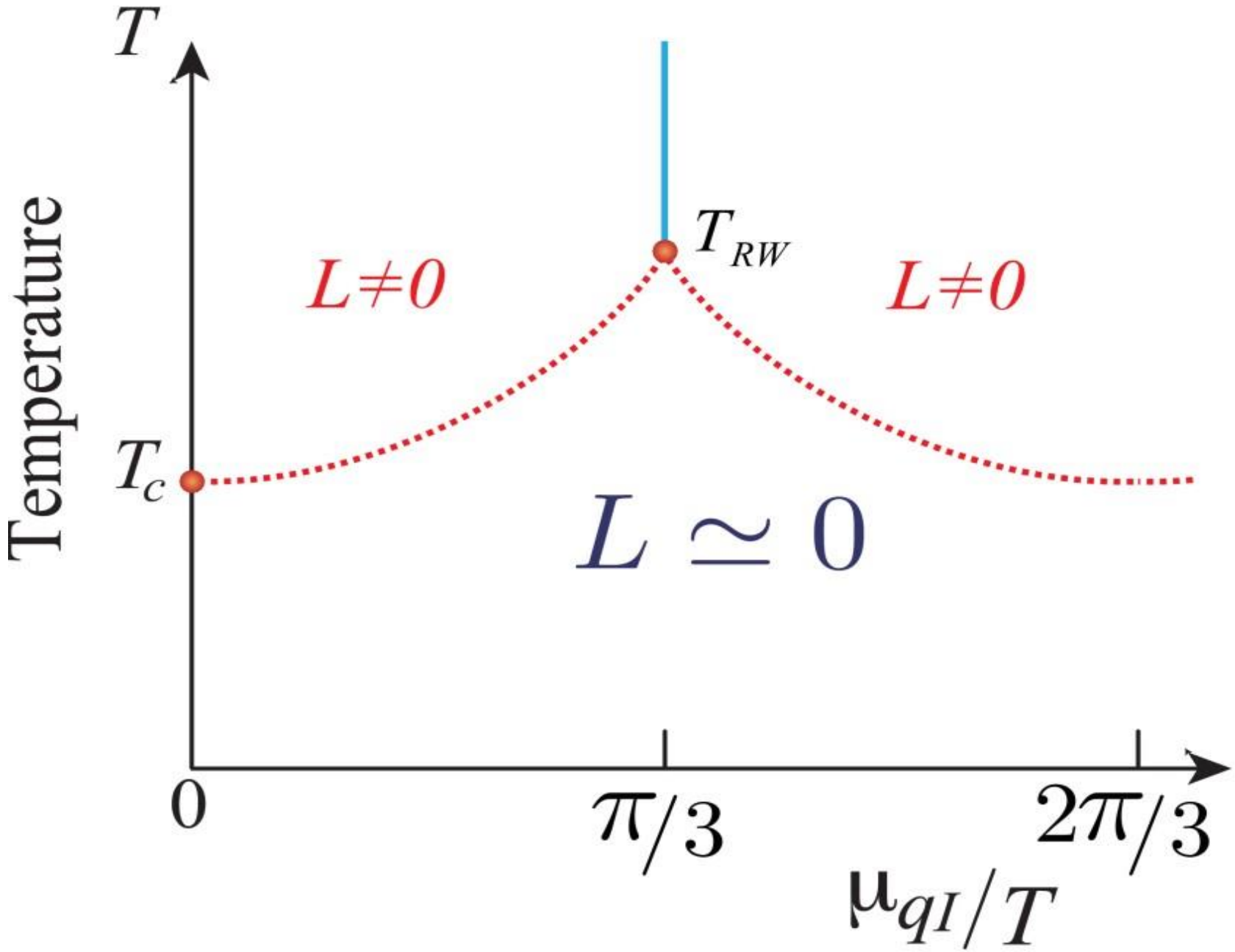
$$Z(\theta) = Z(\theta + 2\pi k/3)$$

There are 1st order phase transitions at $\theta = (2k+1)\frac{\pi}{3}$

This symmetry is called Roberge-Weiss symmetry

Roberge,Weiss, 1986

QCD phase diagram at imaginary μ

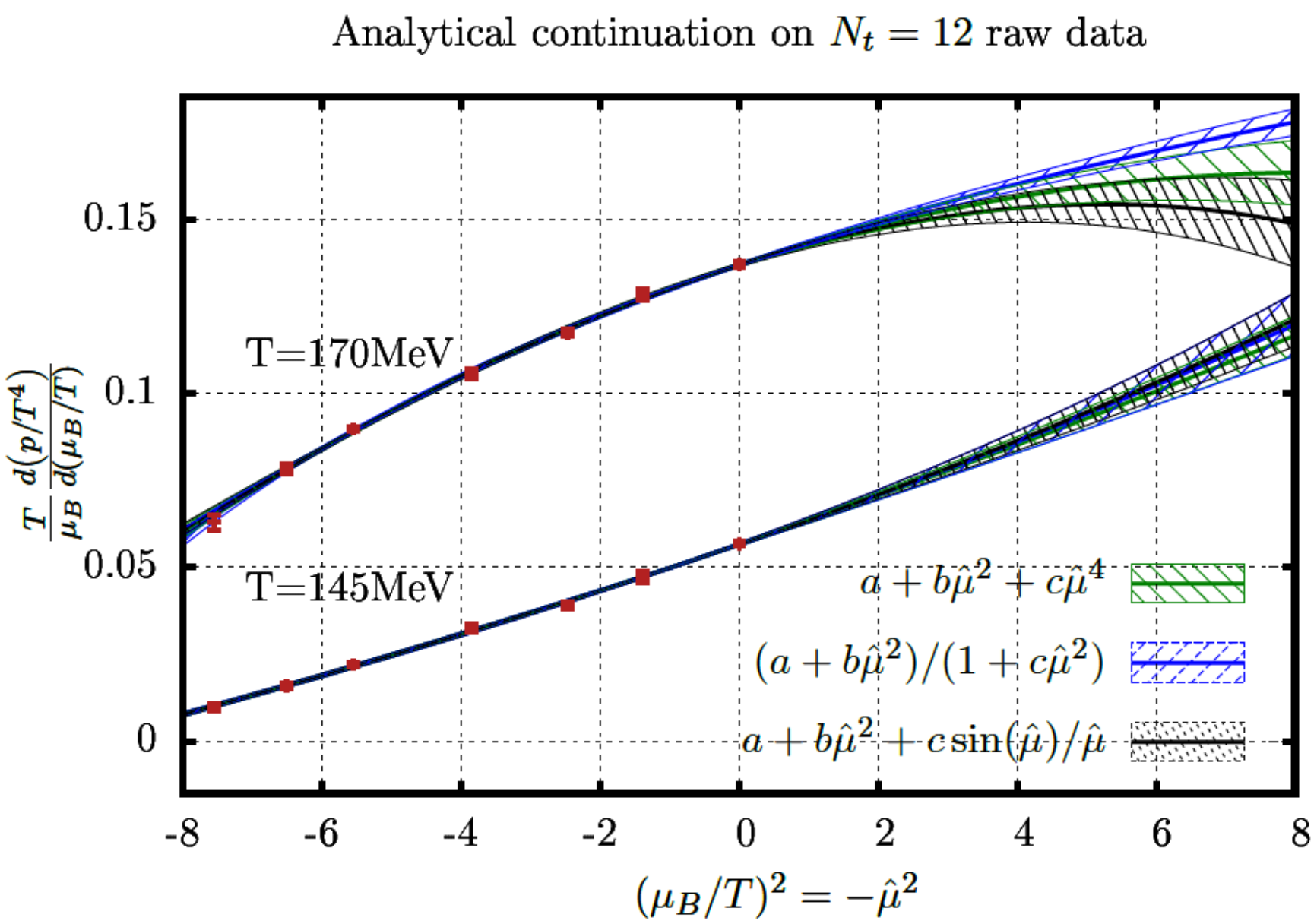


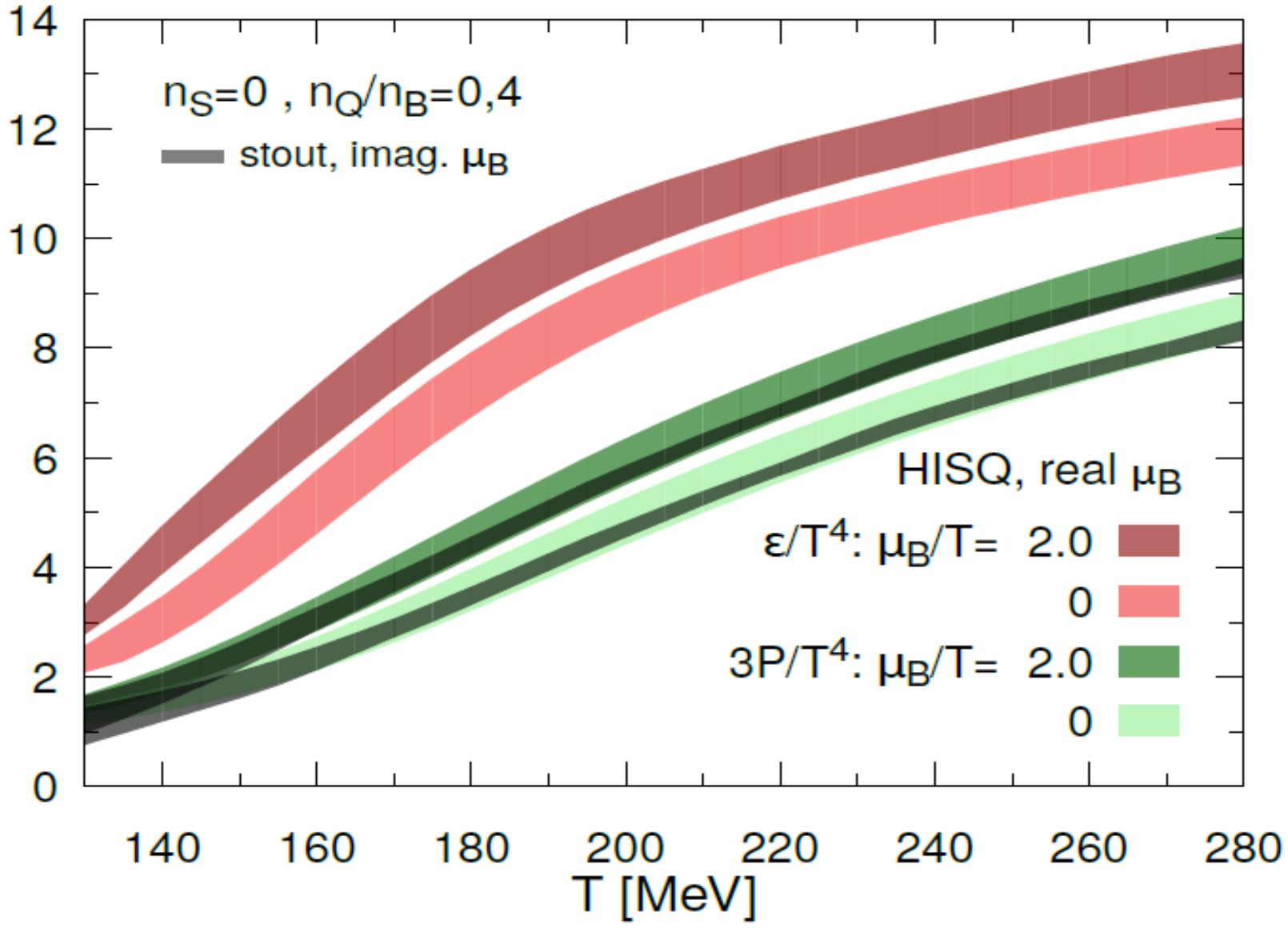
The QCD equation of state at finite density from analytical continuation

EPJ Web Conf. 137 (2017) 07008 ,
Wuppertal-Budapest collaboration

$N_f = 2 + 1$
N_t= 10, 12, 16
N_s=40, 48, 64

Staggered fermions,
physical quark masses


Analytical continuation on $N_t = 12$ raw data
$\frac{T}{\mu_B}\frac{d(p/T^4)}{d(\mu_B/T)}$
0.15
0.1
0.05
0
T=170MeV
T=145MeV
$a + b\hat{\mu}^2 + c\hat{\mu}^4$
$(a + b\hat{\mu}^2)/(1 + c\hat{\mu}^2)$
$a + b\hat{\mu}^2 + c\sin(\hat{\mu})/\hat{\mu}$
-8
-6
-4
-2
0
2
4
6
8
$(\mu_B/T)^2 = -\hat{\mu}^2$



Pressure obtained by two different methods

from Phys.Rev. D95 (2017) by F.Karsch et al.

Canonical approach

The canonical approach is based on the following relations.

- Relation between grand canonical partition function $Z_{GC}(\mu, T, V)$ and canonical one $Z_C(n, T, V)$:

$$Z_{GC}(\mu_q, T, V) = \sum_{n=-\infty}^{\infty} Z_C(n, T, V)\xi^n, \qquad \xi = e^{\mu/T}$$

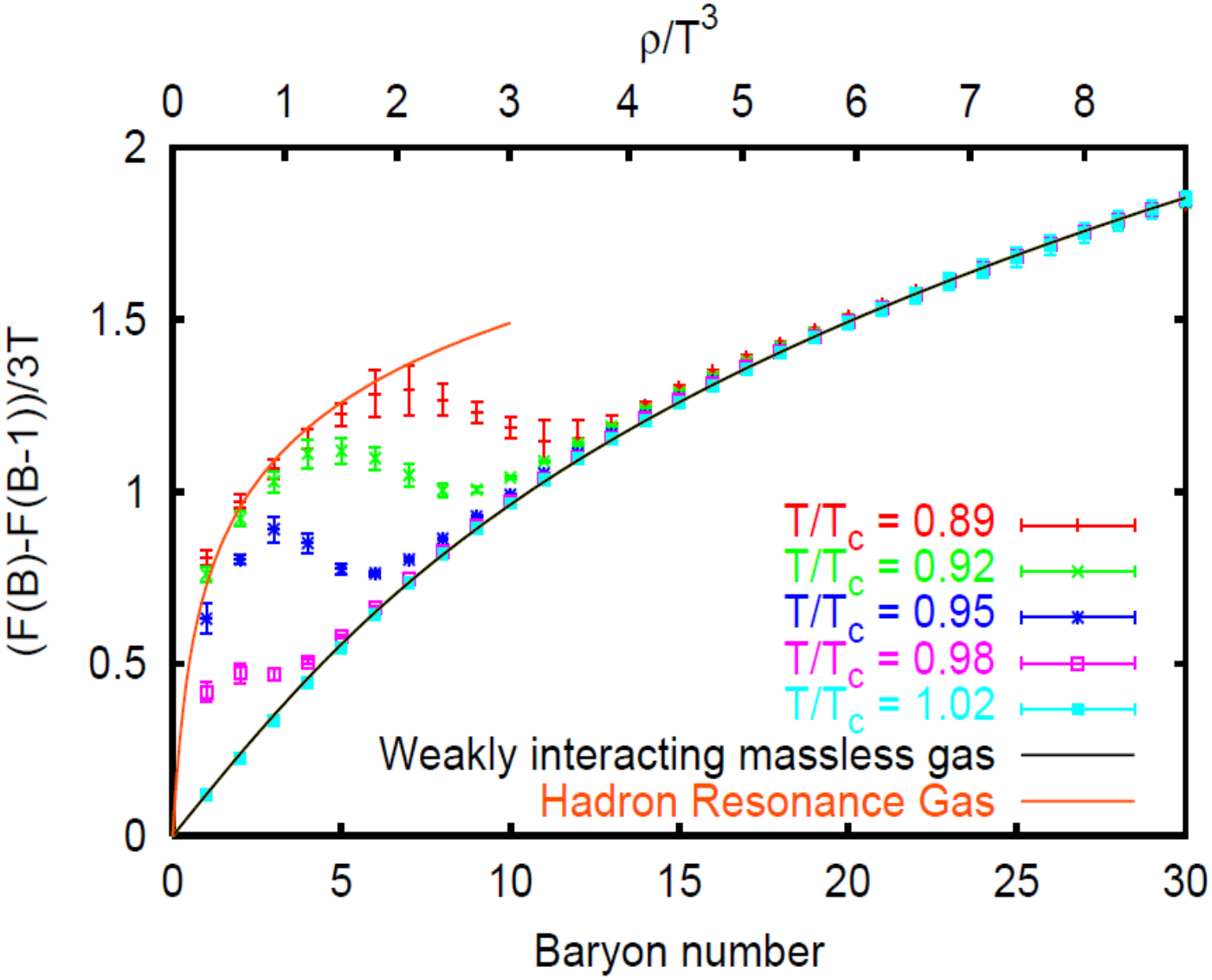
- The inverse of this equation : (Hasenfratz, Toussaint, 1992)

$$Z_C(n, T, V) = \int_0^{2\pi} \frac{d\theta}{2\pi} e^{-in\theta} Z_{GC}(i\theta, T, V).$$

$\theta = \mu_I/T$

Standard Monte Carlo simulations are possible

N_f=4, staggered fermions, m_π=350 MeV
lattice size $4x6^3$



Other approaches

- Complex Langevin

- Density of states

- Dual formulation

- Lefschetz thimble

Problem of LQCD simulations at nonzero μ_B is **not yet solved**!

It is waiting for **you** !