# S-MATRIX APPROACH TO HADRON GAS

POK MAN LO

**University of Wroclaw**

HELMHOLTZ INTERNATIONAL SUMMER SCHOOL
20 AUG-02 SEPT 2017, DUBNA, RUSSIA

# CONTENT

- QCD equation of state
- S-matrix approach to broad resonances
- extension to N-body

# QCD EQUATION OF STATE

# HADRON RESONANCE GAS MODEL

- Confinement

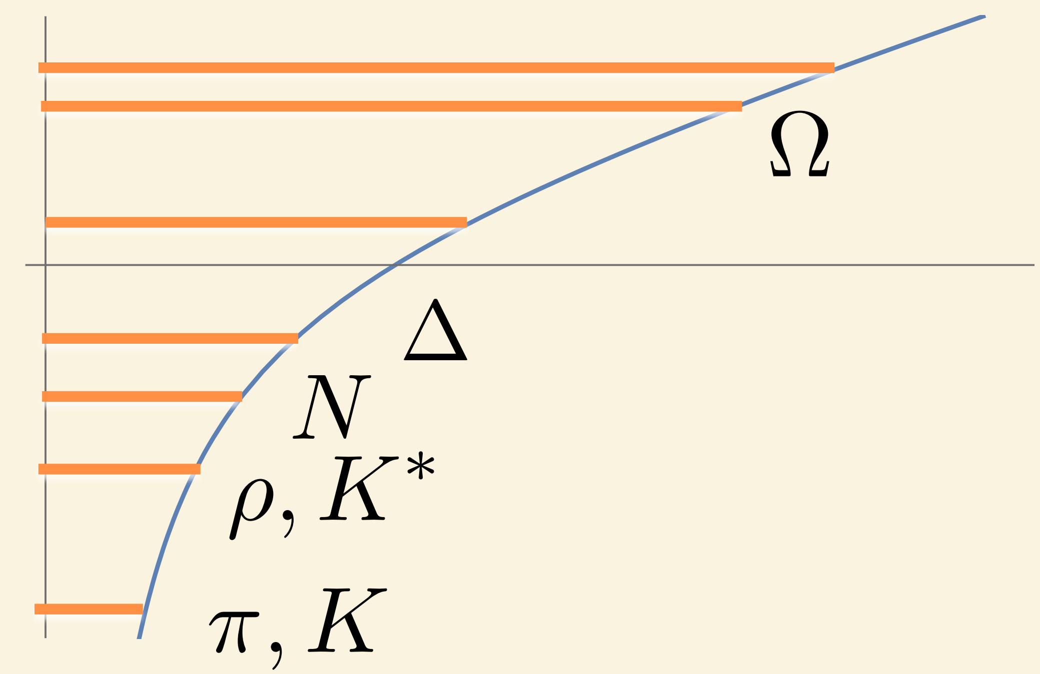
physical quantities → hadronic states representation

$$Z = \sum_{\alpha=B,M} \langle \alpha | e^{-\beta H} | \alpha \rangle$$

# HADRON RESON
# MODEL

- Confinement

physical quantities




$$Z = \sum_{\alpha=B,M} \langle \alpha | e^{-\beta H} | \alpha \rangle$$

# HADRON RESONANCE GAS MODEL

- Ground states $\pi, K, P, N...$

- Resonance formation dominates thermodynamics
- Resonances treated as point-like particles

$$P = T \sum_{\alpha=M,B} g_\alpha \int \frac{d^3k}{(2\pi)^3} \mp \ln(1 \mp e^{-\beta\sqrt{k^2+M_\alpha^2}})$$

# HADRON RESONANCE GAS MODEL

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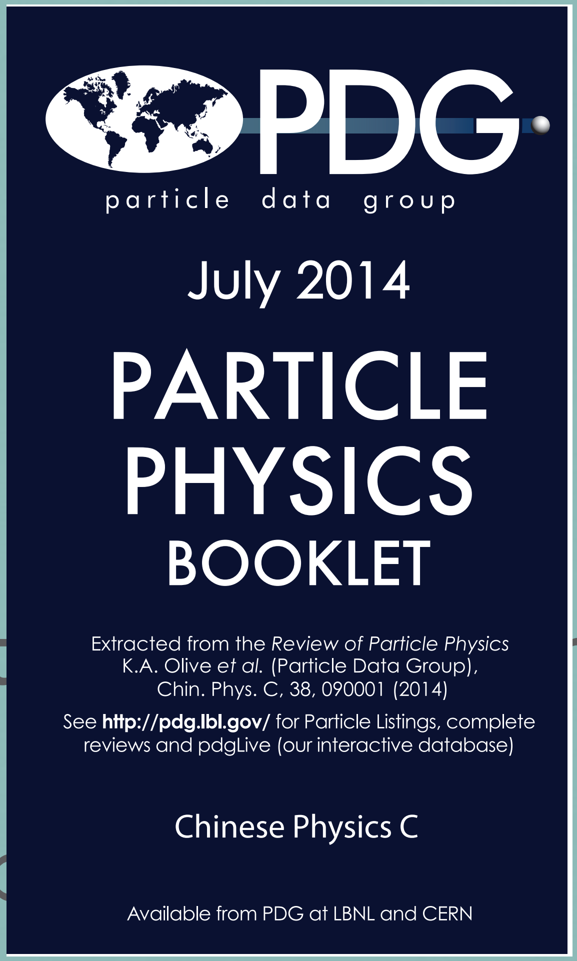


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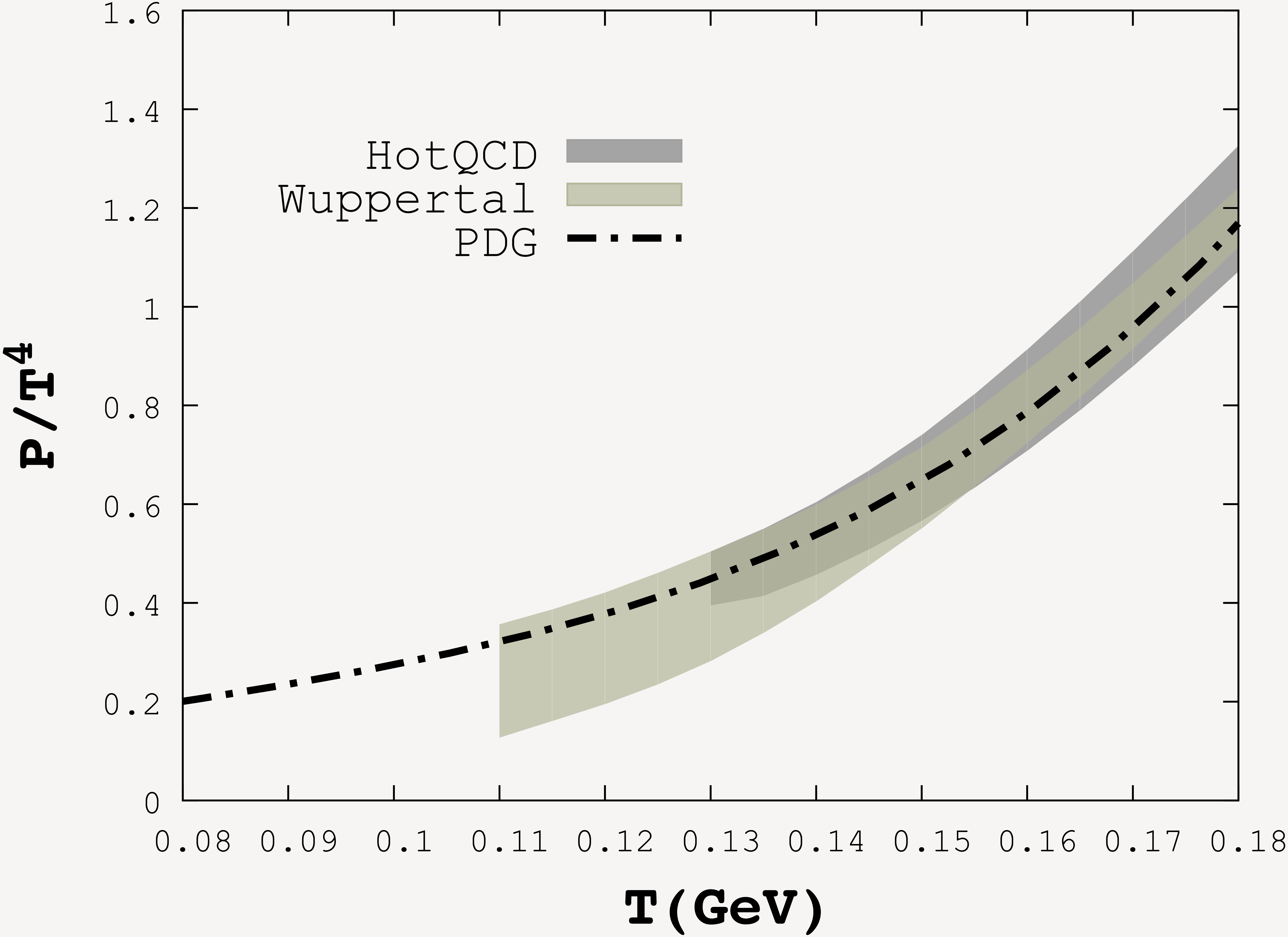
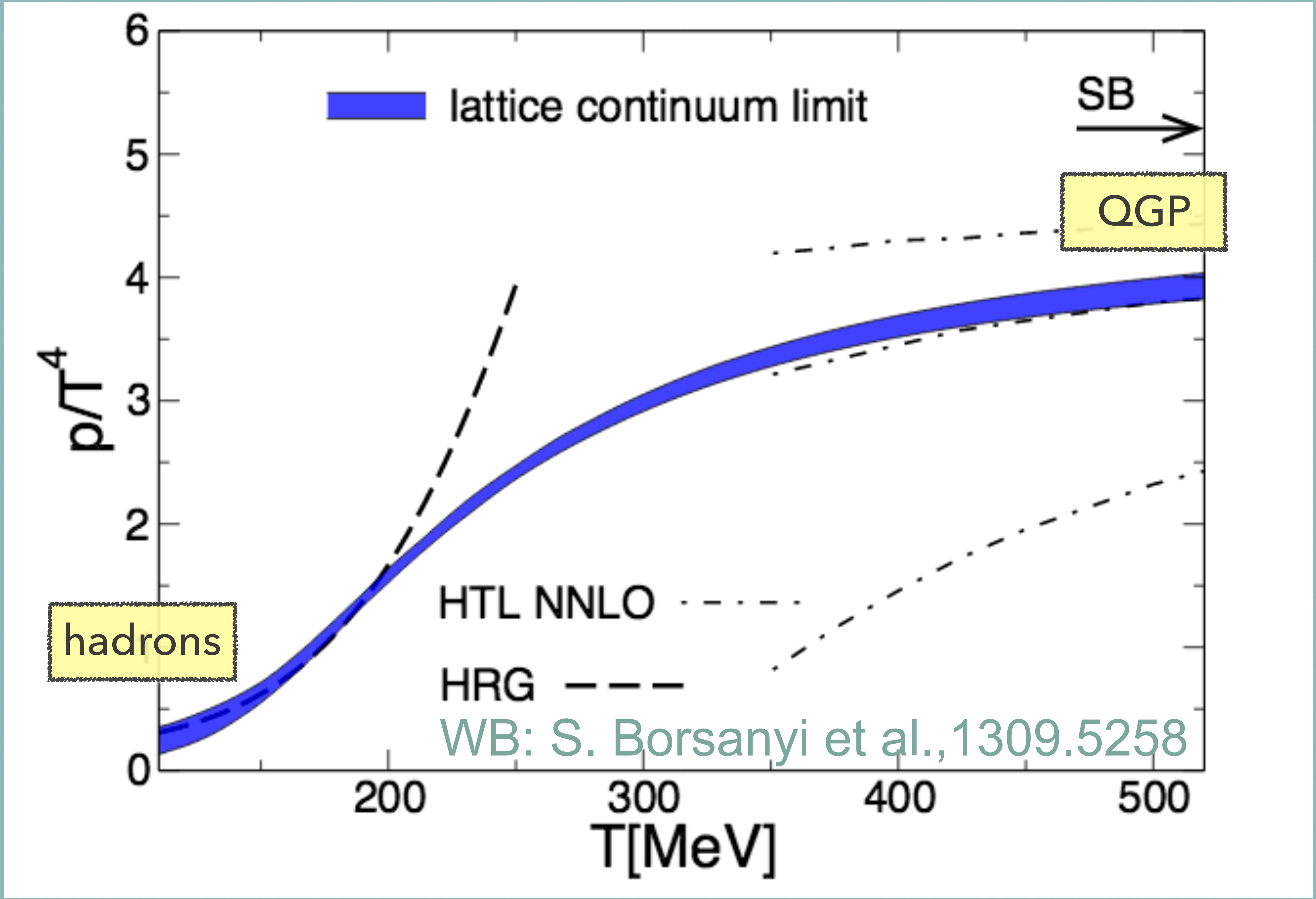
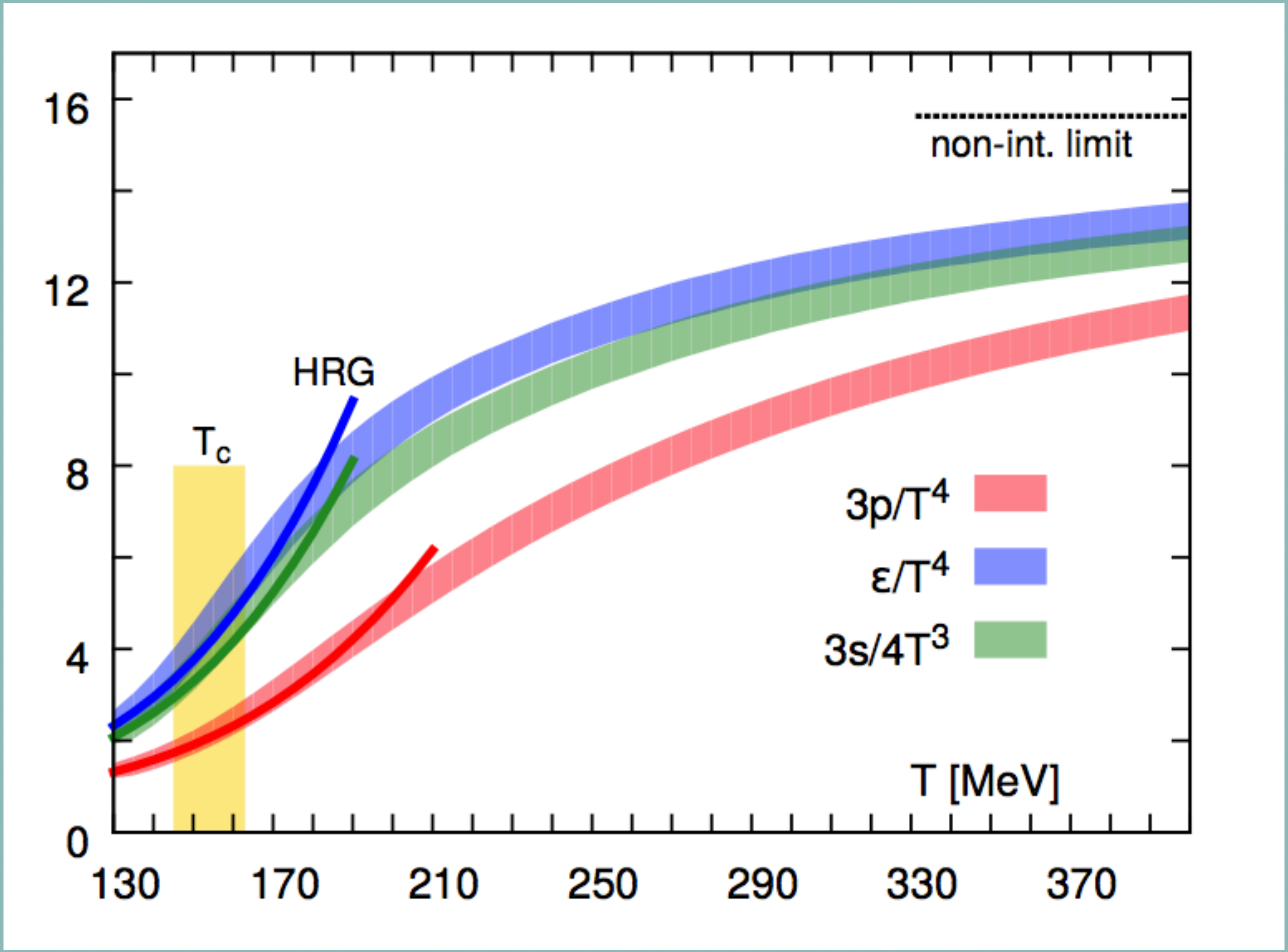

HotQCD
Wuppertal
PDG
$P/T^4$
T(GeV)
0
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6
0.08
0.09
0.1
0.11
0.12
0.13
0.14
0.15
0.16
0.17
0.18



lattice continuum limit
SB
QGP
p/T^4
HTL NNLO
HRG
hadrons
WB: S. Borsanyi et al.,1309.5258
0
2
3
4
5
6
200
300
400
500
T[MeV]




A. Bazavov et al. (hotQCD), PRD90 (2014) 094503

# FLUCTUATIONS

- studying the system by linear response

$$\mu = \mu_B B + \mu_Q Q + \mu_S S$$

$$\chi_{B,S,\ldots} = \frac{1}{\beta V} \frac{\partial^2}{\partial \bar{\mu}_B \partial \bar{\mu}_S \ldots} \ln Z$$

$\mu_B$

$\mu_S$

$\mu_Q$

$m_q$

# FLUCTUATIONS

- Baryon sector

$$P = T \sum_{\alpha=\cancel{M},B} g_\alpha \int \frac{d^3k}{(2\pi)^3} \mp \ln(1 \mp e^{-\beta\sqrt{k^2+M_\alpha^2}})$$

or introduce the chemical potential

$$P = T \sum_{\alpha=B,\bar{B}} g_\alpha \int \frac{d^3k}{(2\pi)^3} \ln(1 + e^{-\beta\sqrt{k^2+M_\alpha^2} \pm \bar{\mu}_B})$$

# FLUCTUATIONS

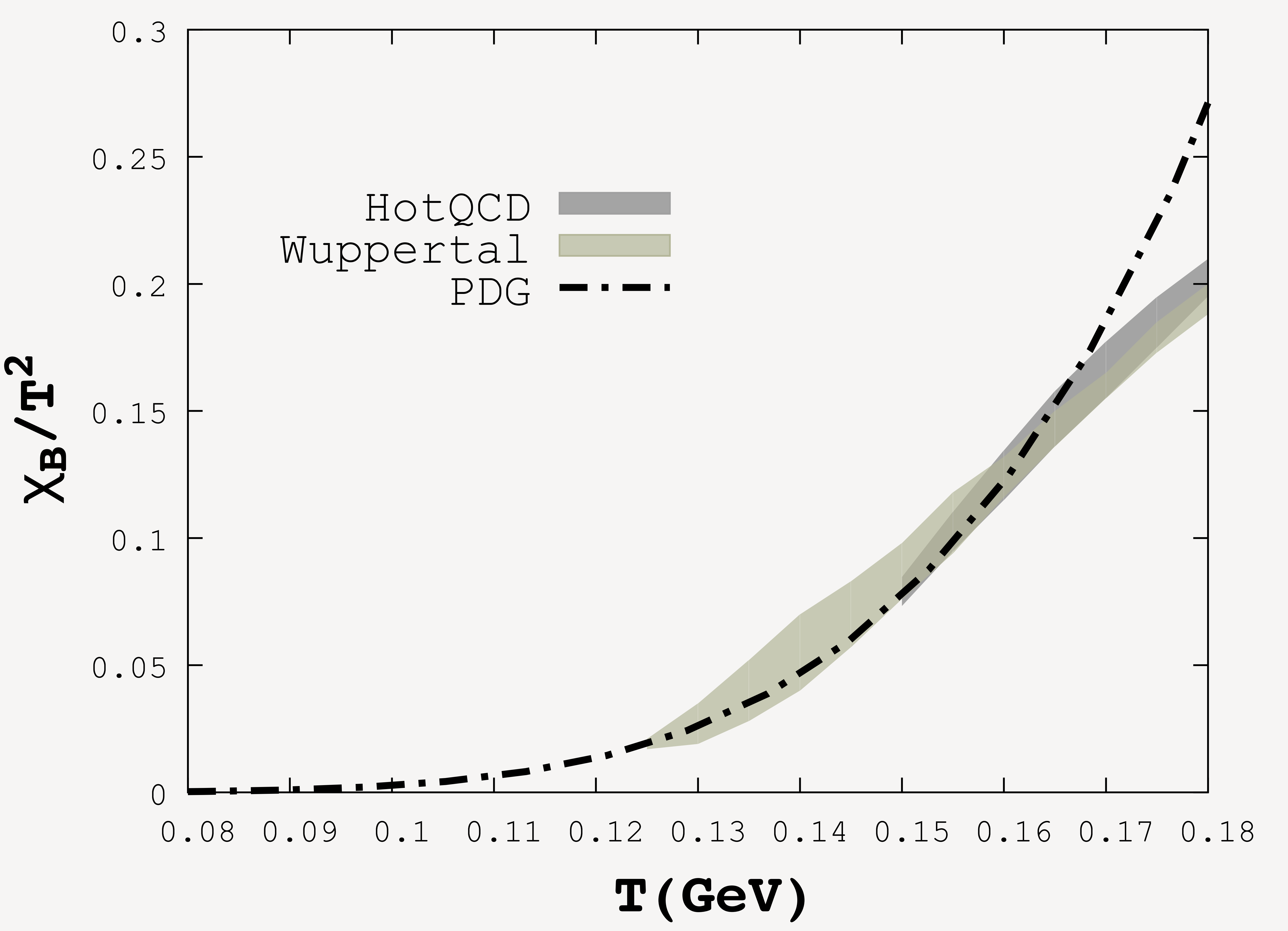
- taking derivative

$$\chi_B = \frac{\partial^2}{\partial\bar{\mu}_B \partial\bar{\mu}_B} P$$ *at the limit* $\mu_B \to 0$

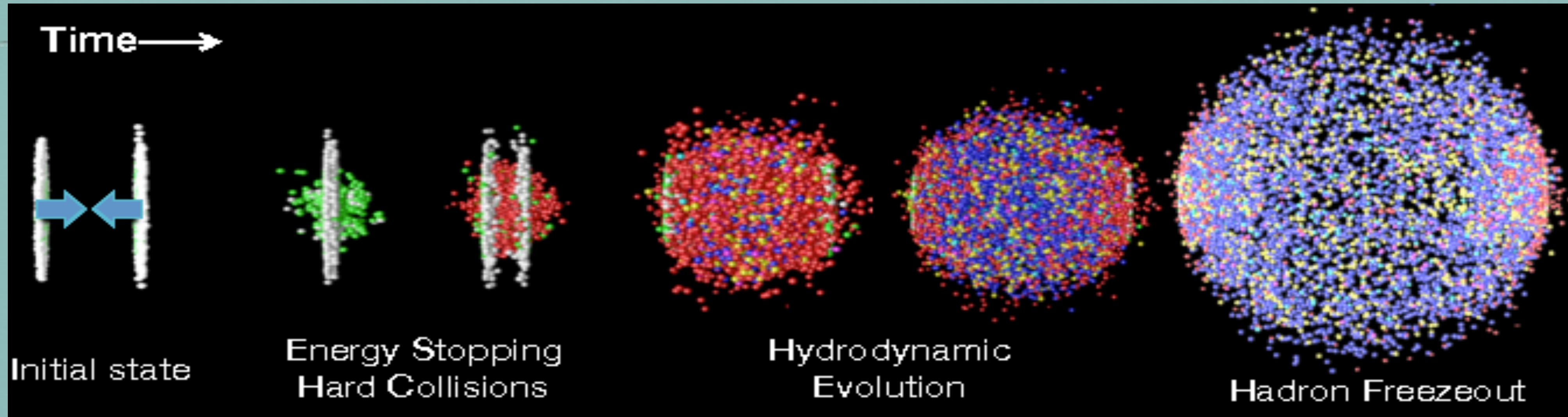
probes fluctuations

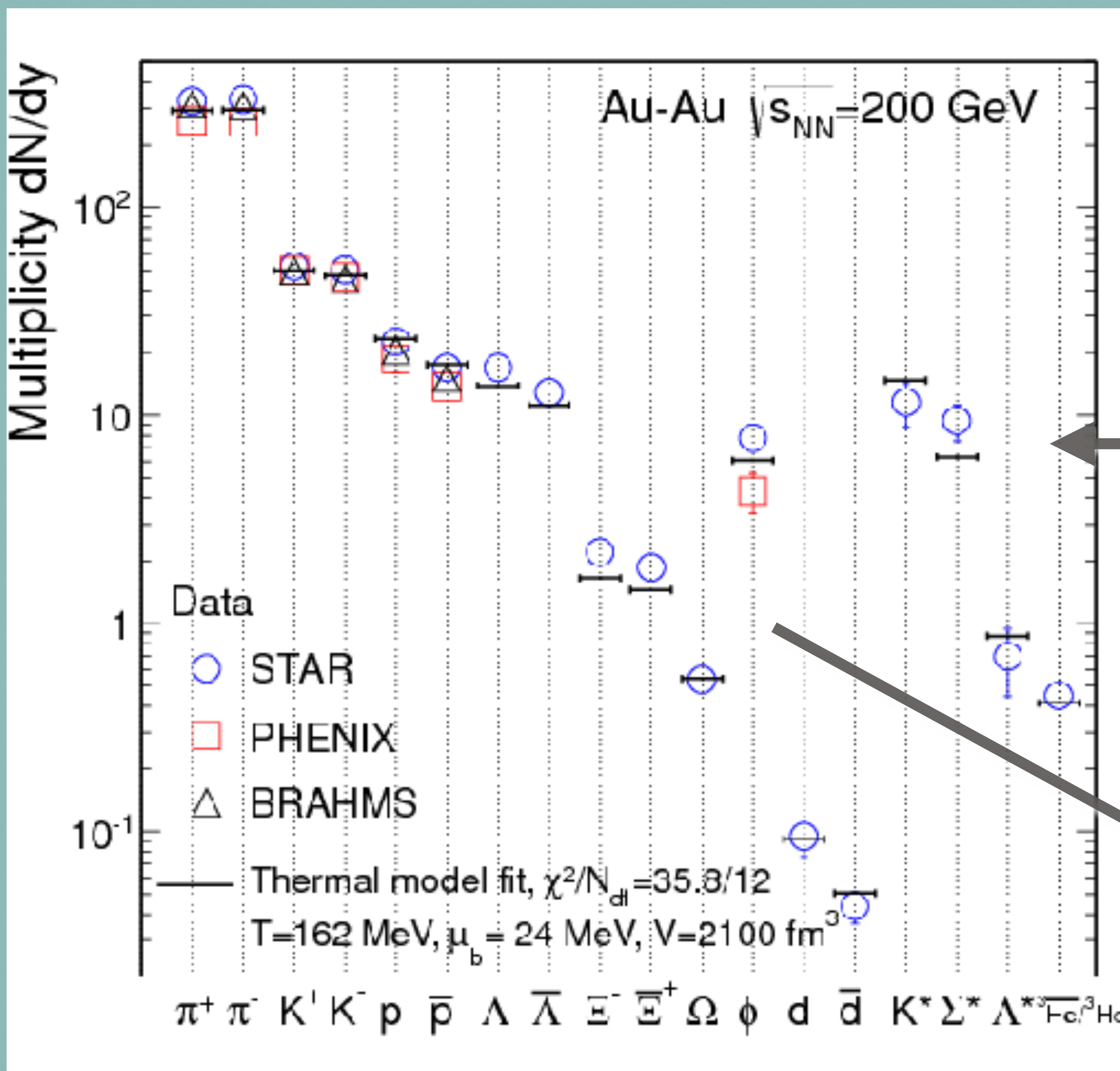
$$\chi_B = \frac{1}{\beta V} \frac{\partial^2}{\partial\bar{\mu}_B \partial\bar{\mu}_B} \ln Z$$

$$= T^2 \langle\langle \int d^4x\, \bar{\psi}(x)\gamma^0\psi(x)\bar{\psi}(0)\gamma^0\psi(0)\rangle\rangle_c$$


HotQCD
Wuppertal
PDG
$\chi_B/T^2$
T(GeV)
0.3
0.25
0.2
0.15
0.1
0.05
0
0.08
0.09
0.1
0.11
0.12
0.13
0.14
0.15
0.16
0.17
0.18


*Central Au+Au 200 GeV/nucleon*
*MADAI*
*Simulation with UrQMD*



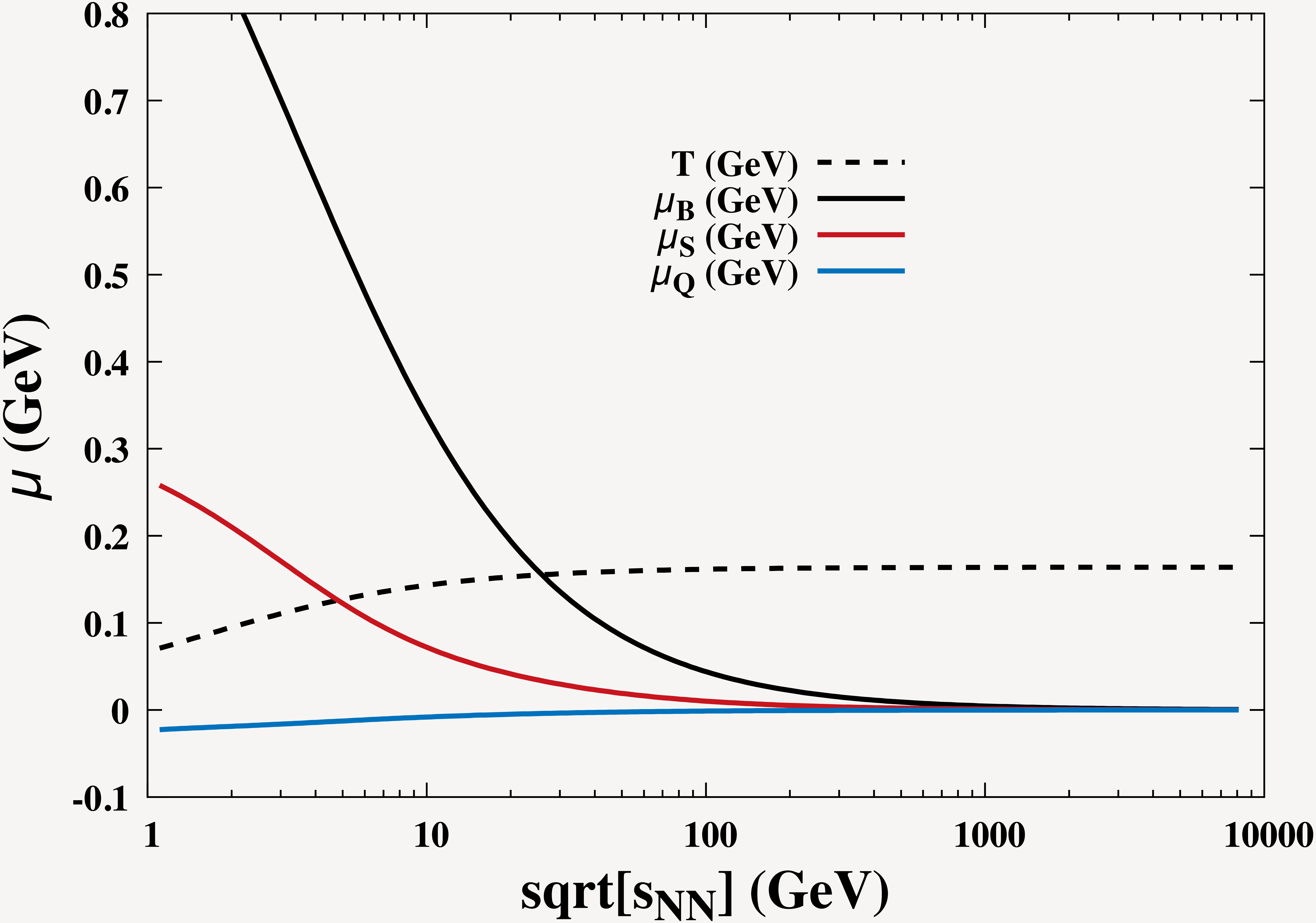




*freezeout hadrons yields described by HRG*

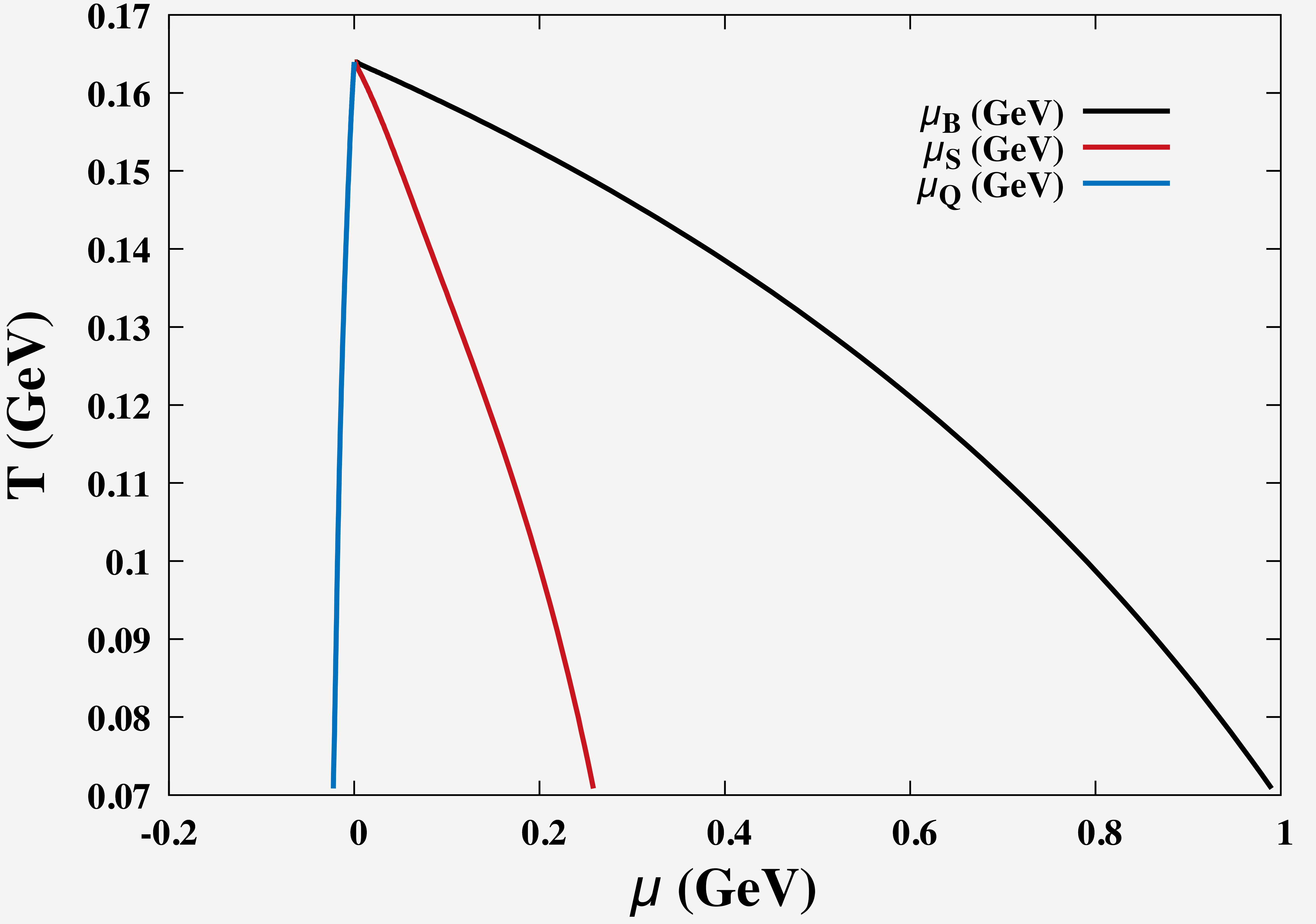
*Freezeout parameters*

$$T^f, \mu_B^f, \mu_S^f, \mu_Q^f, \ldots$$

Andronic, A. *et al.* Nucl.Phys. A904-905 (2013)



0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
-0.1
μ (GeV)
T (GeV)
μB (GeV)
μS (GeV)
μQ (GeV)
1
10
100
1000
10000
sqrt[sNN] (GeV)

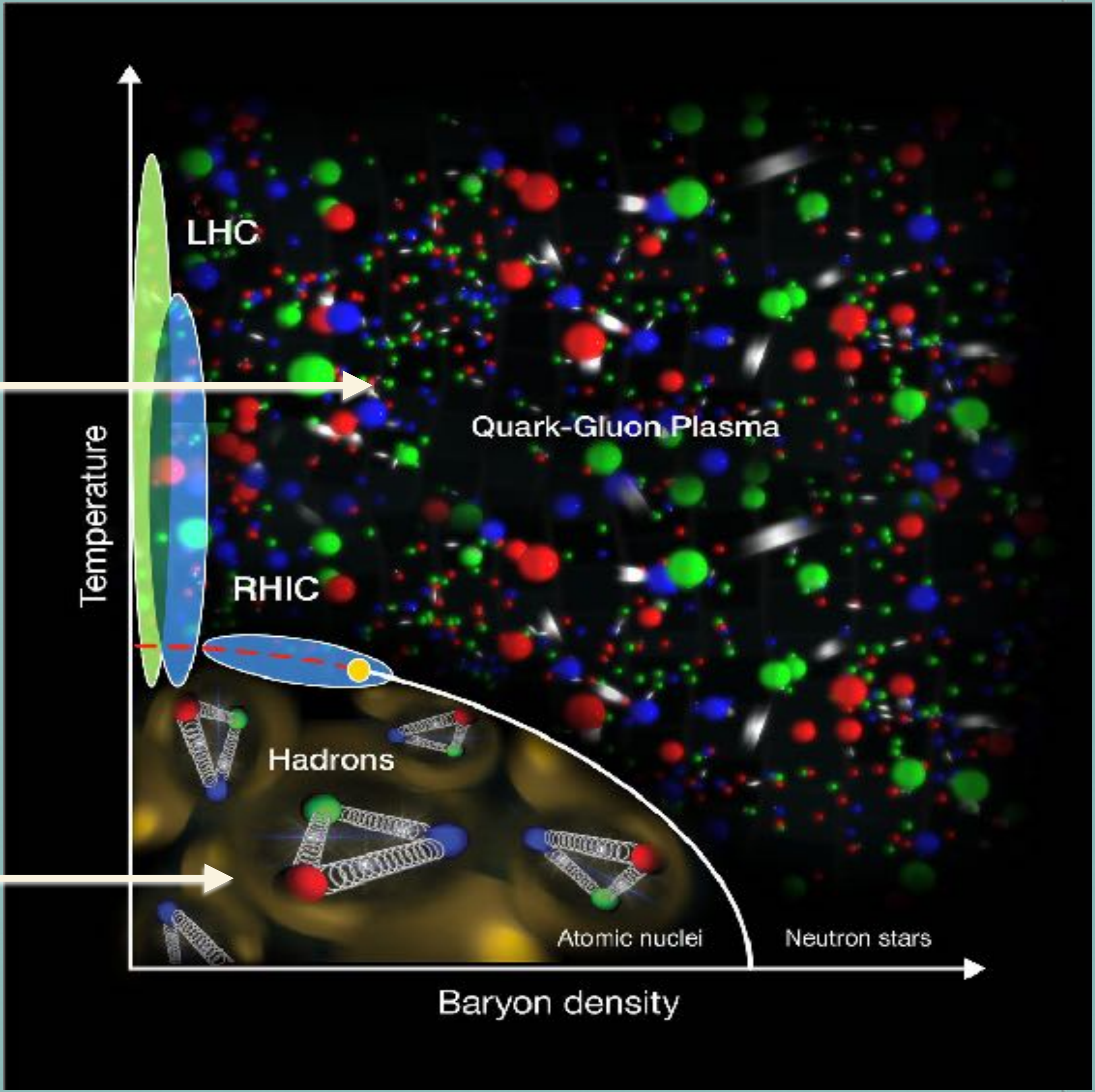


0.17
0.16
0.15
0.14
0.13
0.12
0.11
0.1
0.09
0.08
0.07
T (GeV)
-0.2
0
0.2
0.4
0.6
0.8
1
μ (GeV)
μB (GeV)
μS (GeV)
μQ (GeV)

# QCD Phase Diagram

QGP:
quarks and gluons
are deconfined.

Hadronic phase:
quarks are confined
and massive.

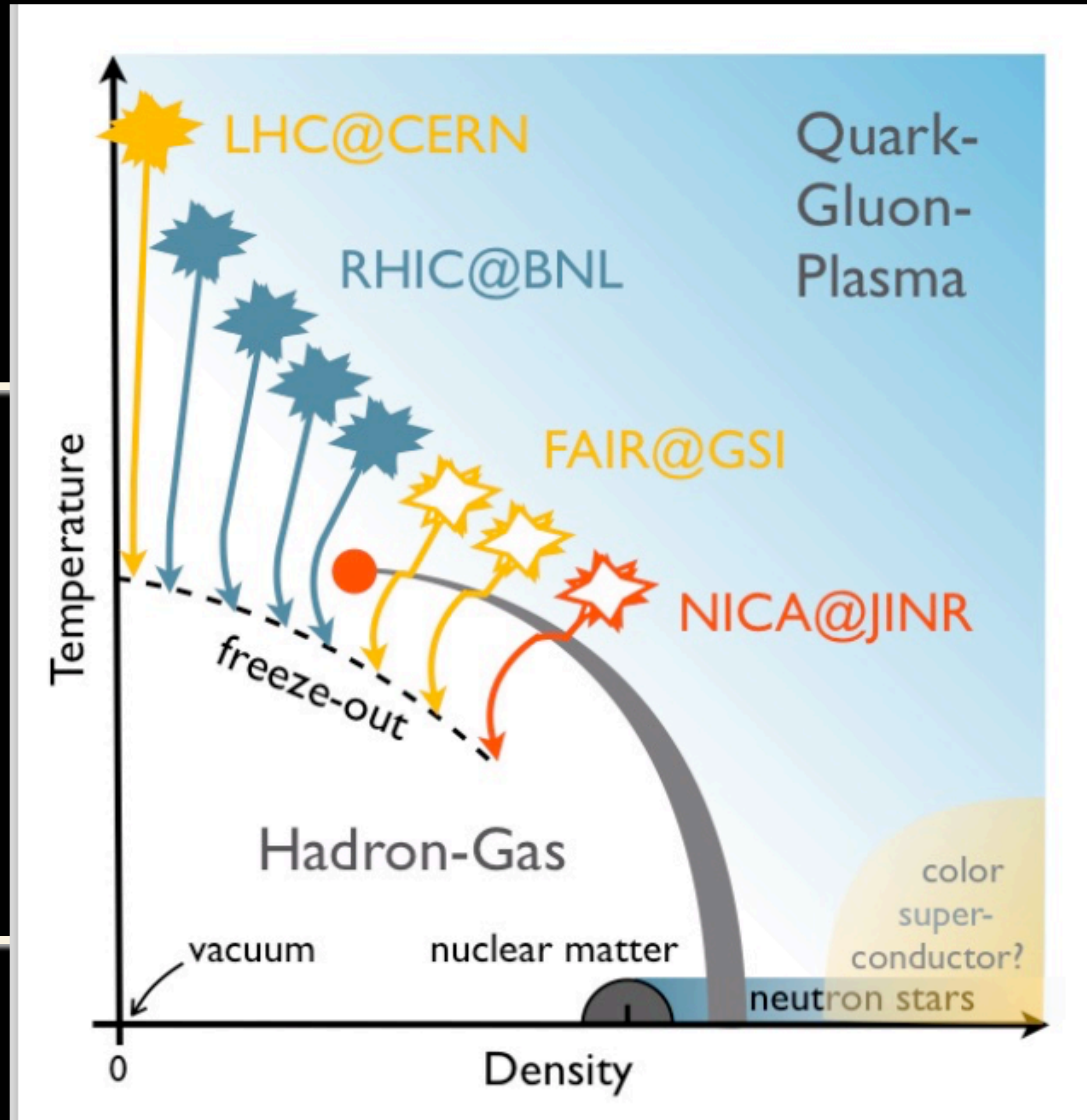



*Courtesy of Brookhaven National Laboratory*

# QCD Phase Diagram

QGP:
quarks and gluons are deconfined.

Hadronic phase:
quarks are confined and massive.



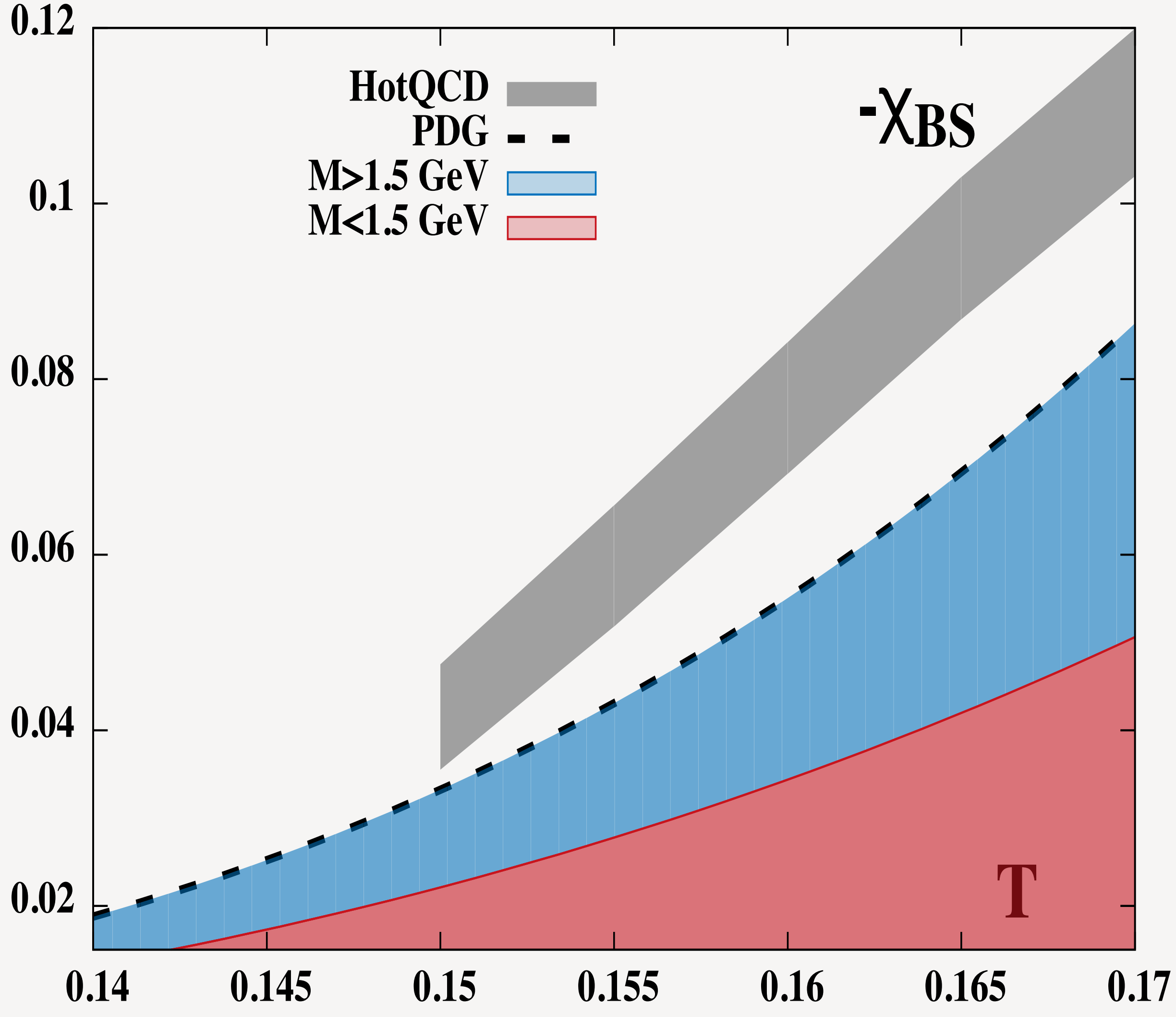
# TOWARDS REAL HADRON GAS

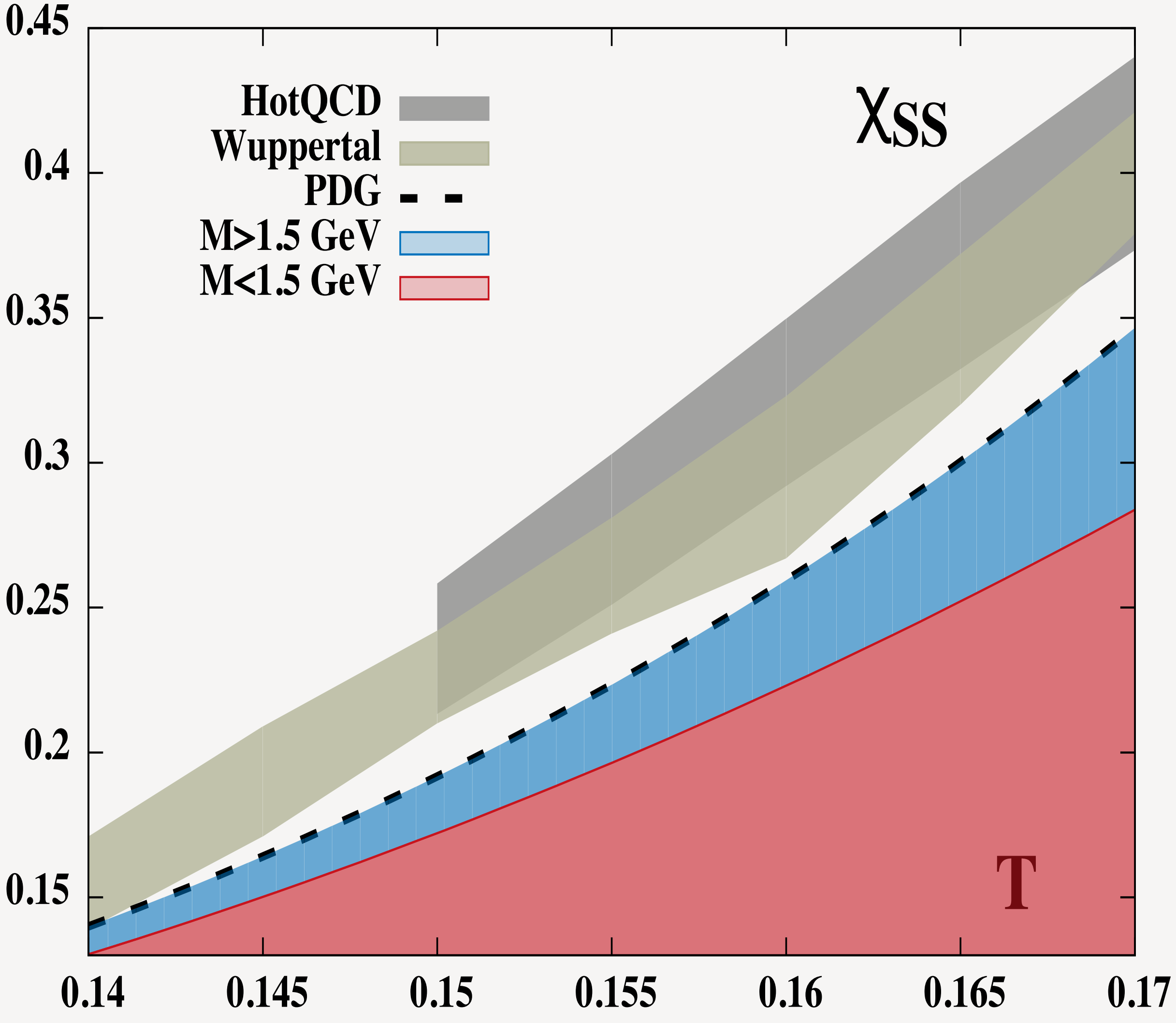
- Hadron contents in individual sectors

  -> the case of missing strange baryons

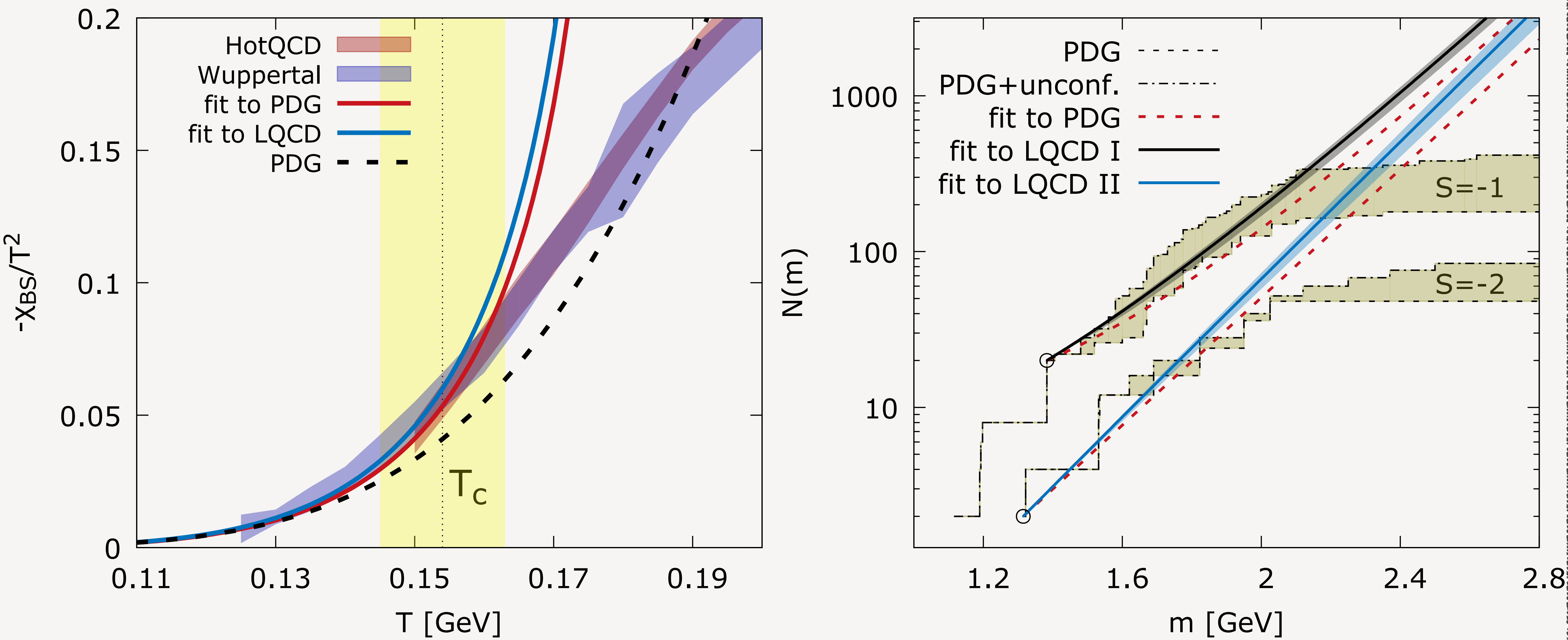
- Question the assumption of HRG treatment for resonances: non-interacting and point-like.

# Missing resonances in the strange sector



# strange mesons to be discovered...




PML, M. Marczenko, K. Redlich and C. Sasaki
Phys.Rev. C92 (2015) no.5, 055206

# THERMODYNAMICS OF BROAD RESONANCES

- unconfirmed light resonances in the strange sector

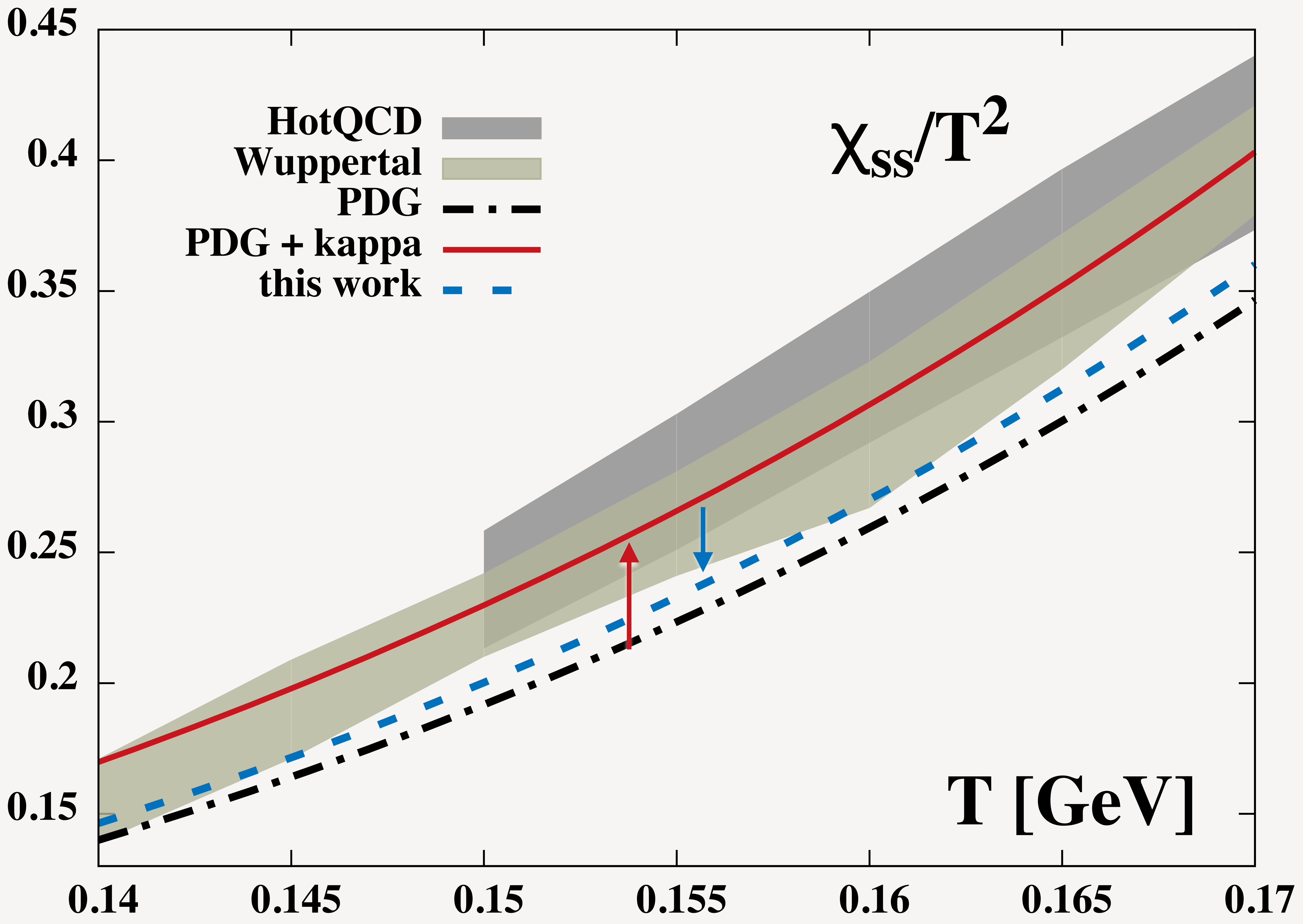
$K_0^*(800)$

or $\kappa$

$I(J^P) = \frac{1}{2}(0^+)$

OMITTED FROM SUMMARY TABLE

Needs confirmation. See the mini-review on scalar mesons under $f_0(500)$ (see the index for the page number).



HotQCD
Wuppertal
PDG
PDG + kappa
this work
$\chi_{ss}/T^2$
T [GeV]
0.45
0.4
0.35
0.3
0.25
0.2
0.15
0.14
0.145
0.15
0.155
0.16
0.165
0.17

# THERMODYNAMICS OF BROAD RESONANCES

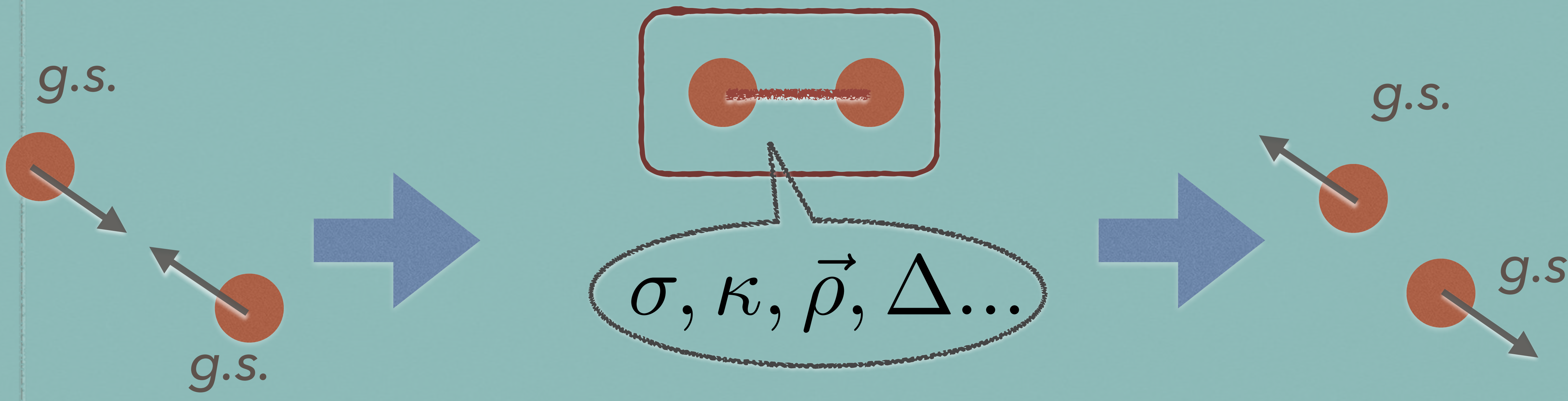
- The $\kappa$ meson has the right mass range.
- But it also has a broad width!

WHAT IS THE EFFECT OF RESONANCE'S WIDTH ON THERMODYNAMICS?

# S-MATRIX APPROACH

R. Dashen, S. K. Ma and H. J. Bernstein,
Phys. Rev. 187 (1969) 345.

# S-MATRIX APPROACH

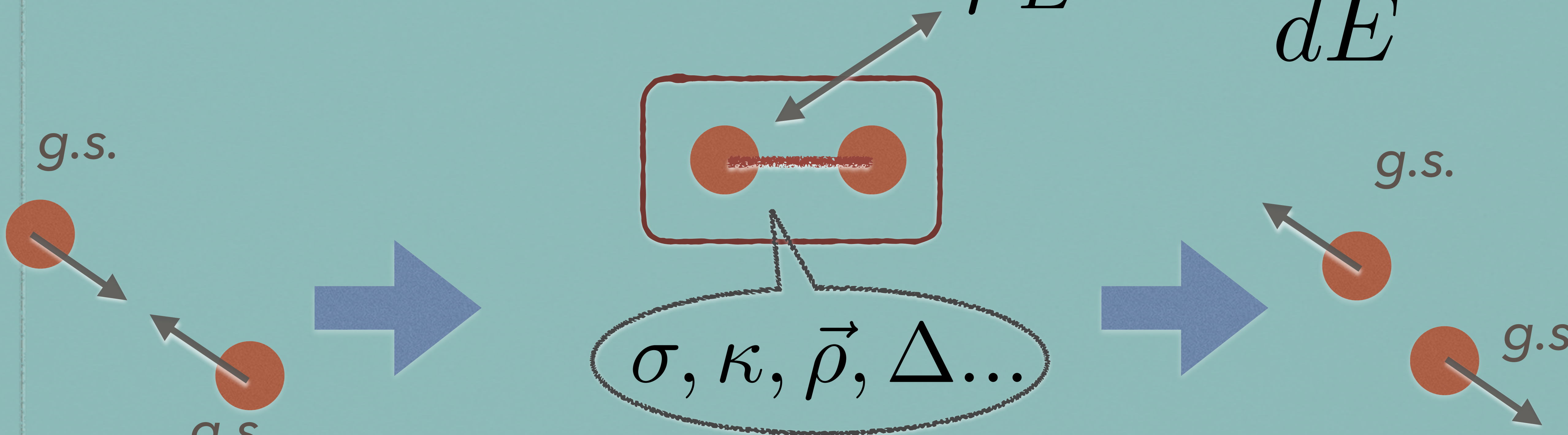



+ *repulsive forces*

consistent treatment of both attractive and repulsive forces

# S-MATRIX APPROACH

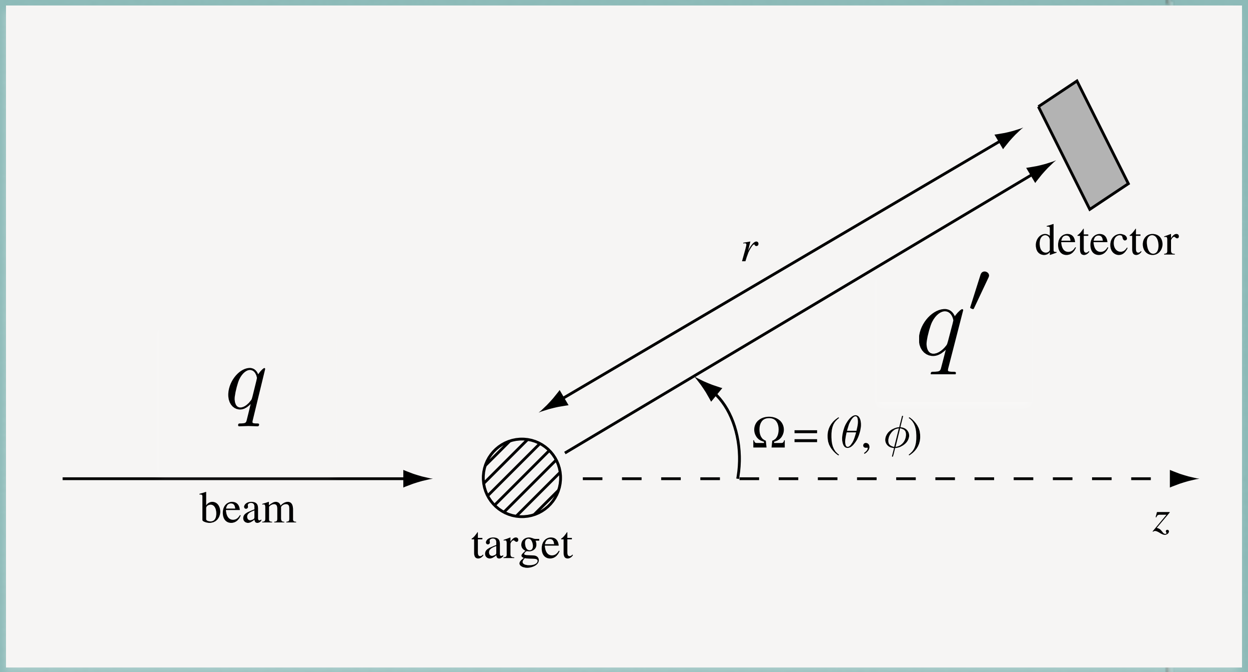
$$\rho_E \sim 2\frac{d\delta}{dE}$$




+ *repulsive forces*

consistent treatment of both attractive and repulsive forces

# FORMULATION




- Starting point:
  hard-core potential in QM

$$V = \infty \qquad r < a$$
$$\;\; = 0 \qquad r > a$$

*Momentum $q$ enters through the scattering Schroedinger equation with a centrifugal term ( $l$ -dependence )*

$$\tan(\delta_l) = \frac{j_l(qa)}{n_l(qa)}$$

$$\psi^q(r \to \infty) \longrightarrow e^{iqr\cos(\theta)} + \frac{e^{iqr}}{r} \sum_l (2l+1)\, P_l\, \frac{e^{i\delta_l}}{q}\, \sin(\delta_l)$$

# FORMULATION

$$\tan(\delta_l) = \frac{j_l(qa)}{n_l(qa)}$$

*for small* $x = qa$ *(near threshold)*

$$\tan(\delta_l) \to \frac{-x^{2l+1}}{(2l+1)((2l-1)!!)^2}$$

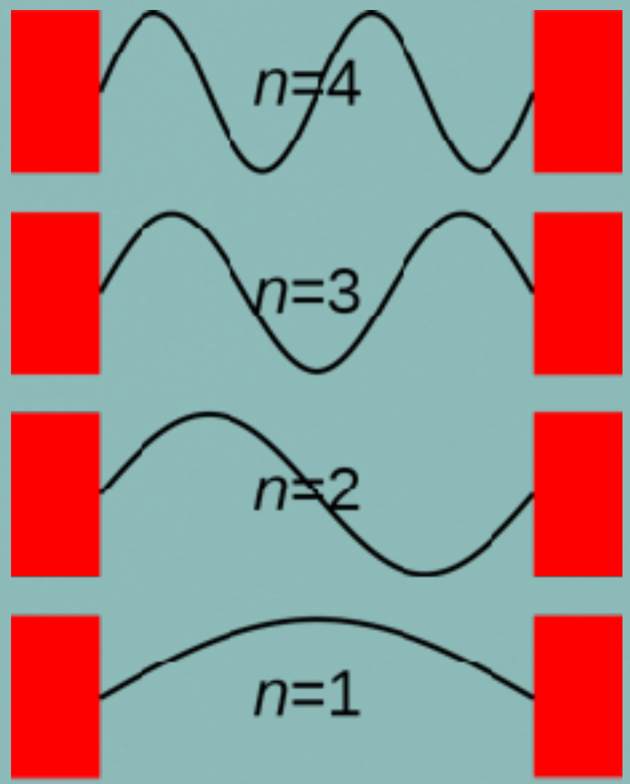
$$\delta_l \propto (q\,a)^{2l+1}$$

*(near threshold)*

# PHASE SHIFT AND DENSITY OF STATES

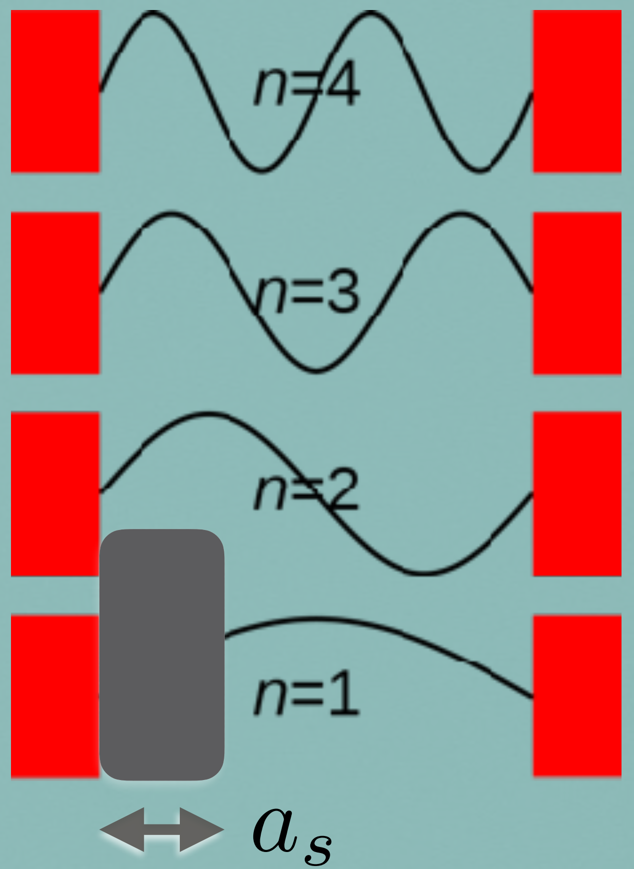
*particle in a box*

$$\psi \sim \sin(k^{(0)}x) \qquad k^{(0)} = \frac{n\pi}{L}$$



# PHASE SHIFT AND DENSITY OF STATES

*particle in a box*




$$\psi \sim \sin(k^{(0)}x) \qquad k^{(0)} = \frac{n\pi}{L}$$

*in the presence of a scattering potential*

$$\psi \sim \sin(kx + \delta(k))$$

density of states

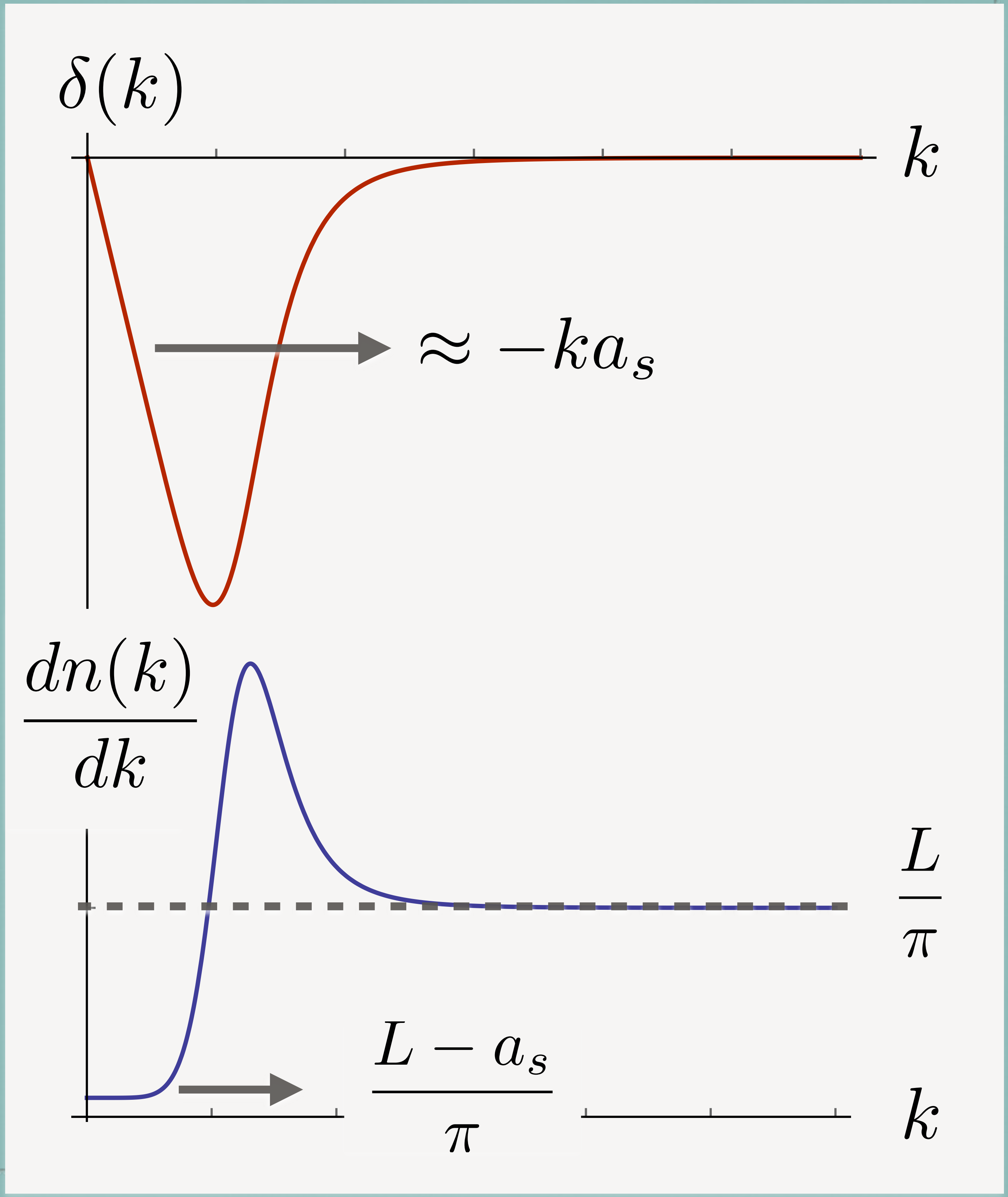
$$kL + \delta(k) = n\pi \quad \Rightarrow \quad \frac{dn(k)}{dk} = \frac{L}{\pi} + \frac{1}{\pi}\frac{d\delta}{dk}$$

# PHASE SHIFT AND DENSITY OF STATES

$$\frac{dn(k)}{dk} = \frac{L}{\pi} + \frac{1}{\pi}\frac{d\delta}{dk}$$

*change in d.o.s. due to int.*

Effect of repulsive interaction: pushing states from low k to high k




*phase shift and d.o.s. (schematics)*

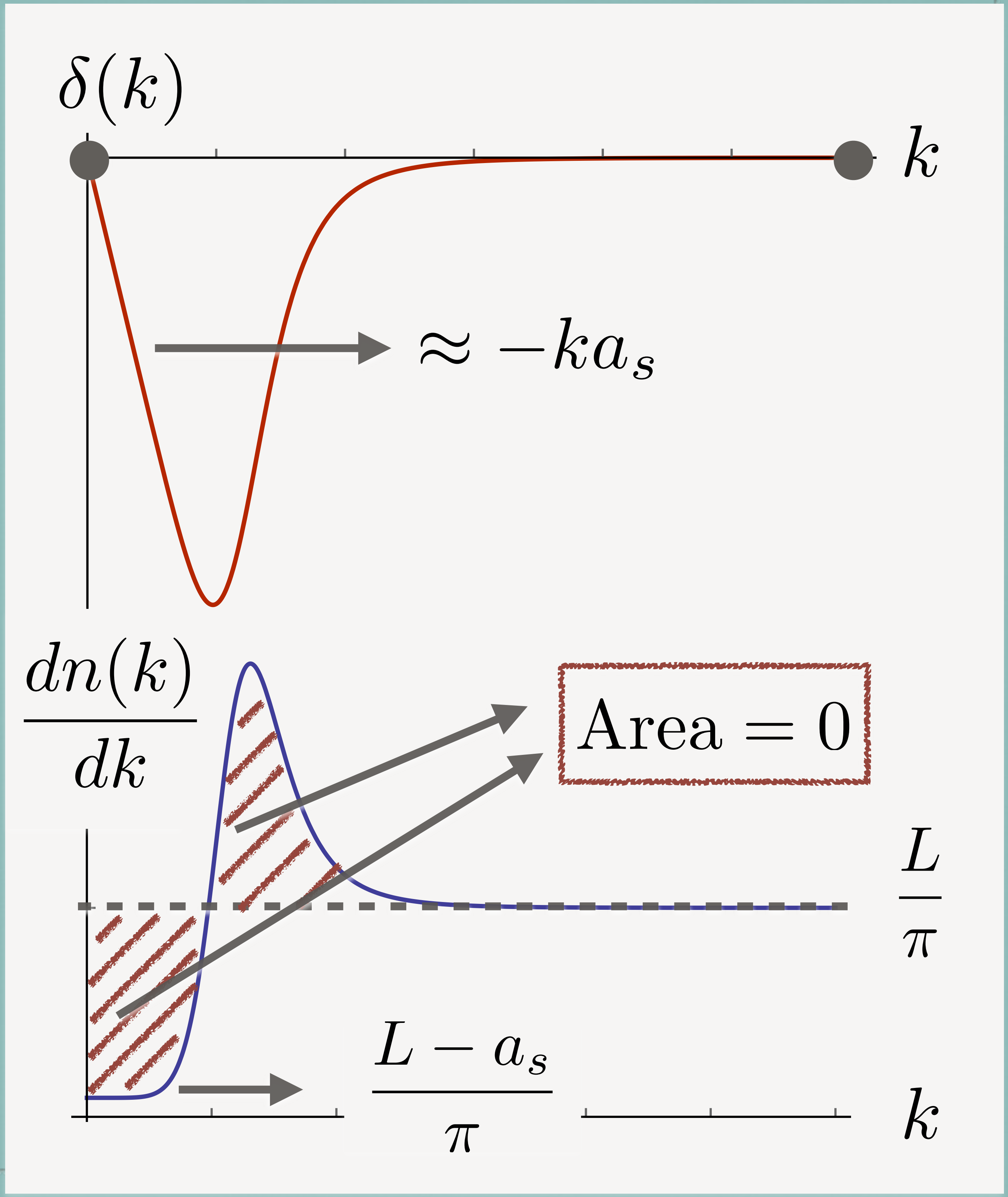
# PHASE SHIFT AND DENSITY OF STATES

$$\frac{dn(k)}{dk} = \frac{L}{\pi} + \frac{1}{\pi}\frac{d\delta}{dk}$$

sum rule
(Levinson's theorem)

$$\int_0^\infty dk\,\frac{1}{\pi}\delta' = \frac{\delta(\infty) - \delta(0)}{\pi}$$

$\to n_{\rm int}$




*phase shift and d.o.s. (schematics)*

# S-MATRIX FORMULATION OF THERMODYNAMICS

$$\Delta \ln Z = \int dE\, e^{-\beta E} \frac{1}{4\pi i} \operatorname{tr} \left\{ S_E^{-1} \overleftrightarrow{\frac{\partial}{\partial E}} S_E \right\}_c$$

R. Dashen, S. K. Ma and H. J. Bernstein,
Phys. Rev. 187 (1969) 345.

# A SIMPLE TRICK

$$\frac{1}{4\pi i}\,\mathrm{tr}\left\{S_E^{-1}\frac{\overleftrightarrow{\partial}}{\partial E}S_E\right\}_c$$

$$= \frac{1}{2\pi}\times 2\,\frac{\partial}{\partial E}\,\frac{1}{2}\,\mathrm{Im}\,\mathrm{tr}\,\{\ln S_E\}$$

$$S_E = e^{2i\delta_E}$$

$$\Delta \ln Z = \int dE\, e^{-\beta E}\times\frac{1}{\pi}\frac{\partial}{\partial E}\,\mathrm{tr}\,(\delta_E)\,.$$

E. Beth and G. Uhlenbeck,
Physica (Amsterdam) 4, 915 (1937).

# A SIMPLE TRICK

$$\frac{1}{4\pi i}\,\mathrm{tr}\left\{S_E^{-1}\frac{\overleftrightarrow{\partial}}{\partial E}S_E\right\}_c$$

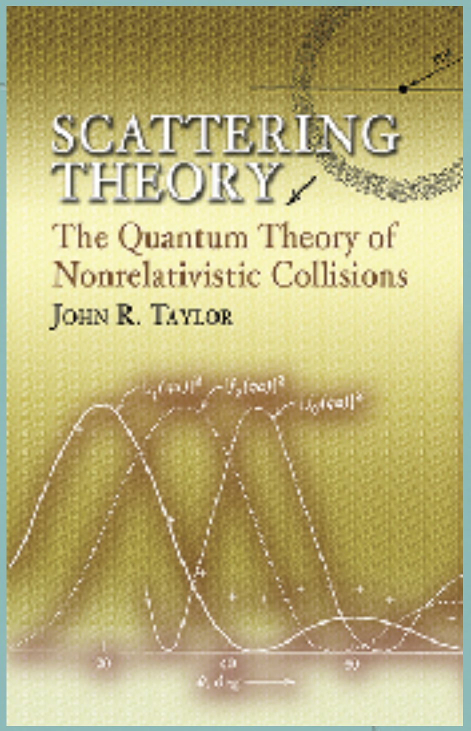
$$S_E = e^{2i\delta_E}$$

$$= \frac{1}{2\pi} \times 2\,\frac{\partial}{\partial E}\,\frac{1}{2}\,\mathrm{Im}\,\mathrm{tr}\,\{\ln S_E\}$$

$\mathcal{Q}(E)$ → *Generalised phase shift*

$B = 2\frac{\partial}{\partial E}\mathcal{Q}(E)$ → *Generalised spectral function*

# EXERCISE: QM SCATTERING OPERATOR




*show that*

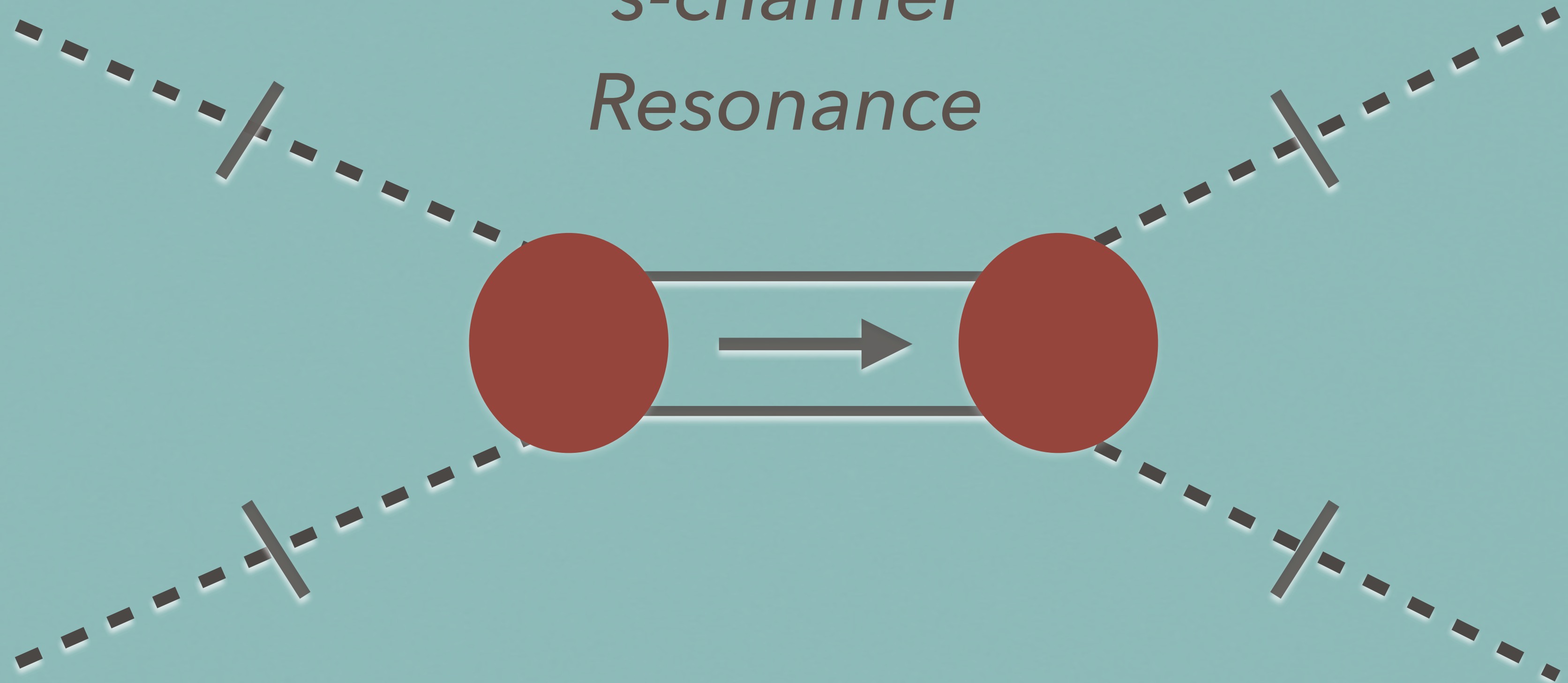
$$S_E = G_0^* \, G^{*\,-1} \, G \, G_0^{-1}$$
$$= 1 - 2\pi i \times \delta(E - H_0) \times T_E$$

$$G = \frac{1}{E - H + i\epsilon}$$

*Verify* $$\Delta \ln Z = \int dE \, e^{-\beta E} \frac{1}{4\pi i} \operatorname{tr} \left\{ S_E^{-1} \frac{\overleftrightarrow{\partial}}{\partial E} S_E \right\}_c$$

*Alternative way to obtain the Beth-Uhlenbeck result!*

# ILLUSTRATION: S-MATRIX FOR RELATIVISTIC RESONANCES




$$i\mathcal{M}_E \approx (-ig)\,\frac{i}{E^2 - m_{\rm res}^2 + iE\,\gamma_E}\,(-ig)$$

$$\mathcal{Q}(E) = \frac{1}{2}\,\mathrm{Im}\,\mathrm{tr}\,\{\ln S_E\}$$
$$= \frac{1}{2}\,\mathrm{Im}\;\ln\,[1 + \int d\phi_2\, i\mathcal{M}_E]$$

$$\int d\phi_2 \, i\mathcal{M}_E = \frac{-i\,2\,E\,\gamma_E}{E^2 - m_{\rm res}^2 + iE\,\gamma_E}$$

$$= 2\,i\,\sin\delta_E\, e^{i\delta_E}$$

*then…*

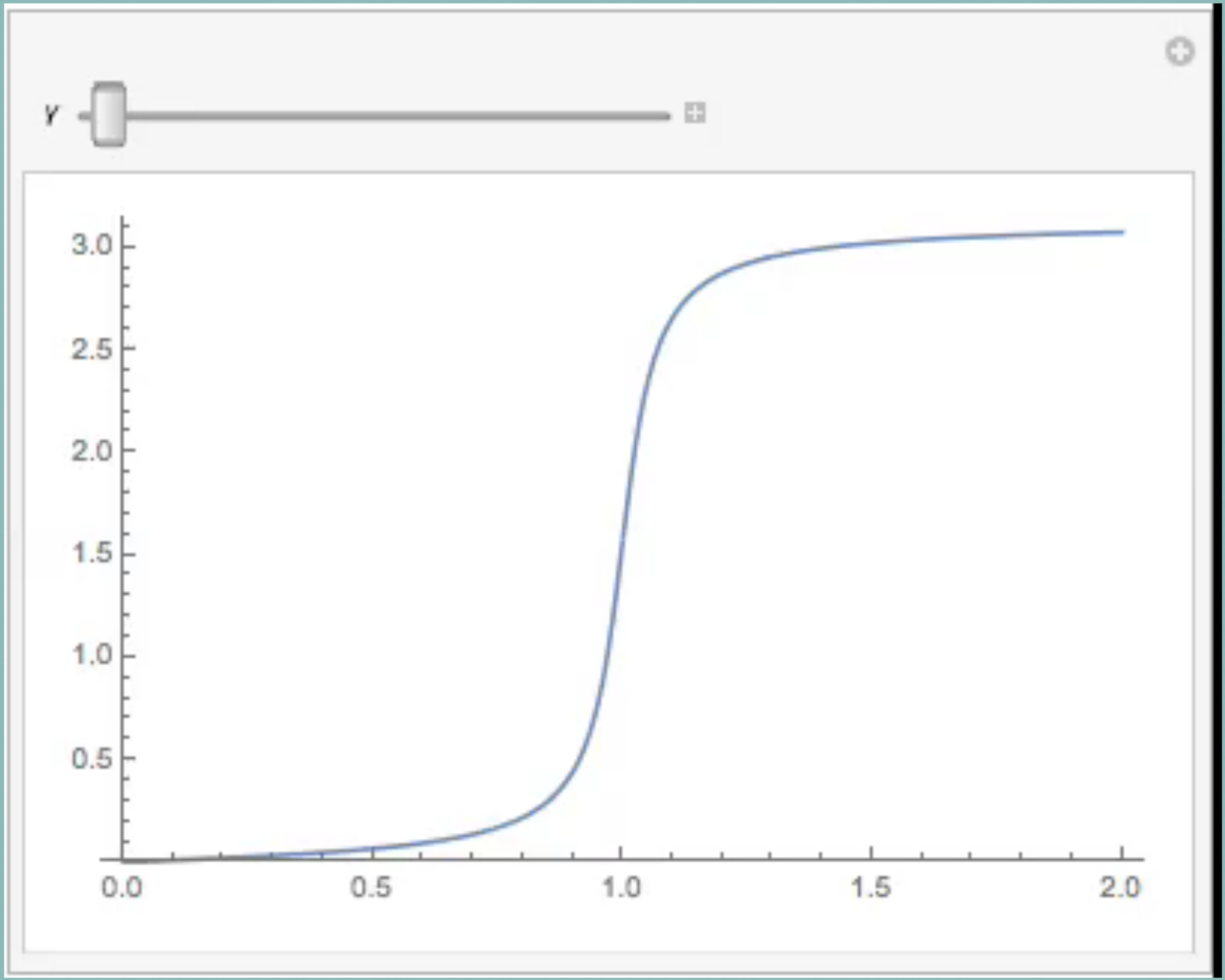
$$\mathcal{Q}(M) = \frac{1}{2}\,\mathrm{Im}\,\left[\ln\left(1 + 2\,i\sin\delta_E e^{i\delta_E}\right)\right]$$

$$= \frac{1}{2}\,\mathrm{Im}\,\ln e^{2i\delta_E}$$

$$= \delta_E$$

*with* $$\delta_E = \tan^{-1}\frac{-E\gamma_E}{E^2 - m_{\rm res}^2}$$

$$\delta_E = \tan^{-1} \frac{-E\gamma_E}{E^2 - m_{\rm res}^2}$$




**HRG approx.**

$$\delta_E = \pi \times \theta(E - m_{\rm res})$$

# FORMULATION

*given the exact phase shift* $\delta_l$

from theory
or
from experiment

thermodynamics

$$B_l = 2\frac{d}{dq}\delta_l$$

eff. spectral function

$$P = P^{(0)} + \Delta P^{B.U.}$$

**free gas + interaction**

# FORMULATION

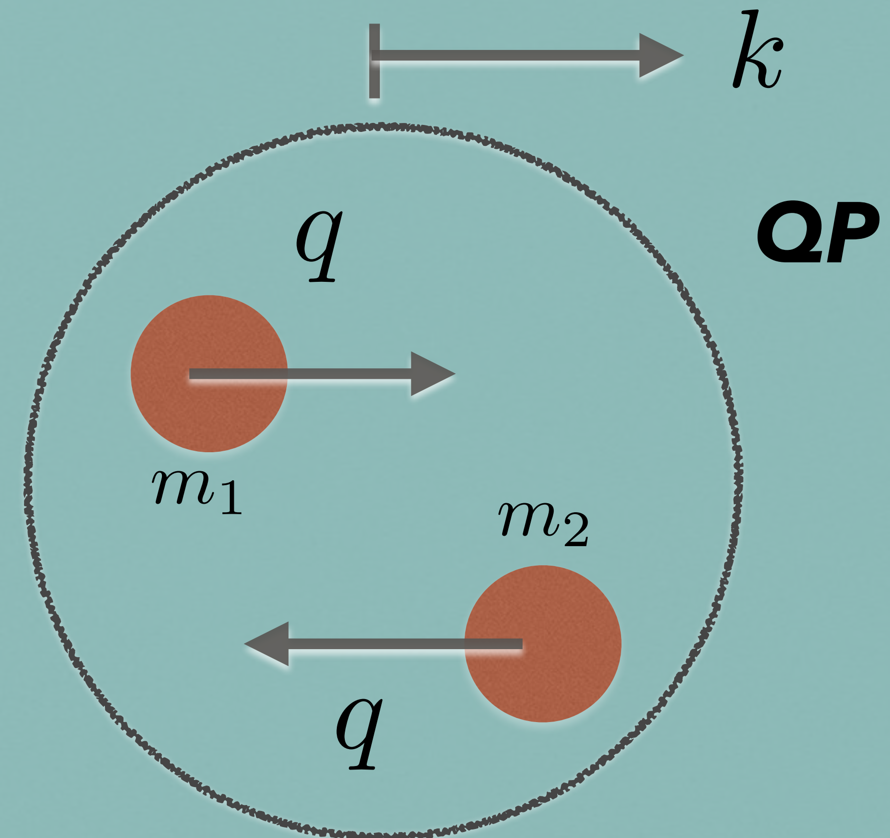
***dynamical*** ***statistical (thermal weight)***

$$\Delta P^{\mathrm{B.U.}} = (2l+1)\int \frac{dq}{2\pi} B_l(q) \int \frac{d^3k}{(2\pi)^3} T \ln(1+e^{-\beta E(k,q,m_i)})$$

$$E = \sqrt{k^2 + M(q)^2}$$

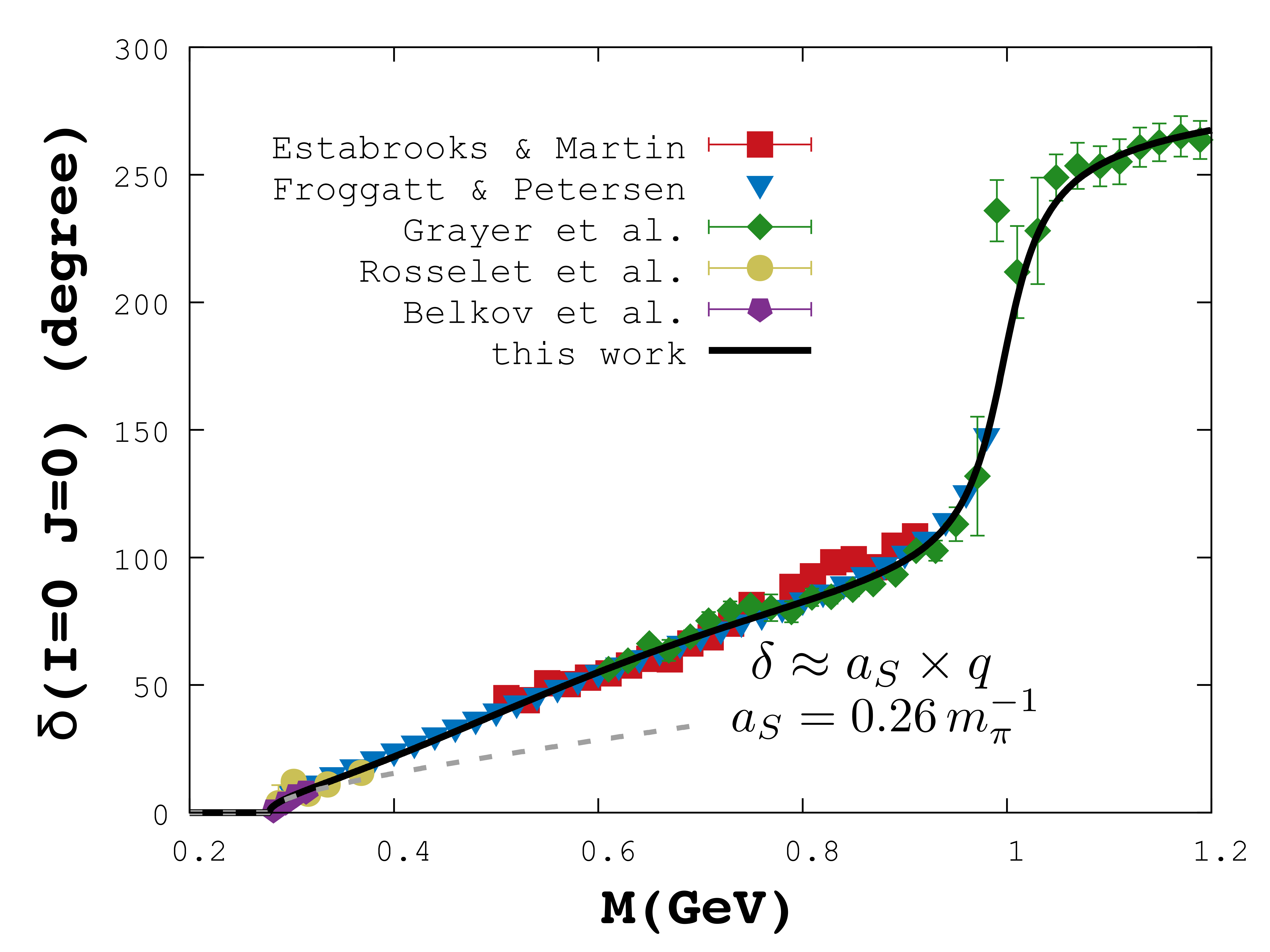
$$M(q) = \sqrt{q^2+m_1^2} + \sqrt{q^2+m_2^2}$$

$$B_l = 2\frac{d}{dq}\delta_l$$

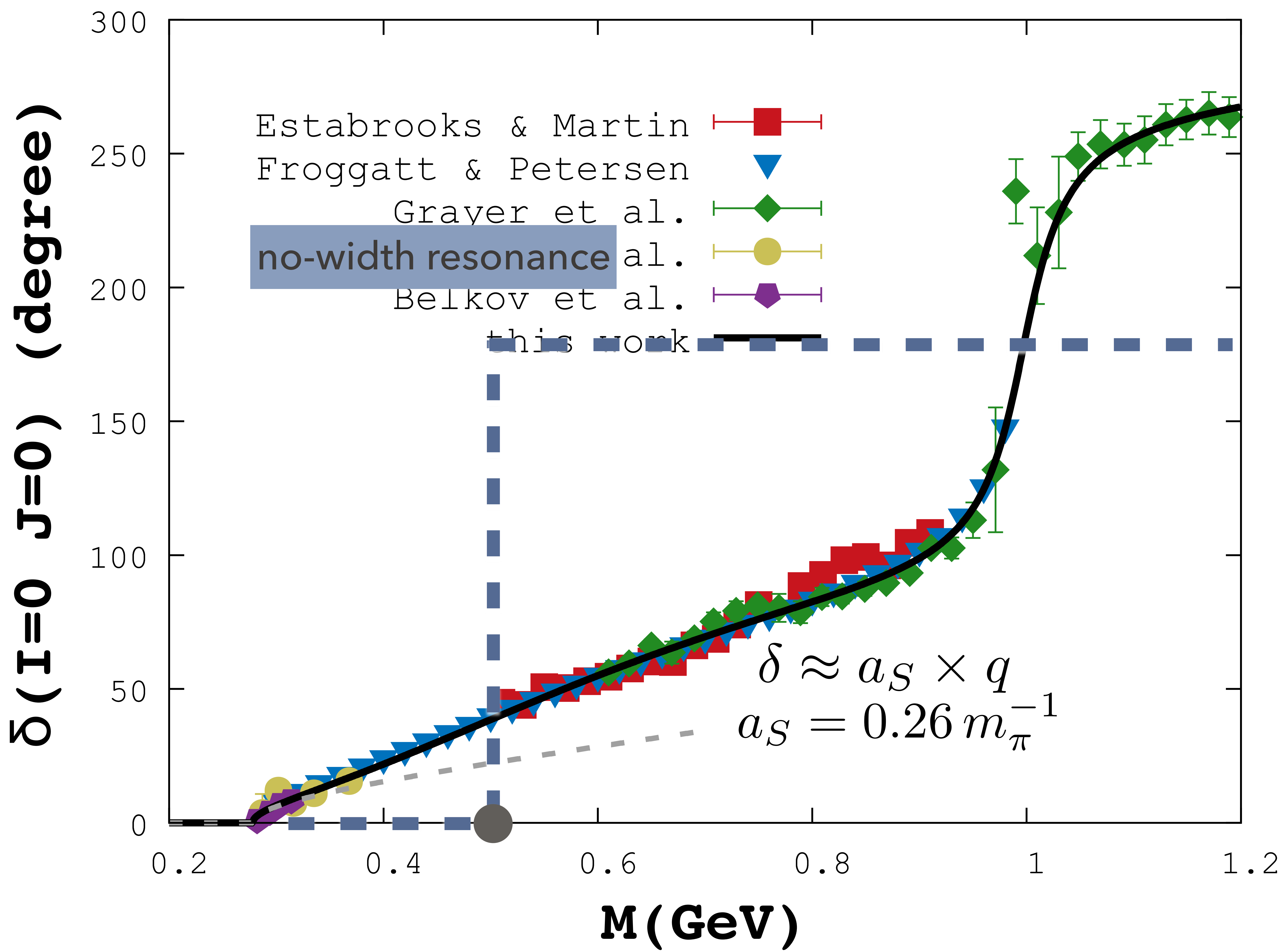



$$M(q) = \sqrt{q^2+m_1^2} + \sqrt{q^2+m_2^2}$$

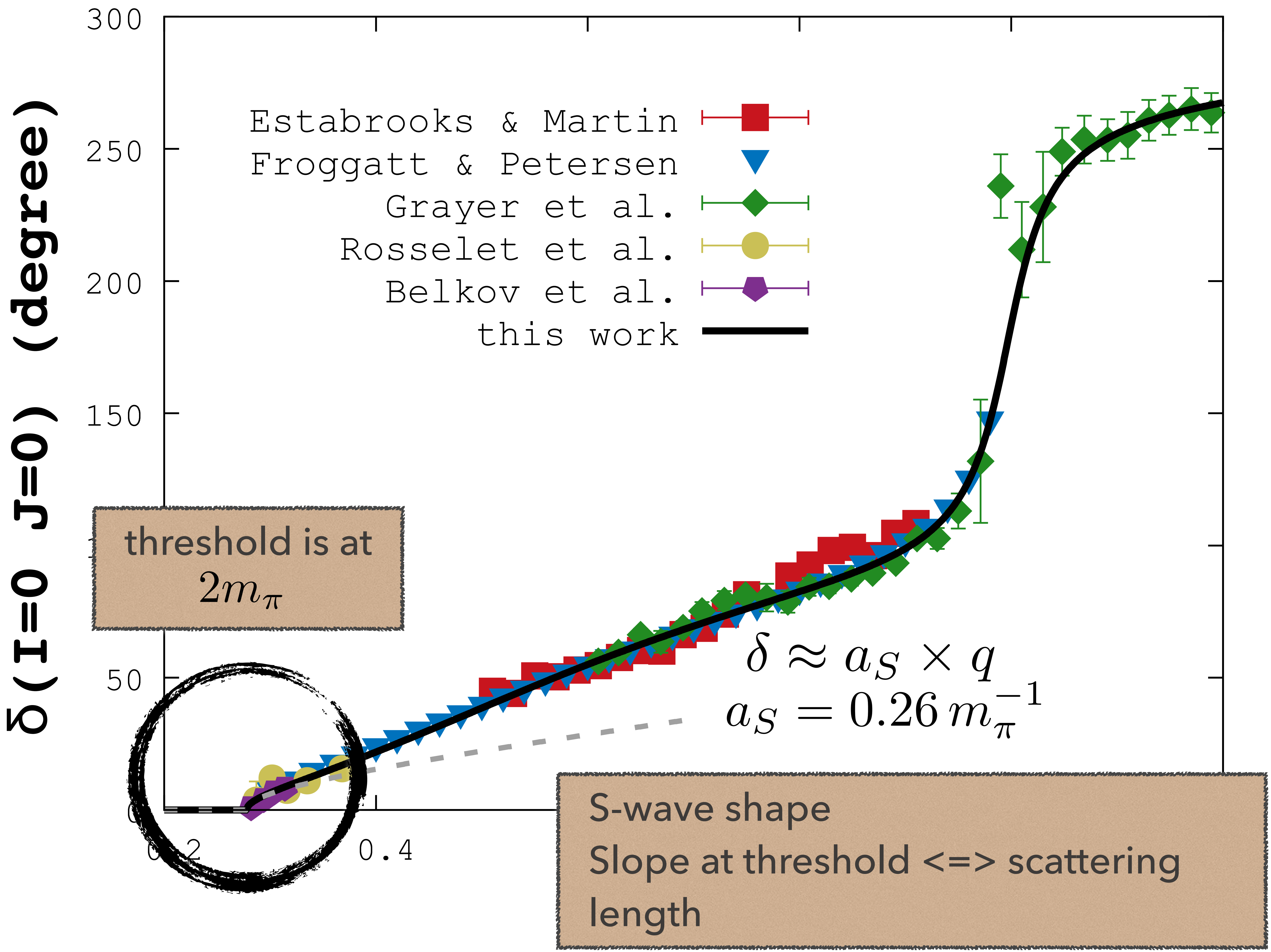
# PI PI SCATTERING (S-WAVE)



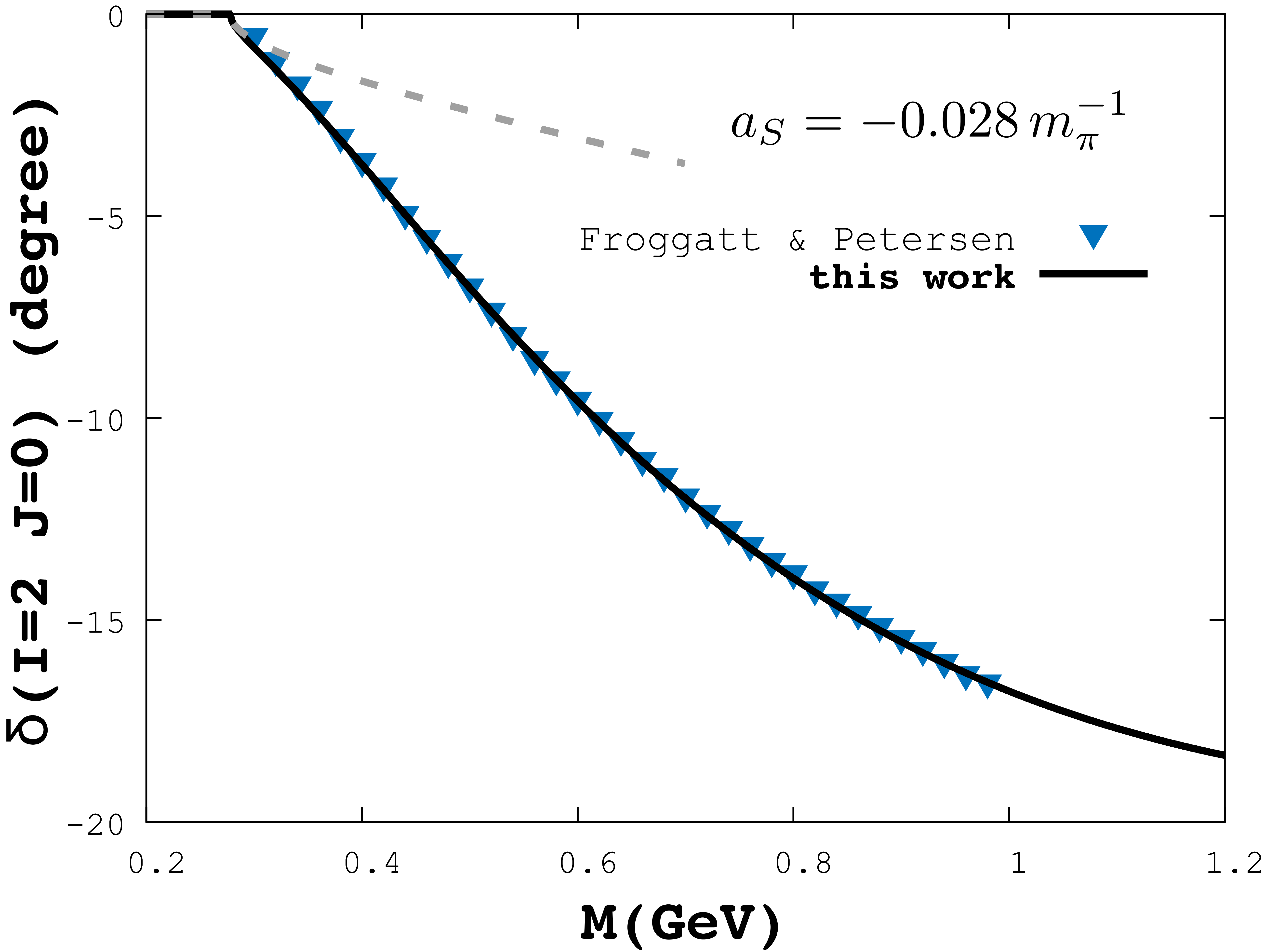
Estabrooks & Martin
Froggatt & Petersen
Grayer et al.
Rosselet et al.
Belkov et al.
this work
$\delta \approx a_S \times q$
$a_S = 0.26\, m_\pi^{-1}$
δ(I=0 J=0) (degree)
M(GeV)
0
50
100
150
200
250
300
0.2
0.4
0.6
0.8
1
1.2



300
250
200
150
100
50
0
0.2
0.4
0.6
0.8
1
1.2
M(GeV)
δ(I=0 J=0) (degree)
Estabrooks & Martin
Froggatt & Petersen
Grayer et al.
no-width resonance
al.
Belkov et al.
this work
$\delta \approx a_S \times q$
$a_S = 0.26\, m_\pi^{-1}$



δ(I=0 J=0) (degree)
300
250
200
150
50
0
0.2
0.4
Estabrooks & Martin
Froggatt & Petersen
Grayer et al.
Rosselet et al.
Belkov et al.
this work
threshold is at
$2m_\pi$
$\delta \approx a_S \times q$
$a_S = 0.26\, m_\pi^{-1}$
S-wave shape
Slope at threshold <=> scattering length



0
-5
-10
-15
-20
0.2
0.4
0.6
0.8
1
1.2
δ(I=2 J=0) (degree)
M(GeV)
$a_S = -0.028\, m_\pi^{-1}$
Froggatt & Petersen
this work

# CHIRAL SYMMETRY

# CHIRAL SYMMETRY

- Chiral partners

$$\sigma \leftrightarrow \pi \qquad \kappa \leftrightarrow K$$

- NJL model offers a good description for low mass spectrum

$$m_\sigma \approx 2M_q$$

$$m_\pi^2 \approx -\frac{m_q}{f_\pi^2}\langle \bar{q}q \rangle$$

# CHIRAL SYMMETRY

- but fails to explain the threshold.

$\sigma \to \bar{q}q$ *instead of* $\sigma \to \pi\pi$

lack of confinement

?? to be cured by pion / other loop corrections ??

P-wave

scattering length constrained by PCAC

*Weinberg*

# CHIRAL SYMMETRY

- Linear sigma model

$$U_{eff}(\sigma, \pi) = -\mu^2(\sigma^2 + \pi^2) + \lambda(\sigma^2 + \pi^2)^2$$

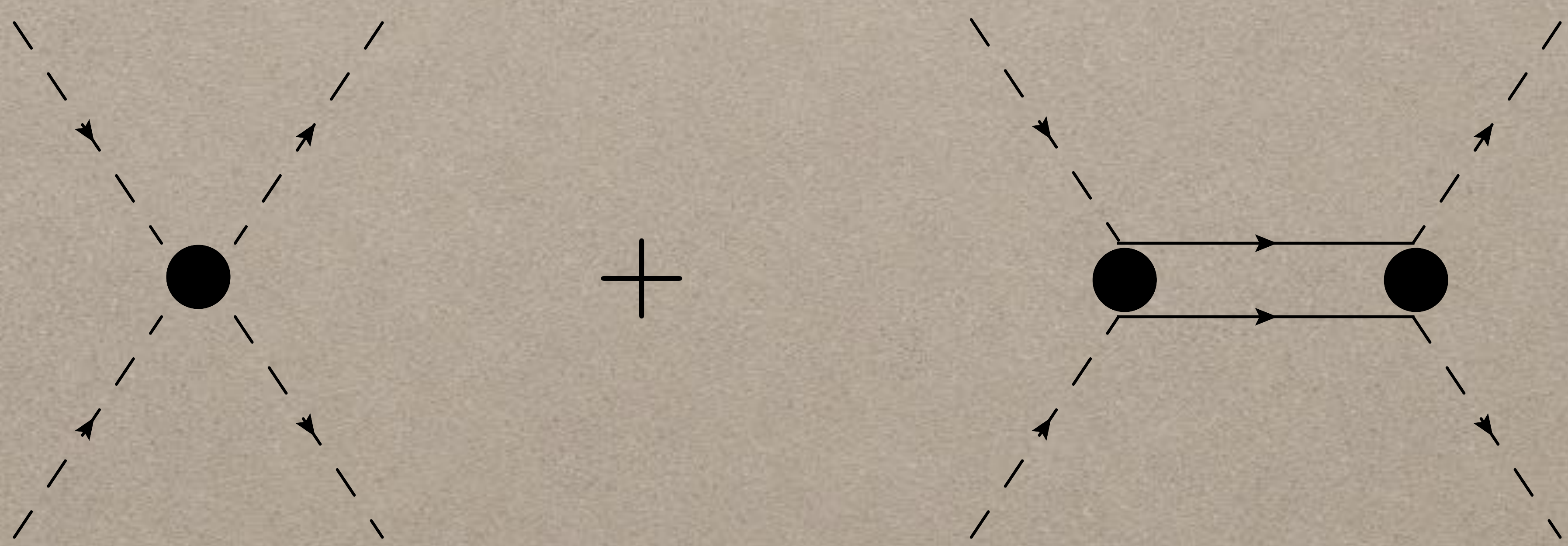
$$m_\pi = 0$$

$$m_\sigma \neq 0$$

# CHIRAL SYMMETRY

- Linear sigma model

$$U_{eff}(\sigma, \pi) = -\mu^2(\sigma^2 + \pi^2) + \lambda(\sigma^2 + \pi^2)^2$$



# WIDTH AND PHASE SHIFT

- Width => particle can decay =>
  existence of an imaginary part in the self energy

$$G(t) \propto e^{-i\Sigma_R t + \Sigma_I t}$$

$$|G(t)|^2 \propto e^{2\Sigma_I t} => e^{-\Gamma t}$$

*N.R.*

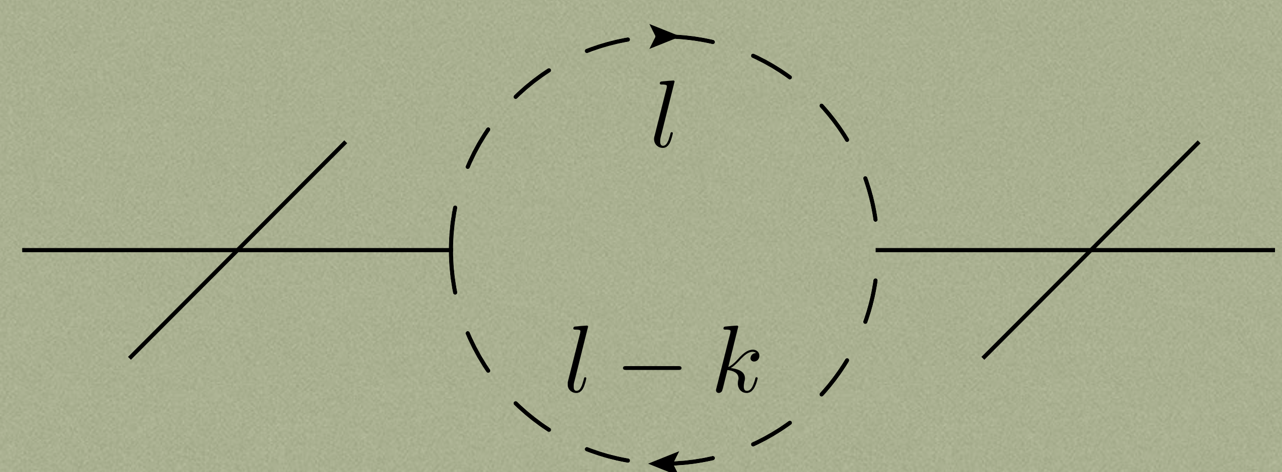
$$\Gamma = -2\Sigma_I$$

# WIDTH AND PHASE SHIFT

- Width comes from interactions.
- illustration:

$$\mathcal{L}_{int} = -g\sigma\phi_\pi^2$$

$$-i\Sigma_\sigma =$$



# WIDTH AND PHASE SHIFT

$$\Sigma_\sigma(k^2) = 2g^2\, i \times \int \frac{d^4 l}{(2\pi)^4} \frac{1}{l^2 - m_\pi^2 + i\epsilon} \frac{1}{(l-k)^2 - m_\pi^2 + i\epsilon}$$

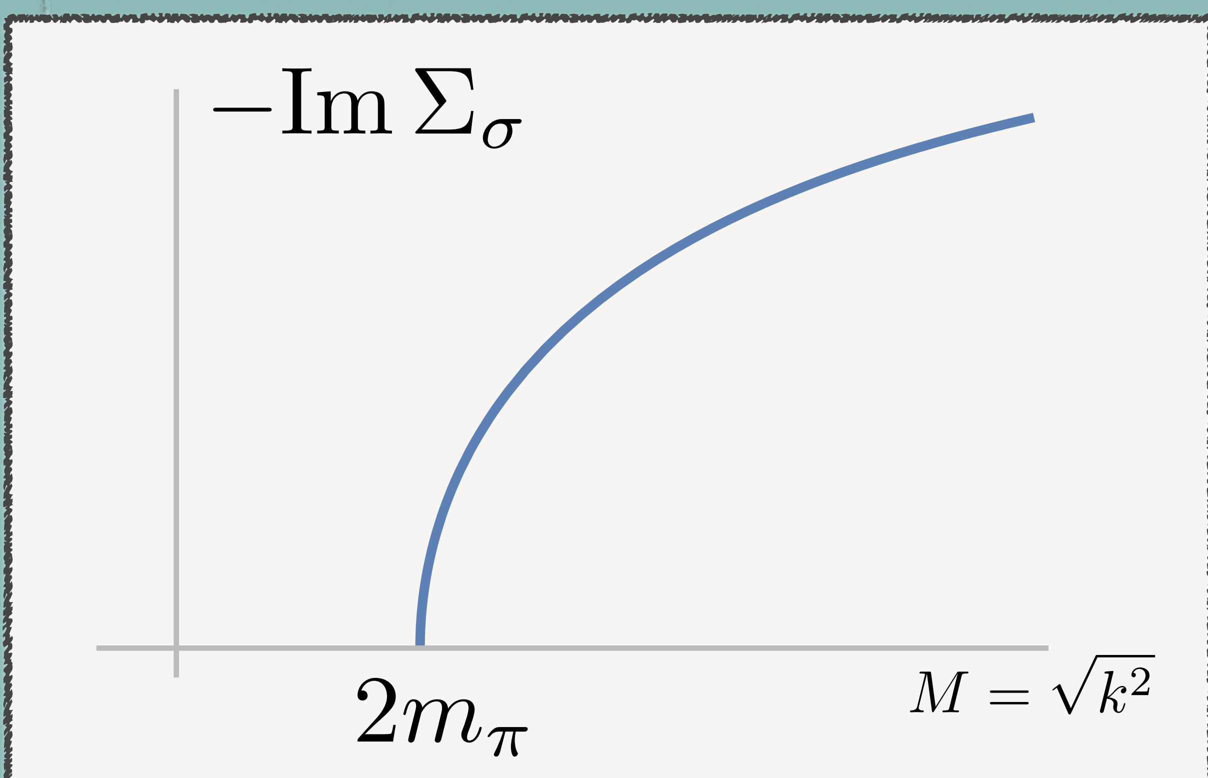
Dim. Reg. $$= -\frac{2g^2}{16\pi^2} \int_0^1 dx \left( \frac{2}{4-d} - \gamma_{\text{Euler}} + \ln(4\pi) - \ln \frac{\Delta(k^2)}{\mu^2} \right)$$

$$\Delta = m_\pi^2 - x(1-x)k^2 - i\epsilon$$

develops an imaginary part if

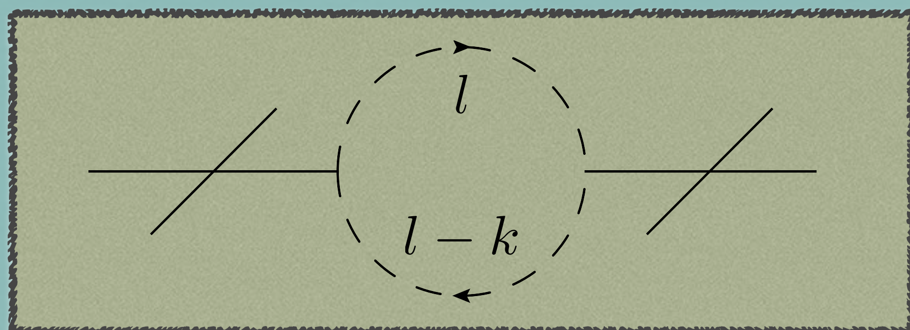
$k^2 \geq (2m_\pi)^2$ threshold

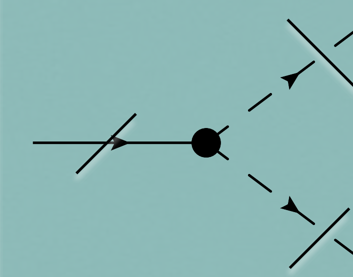
$$\ln(-1) = \pm i\pi$$




*Rel.* $$\Gamma = \frac{-\mathrm{Im}\,\Sigma_\sigma}{M}$$

# CUTKOSKY'S CUTTING RULES

**Im** 


= $\int d\phi \,\Big|$  $\Big|^2$

*Phase space approach*

$$-\mathrm{Im}\,\Sigma_\sigma = M\Gamma = \frac{1}{2}\int d\phi_2\,|\Gamma_{\sigma\to\pi\pi}|^2$$

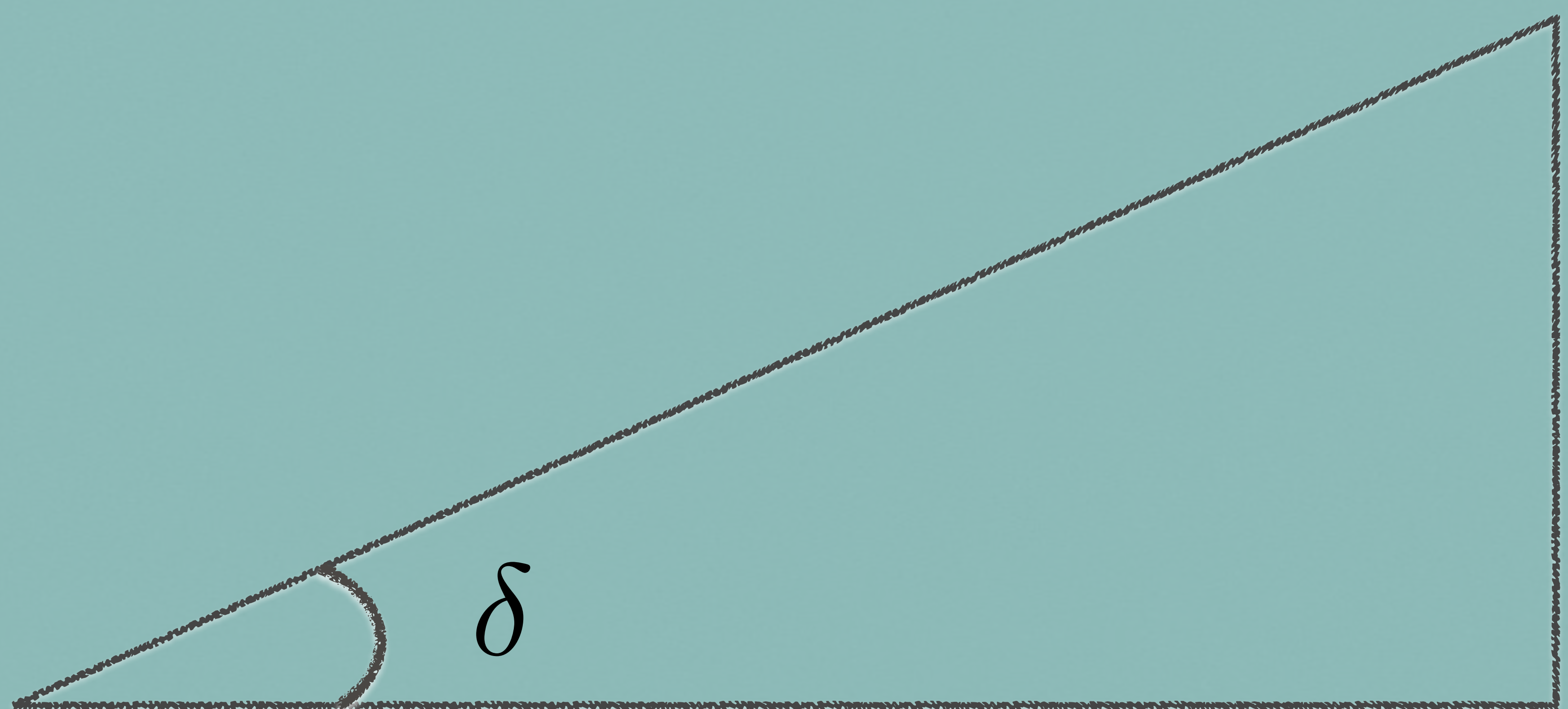
=> $\frac{1}{2} \times \frac{q}{4\pi M} \times g^2 \times 2$

# WIDTH AND PHASE SHIFT

- Field theory knows about the kinematics and phase space
- Width arises from interaction
- Angular momentum dependence $\propto k^{2l+1}$

# WIDTH AND PHASE SHIFT

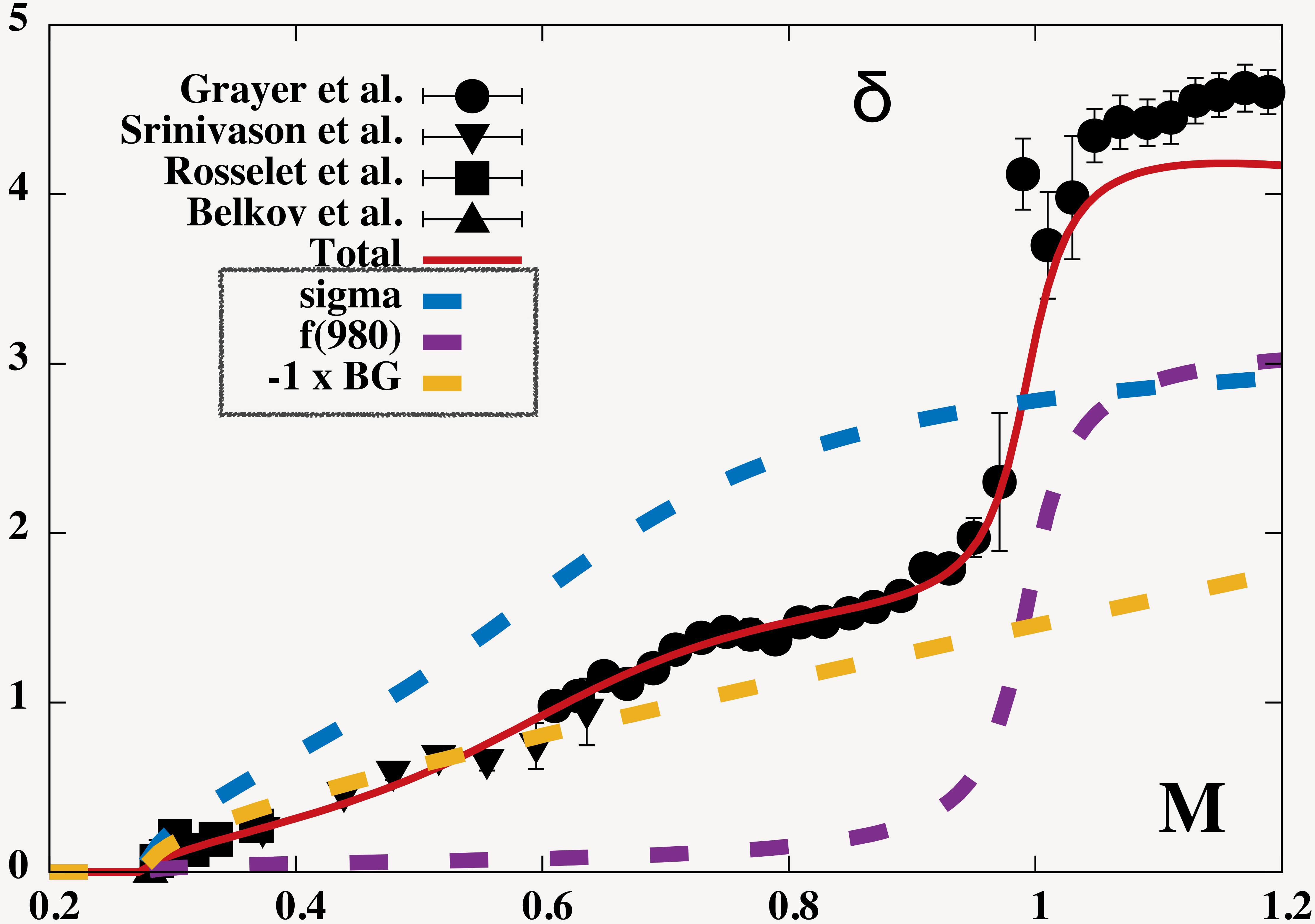
*Green's function (QFT)* ⟷ *Scattering theory*




$$Im(G(k)^{-1})$$

$$M \times \gamma_E$$

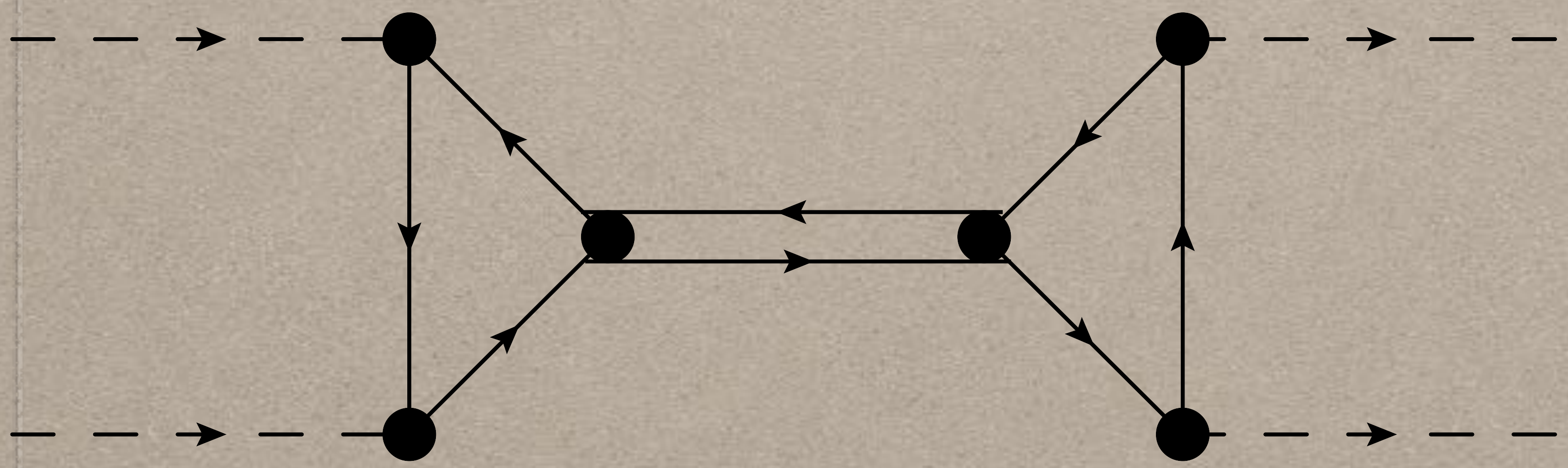
$$-Re(G(k)^{-1}) \longrightarrow -(M^2 - m_{\rm res}^2)$$



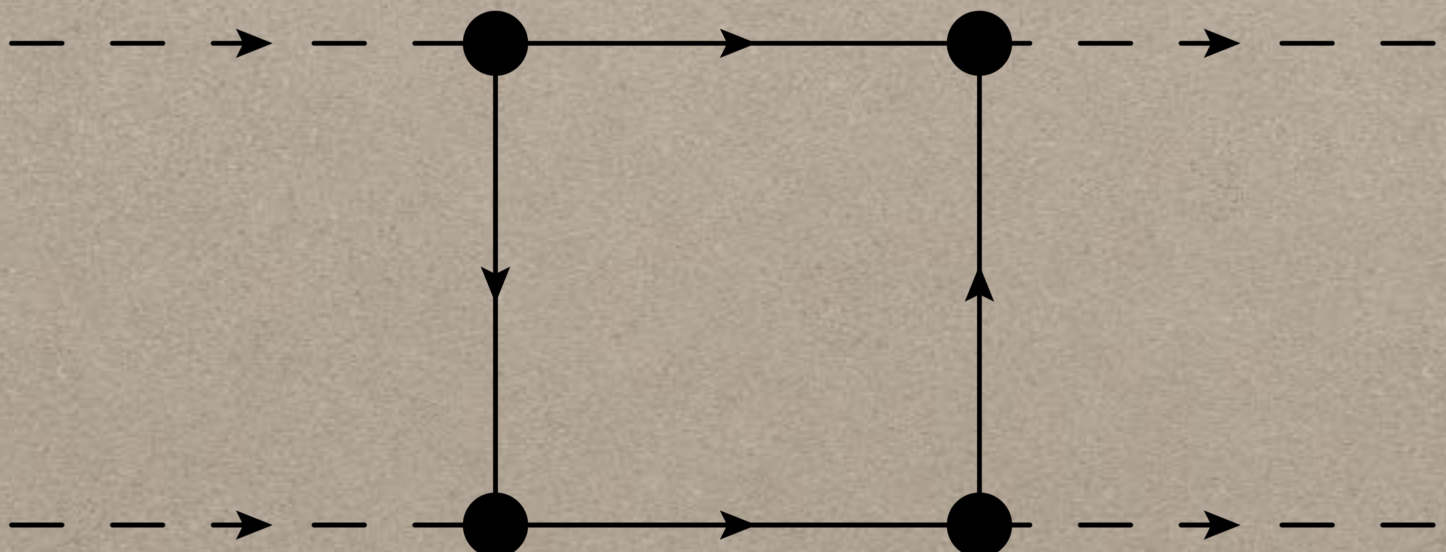
Grayer et al.
Srinivason et al.
Rosselet et al.
Belkov et al.
Total
sigma
f(980)
-1 x BG
δ
M
0
1
2
3
4
5
0.2
0.4
0.6
0.8
1
1.2

# CHIRAL SYMMETRY

- NJL description

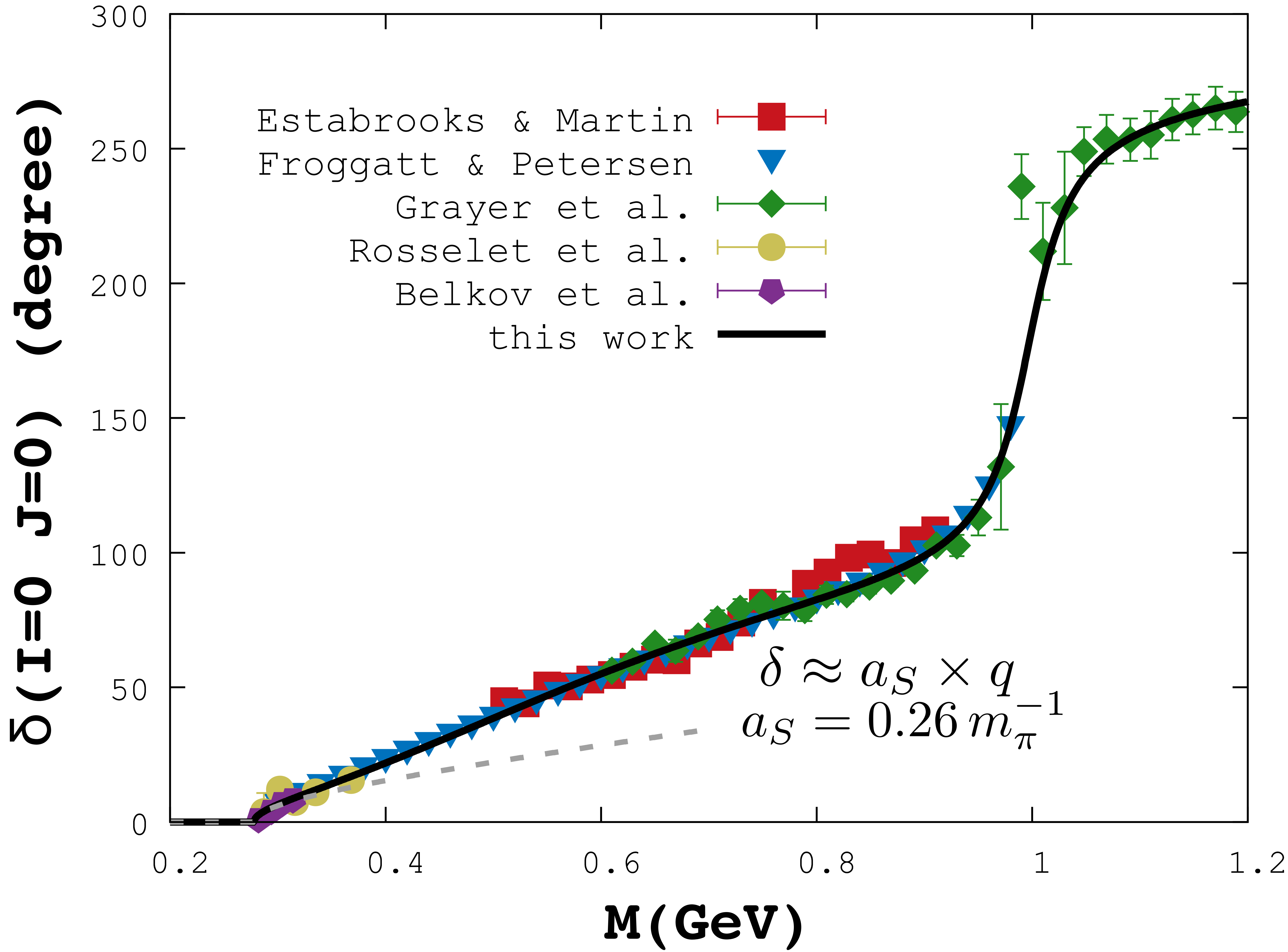


direct term

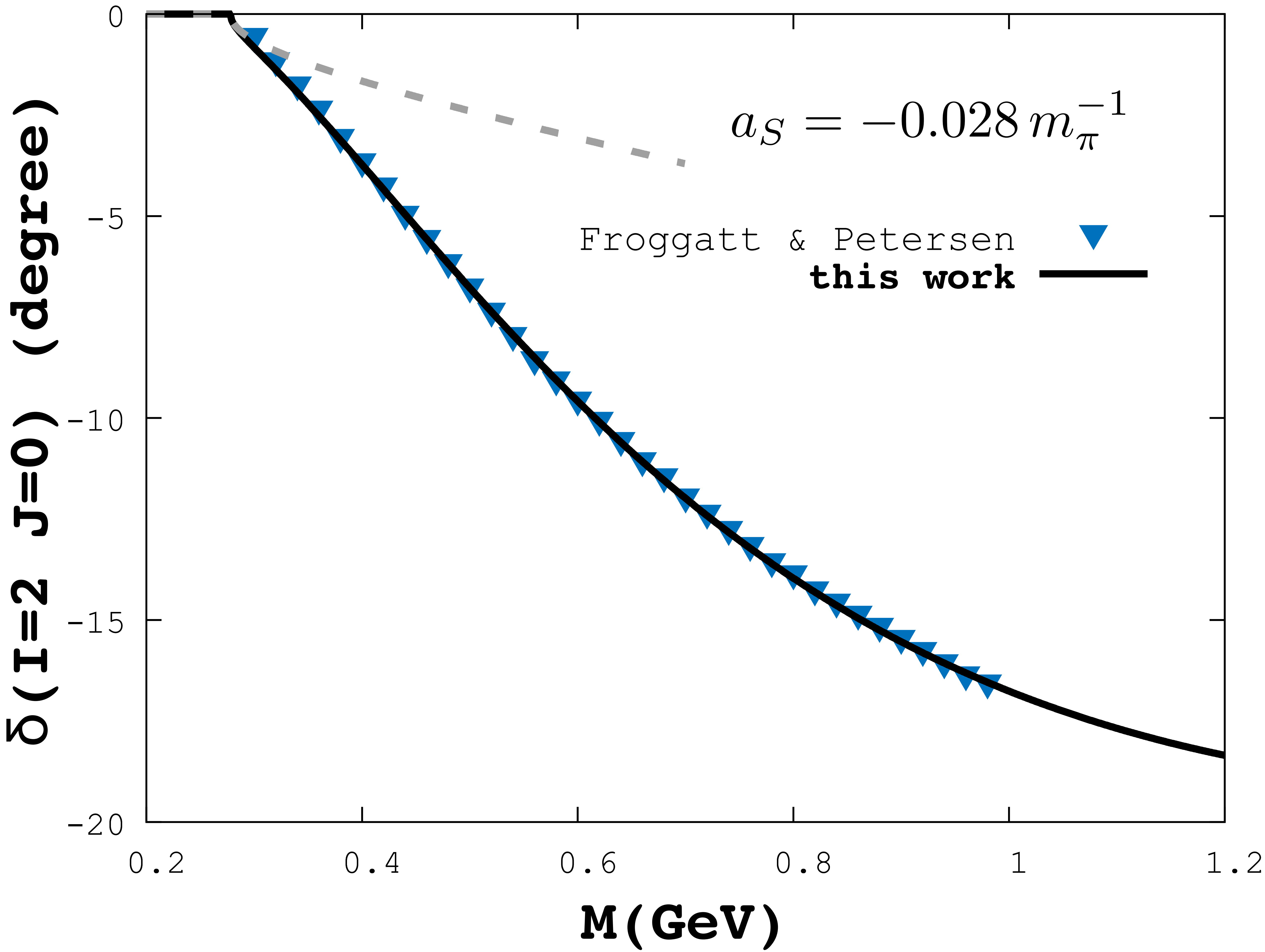


box diagram

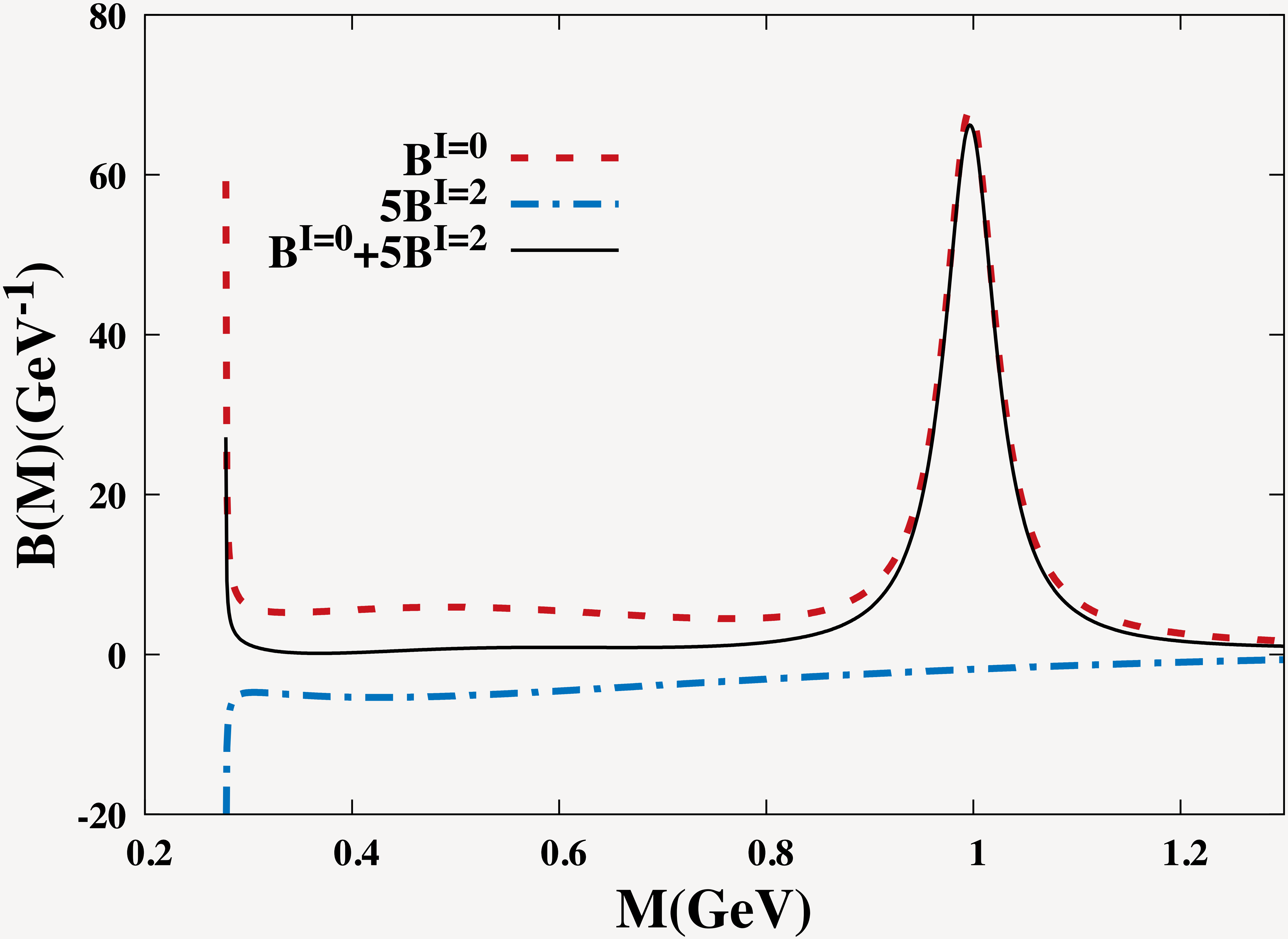
negative scattering
length -> B.G.?



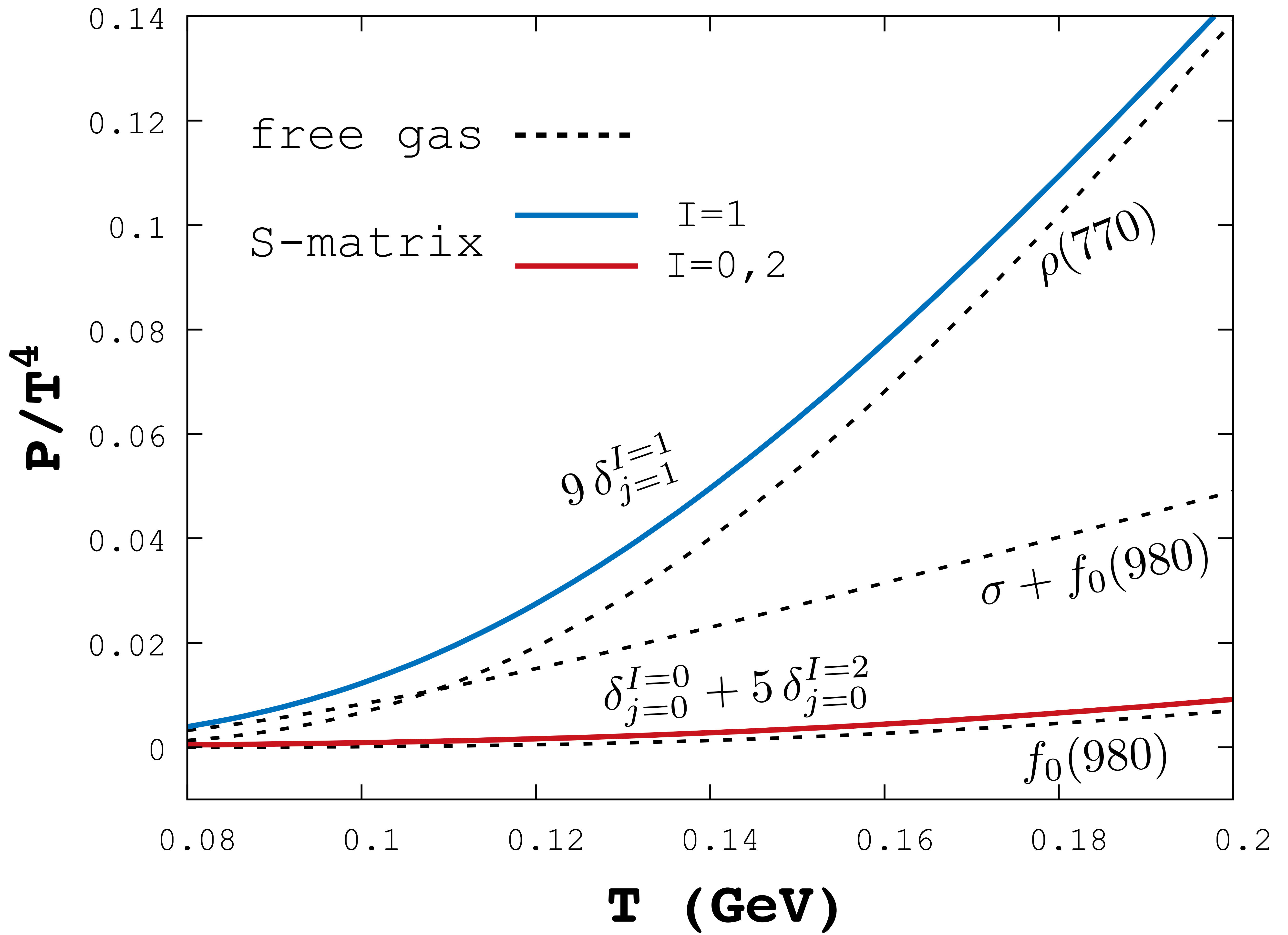
Estabrooks & Martin
Froggatt & Petersen
Grayer et al.
Rosselet et al.
Belkov et al.
this work
$\delta \approx a_S \times q$
$a_S = 0.26\, m_\pi^{-1}$
δ(I=0 J=0) (degree)
M(GeV)
0
50
100
150
200
250
300
0.2
0.4
0.6
0.8
1
1.2



$a_S = -0.028\, m_\pi^{-1}$
Froggatt & Petersen
this work
0
-5
-10
-15
-20
0.2
0.4
0.6
0.8
1
1.2
M(GeV)
δ(I=2 J=0) (degree)



80
60
40
20
0
-20
0.2
0.4
0.6
0.8
1
1.2
M(GeV)
B(M)(GeV-1)
BI=0
5BI=2
BI=0+5BI=2



P/T^4
T (GeV)
free gas
S-matrix
I=1
I=0,2
ρ(770)
$9\,\delta^{I=1}_{j=1}$
$\sigma + f_0(980)$
$\delta^{I=0}_{j=0} + 5\,\delta^{I=2}_{j=0}$
$f_0(980)$

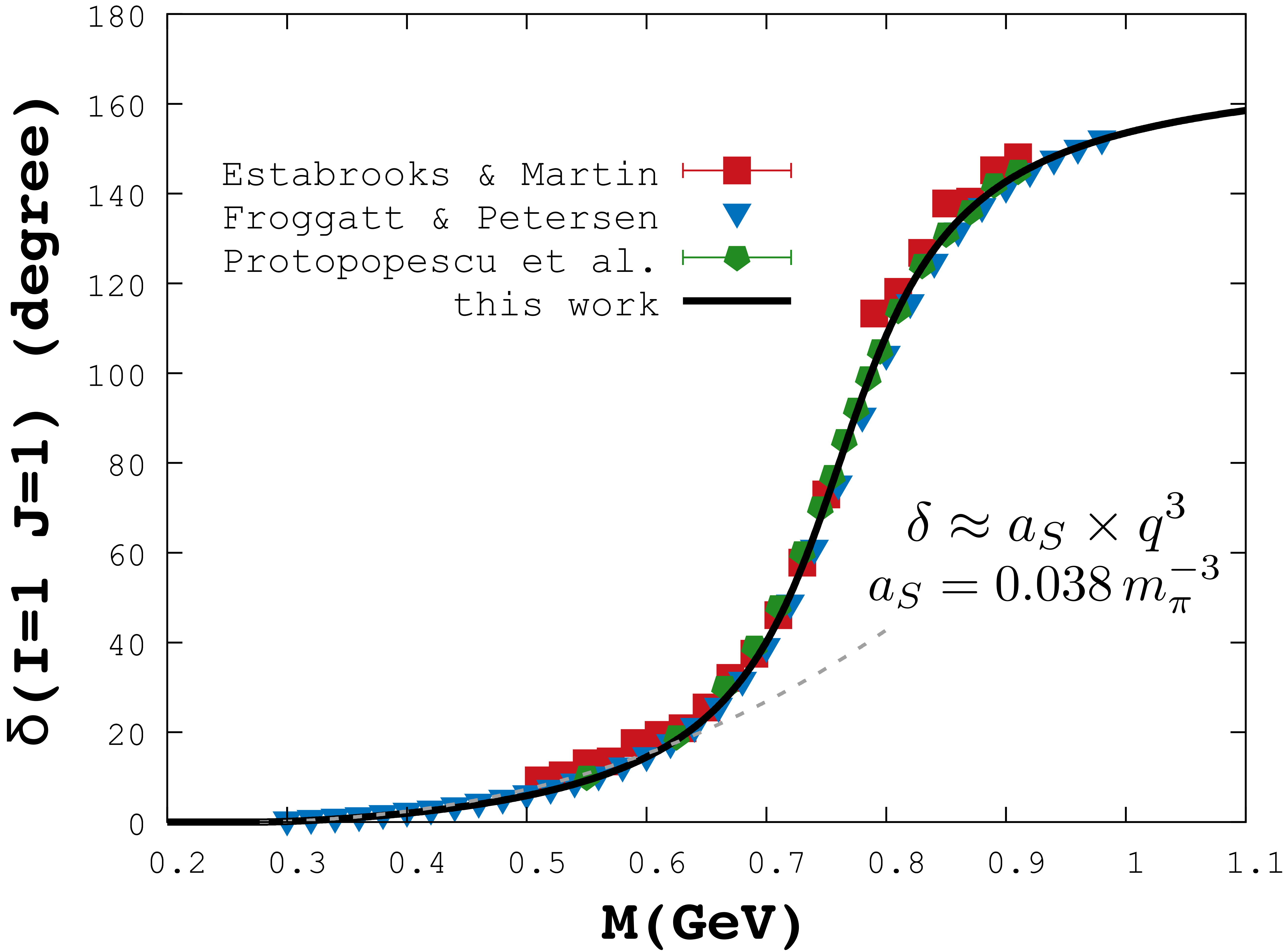
# WHAT'S IN A NAME? THAT WHICH WE CALL A RESONANCES?

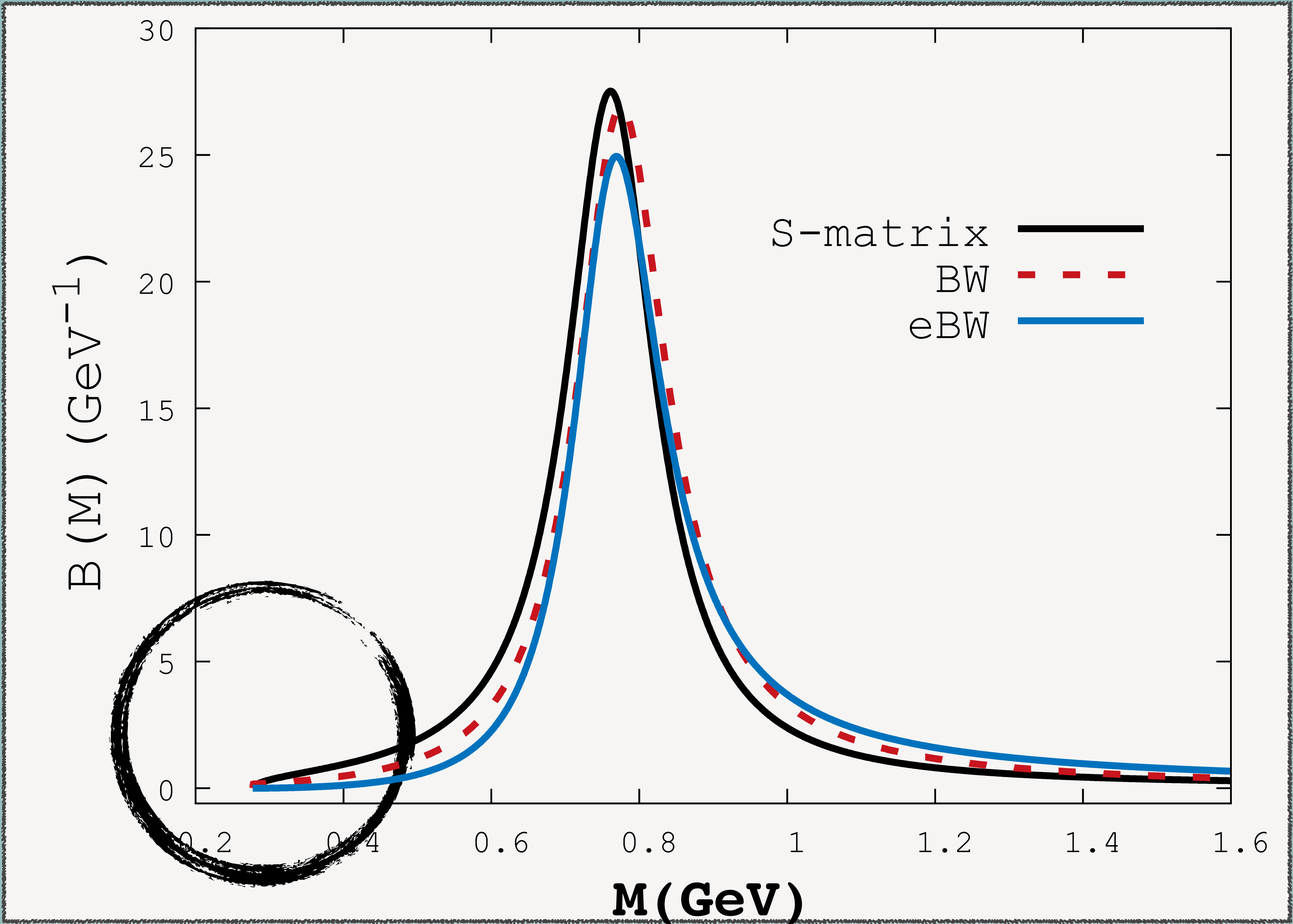
- A resonance is MORE than a MASS and a **WIDTH**

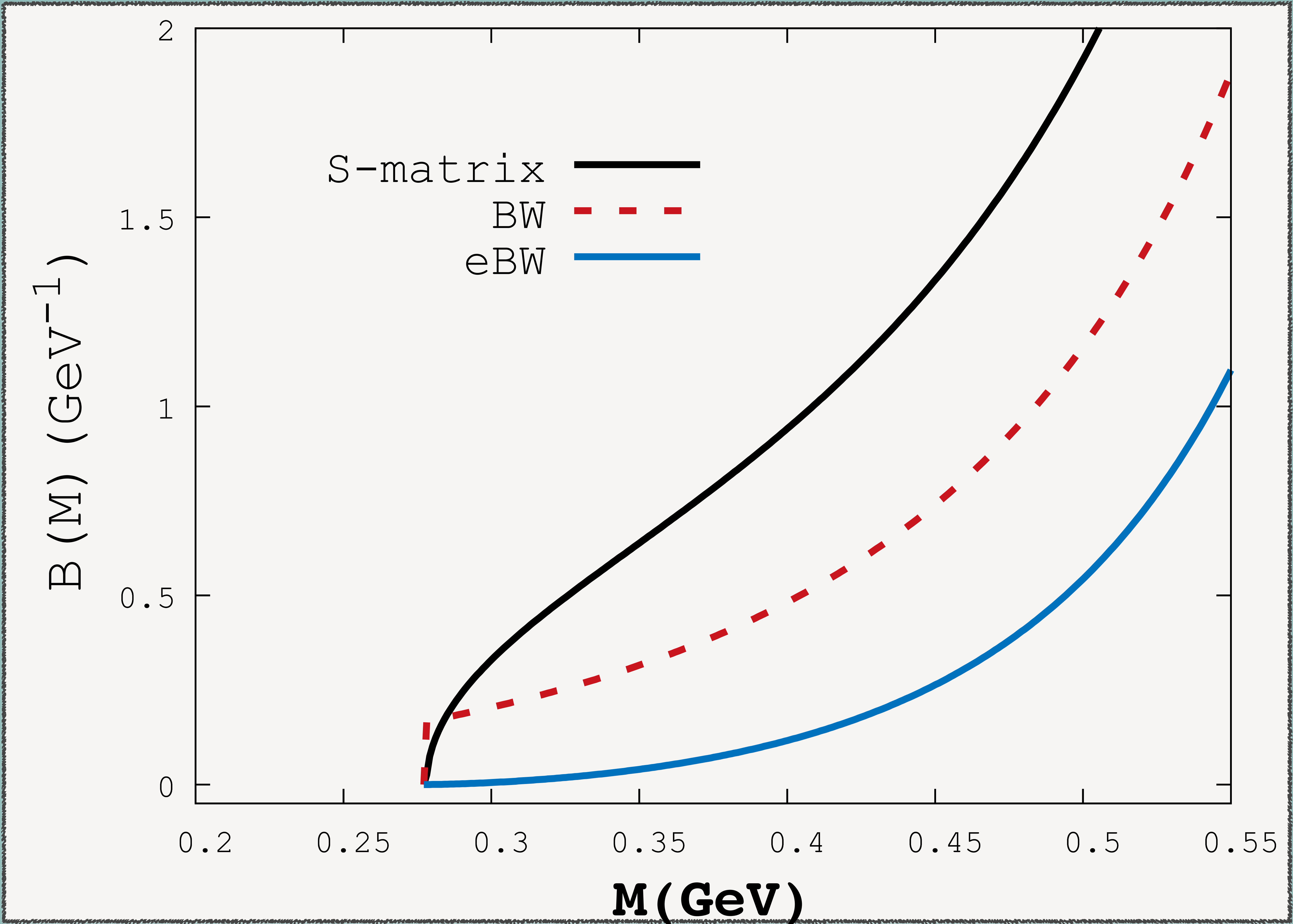
**$\rho(770)$** [h]

$I^G(J^{PC}) = 1^+(1^{--})$

Mass $m = 775.26 \pm 0.25$ MeV
Full width $\Gamma = 149.1 \pm 0.8$ MeV
$\Gamma_{ee} = 7.04 \pm 0.06$ keV


Estabrooks & Martin
Froggatt & Petersen
Protopopescu et al.
this work
$\delta \approx a_S \times q^3$
$a_S = 0.038\, m_\pi^{-3}$
δ(I=1 J=1) (degree)
M(GeV)
0
20
40
60
80
100
120
140
160
180
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
1.1


30
25
20
15
10
5
0
B(M) (GeV$^{-1}$)
S-matrix
BW
eBW
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6
M(GeV)



S-matrix
BW
eBW
B(M) (GeV$^{-1}$)
M(GeV)
0
0.5
1
1.5
2
0.2
0.25
0.3
0.35
0.4
0.45
0.5
0.55

# BETH-UHLENBECK APPROXIMATION

$$\delta = -\mathrm{Im}\,\mathrm{Tr}\ln G_\rho^{-1}$$

*physical interpretation:*

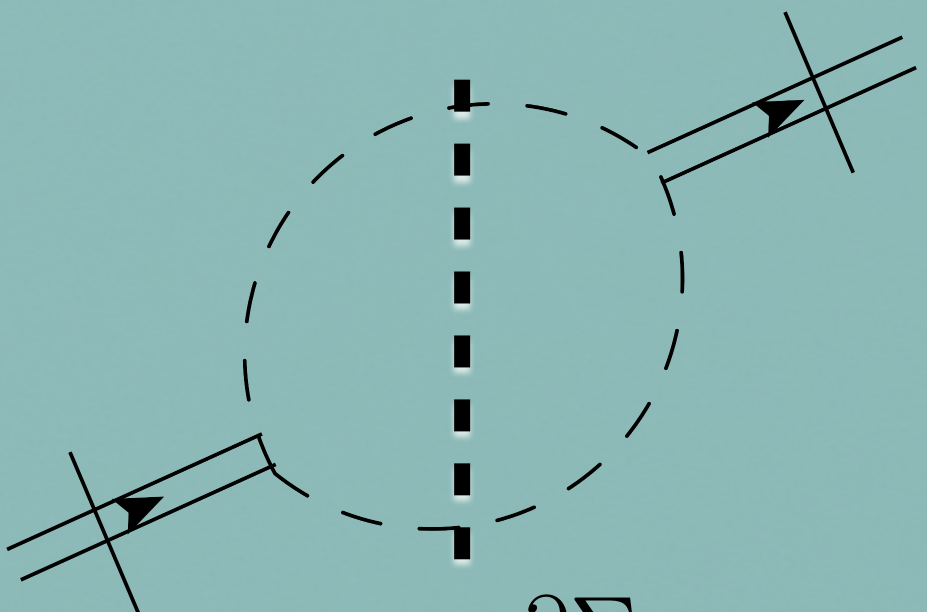
$$B = 2\,\frac{\partial}{\partial E}\delta$$

$$= -2\,\mathrm{Im}\frac{\partial}{\partial E}\ln G_\rho^{-1}$$

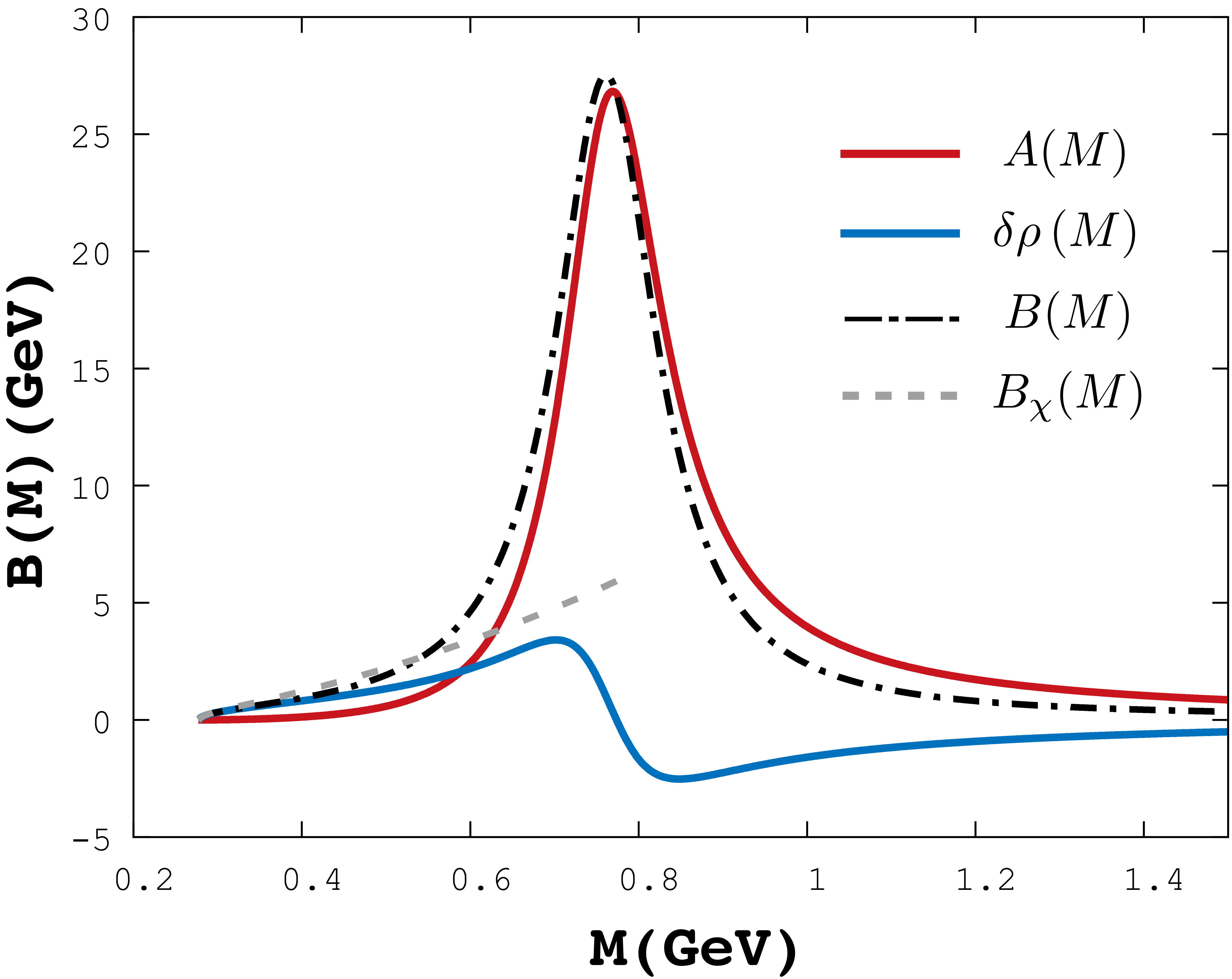
$$= -2\,\mathrm{Im}[G_\rho](2E) + 2\,\mathrm{Im}[\frac{\partial \Sigma_\rho}{\partial E}G_\rho]$$

*contribution from correlated pi pi pair*

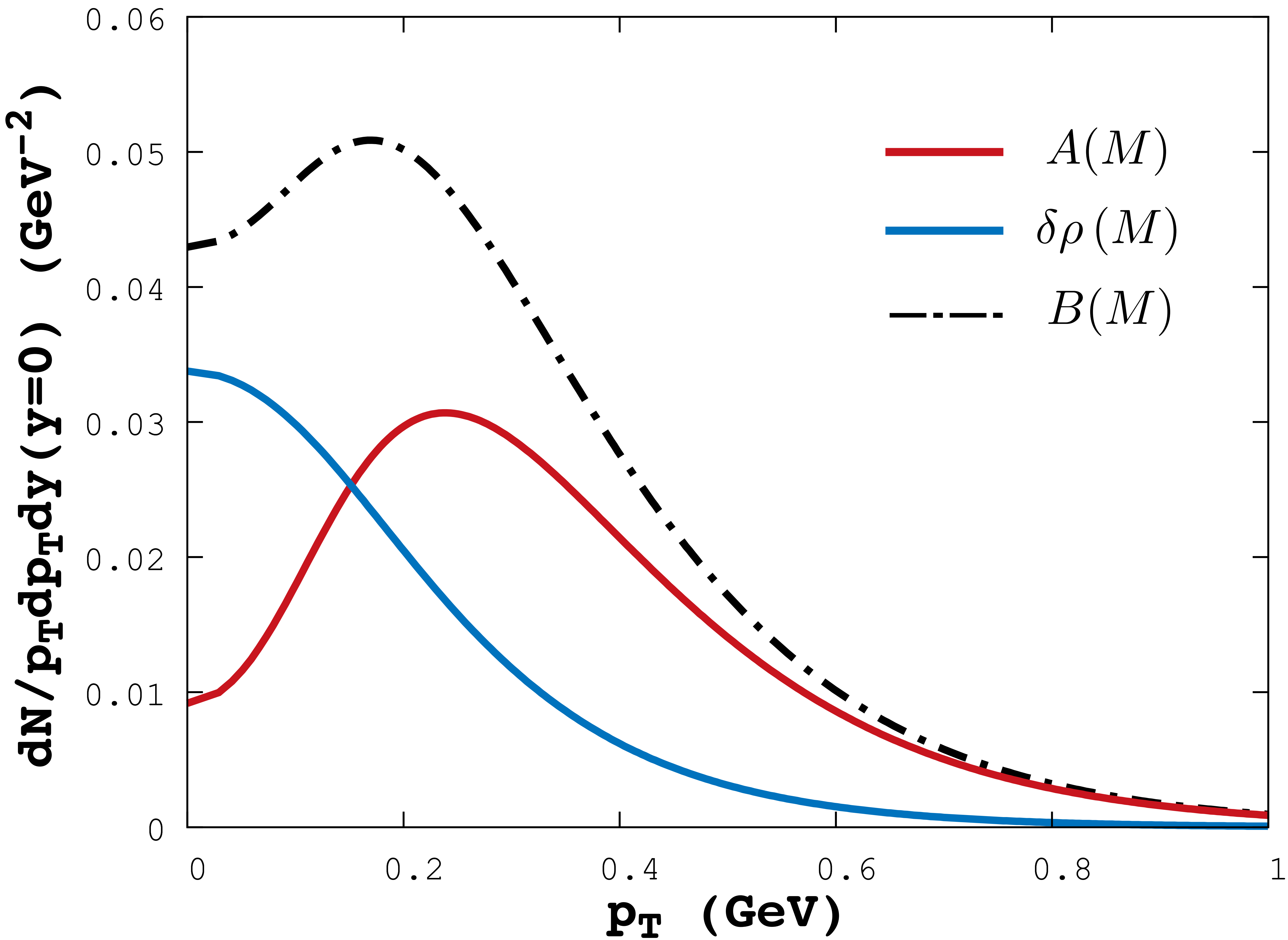
$$=> \rho_\rho(E) + \delta\rho_\rho(E)$$



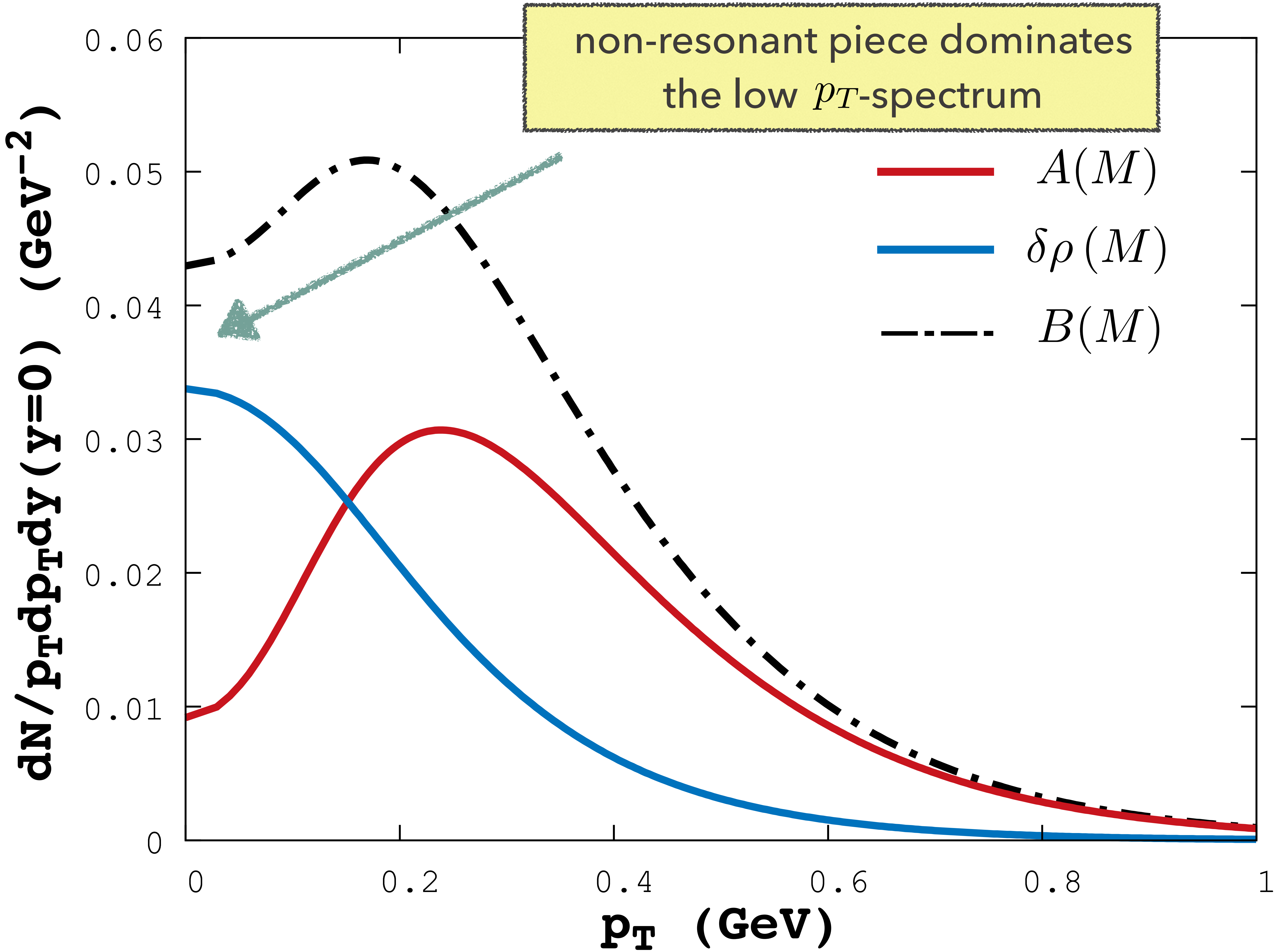
$$\frac{\partial \Sigma_\rho}{\partial E}$$

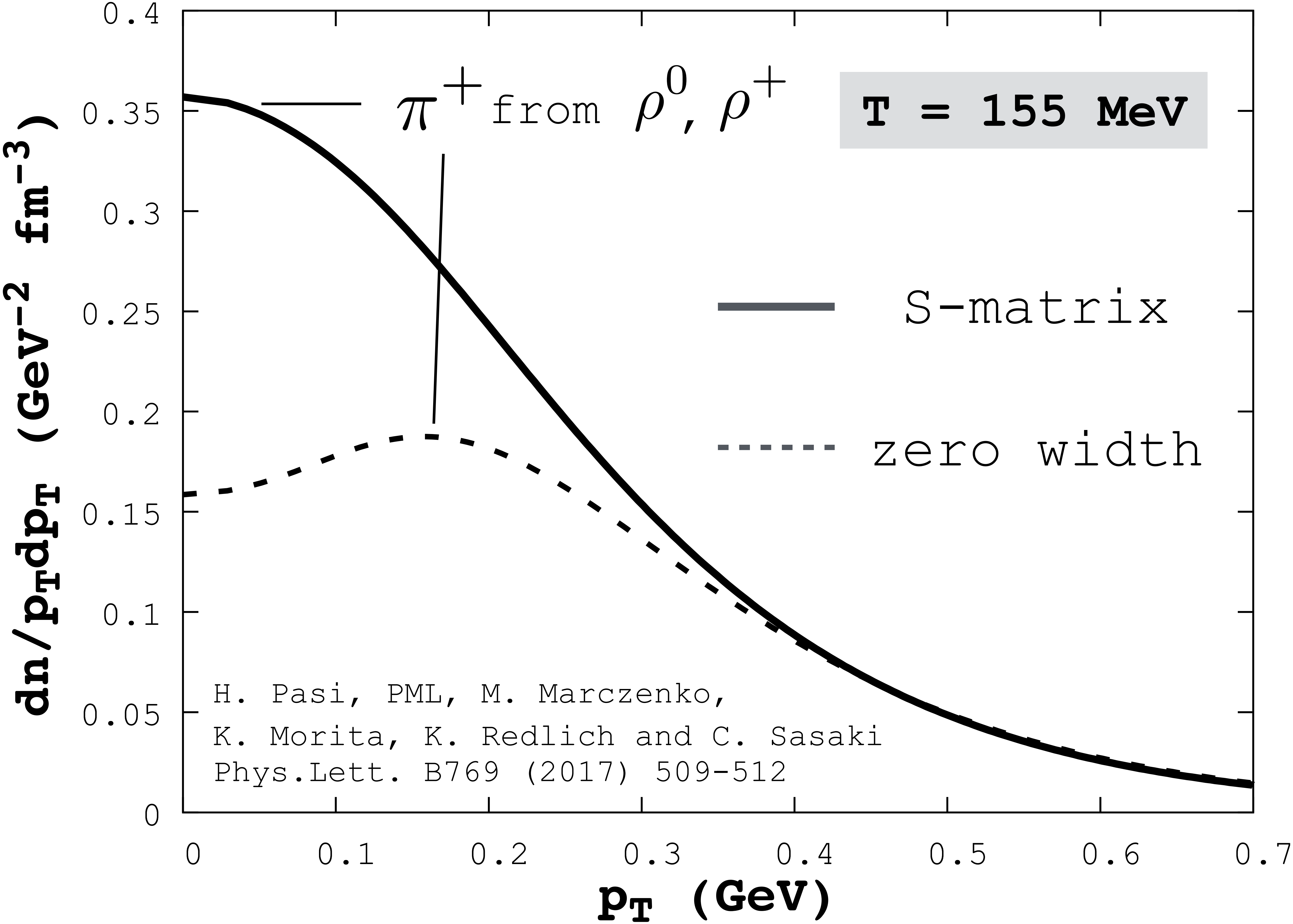


30
25
20
15
10
5
0
-5
0.2
0.4
0.6
0.8
1
1.2
1.4
B(M) (GeV)
M(GeV)
$A(M)$
$\delta\rho(M)$
$B(M)$
$B_\chi(M)$

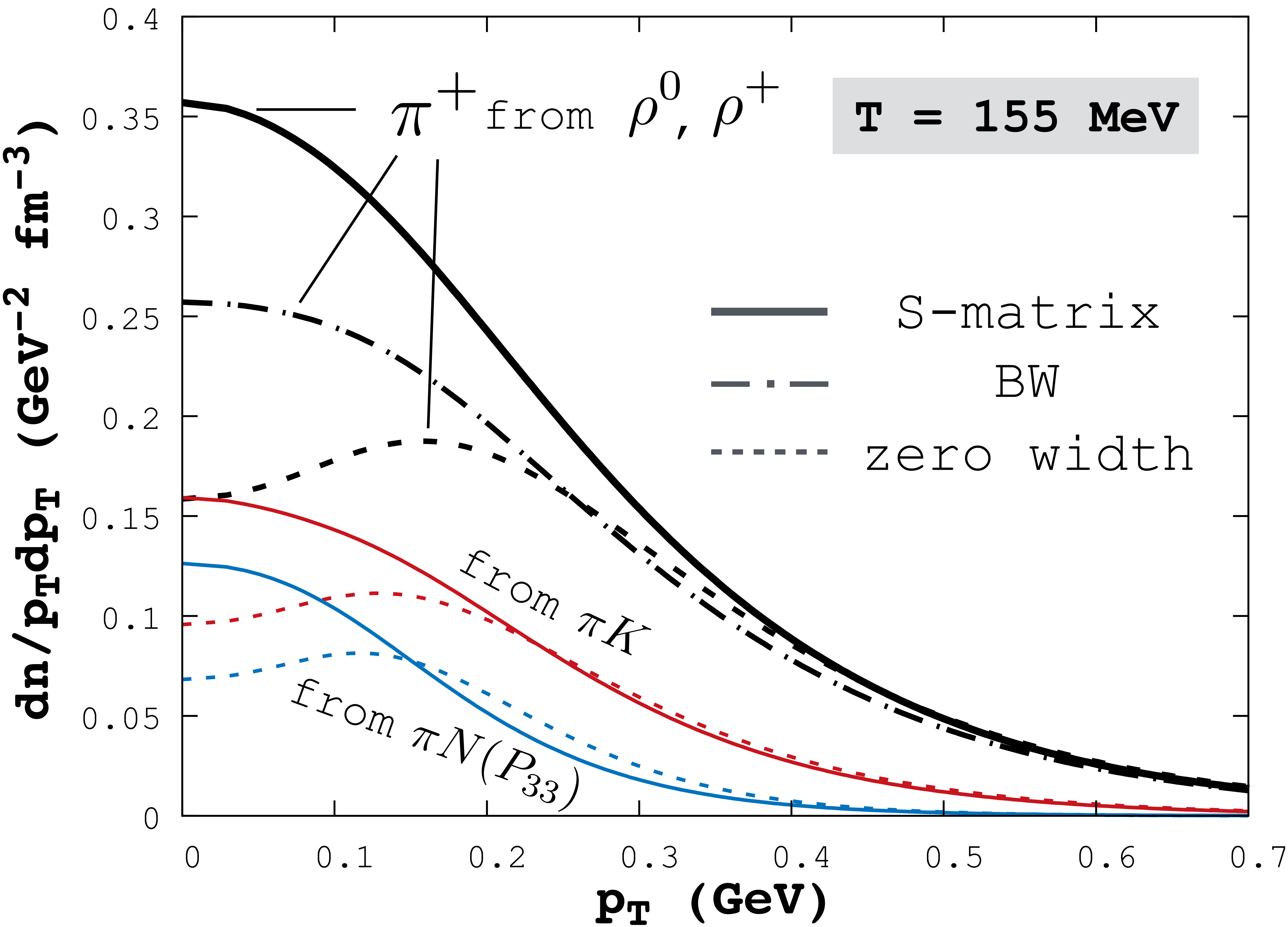


$dN/p_T dp_T dy(y=0)$ $(GeV^{-2})$
0.06
0.05
0.04
0.03
0.02
0.01
0
0
0.2
0.4
0.6
0.8
1
$p_T$ (GeV)
$A(M)$
$\delta\rho(M)$
$B(M)$

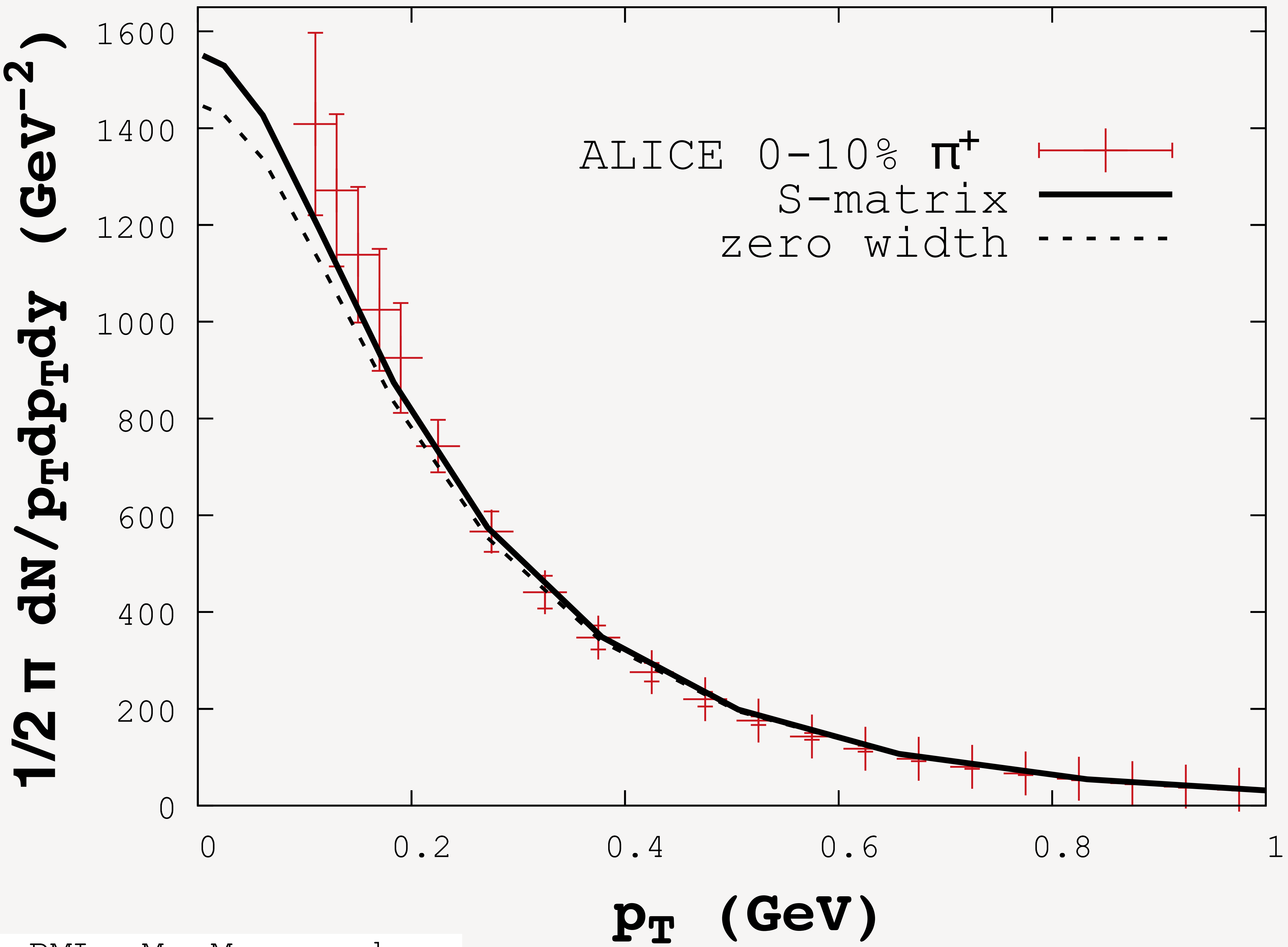

non-resonant piece dominates
the low $p_T$-spectrum
$A(M)$
$\delta\rho(M)$
$B(M)$
0.06
0.05
0.04
0.03
0.02
0.01
0
0
0.2
0.4
0.6
0.8
1
$dN/p_T dp_T dy(y=0)$ $(GeV^{-2})$
$p_T$ (GeV)



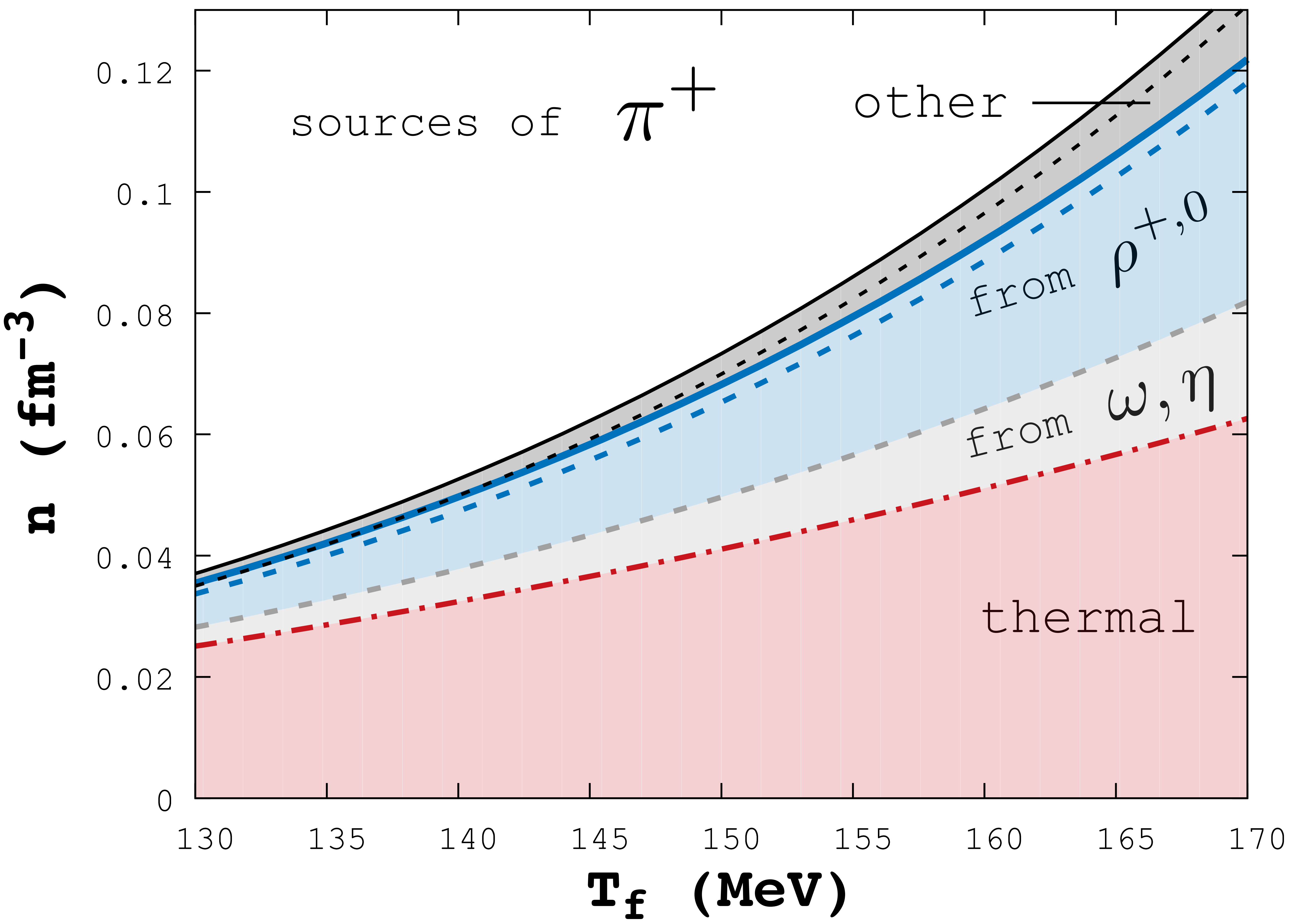
$\pi^+$ from $\rho^0, \rho^+$
T = 155 MeV
S-matrix
zero width
H. Pasi, PML, M. Marczenko,
K. Morita, K. Redlich and C. Sasaki
Phys.Lett. B769 (2017) 509-512
$dn/p_T dp_T$ ($GeV^{-2}$ $fm^{-3}$)
$p_T$ (GeV)
0
0.05
0.1
0.15
0.2
0.25
0.3
0.35
0.4
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7



T = 155 MeV
$\pi^+$ from $\rho^0$, $\rho^+$
S-matrix
BW
zero width
from $\pi K$
from $\pi N(P_{33})$
dn/p_Tdp_T (GeV^-2 fm^-3)
p_T (GeV)
0
0.05
0.1
0.15
0.2
0.25
0.3
0.35
0.4
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7



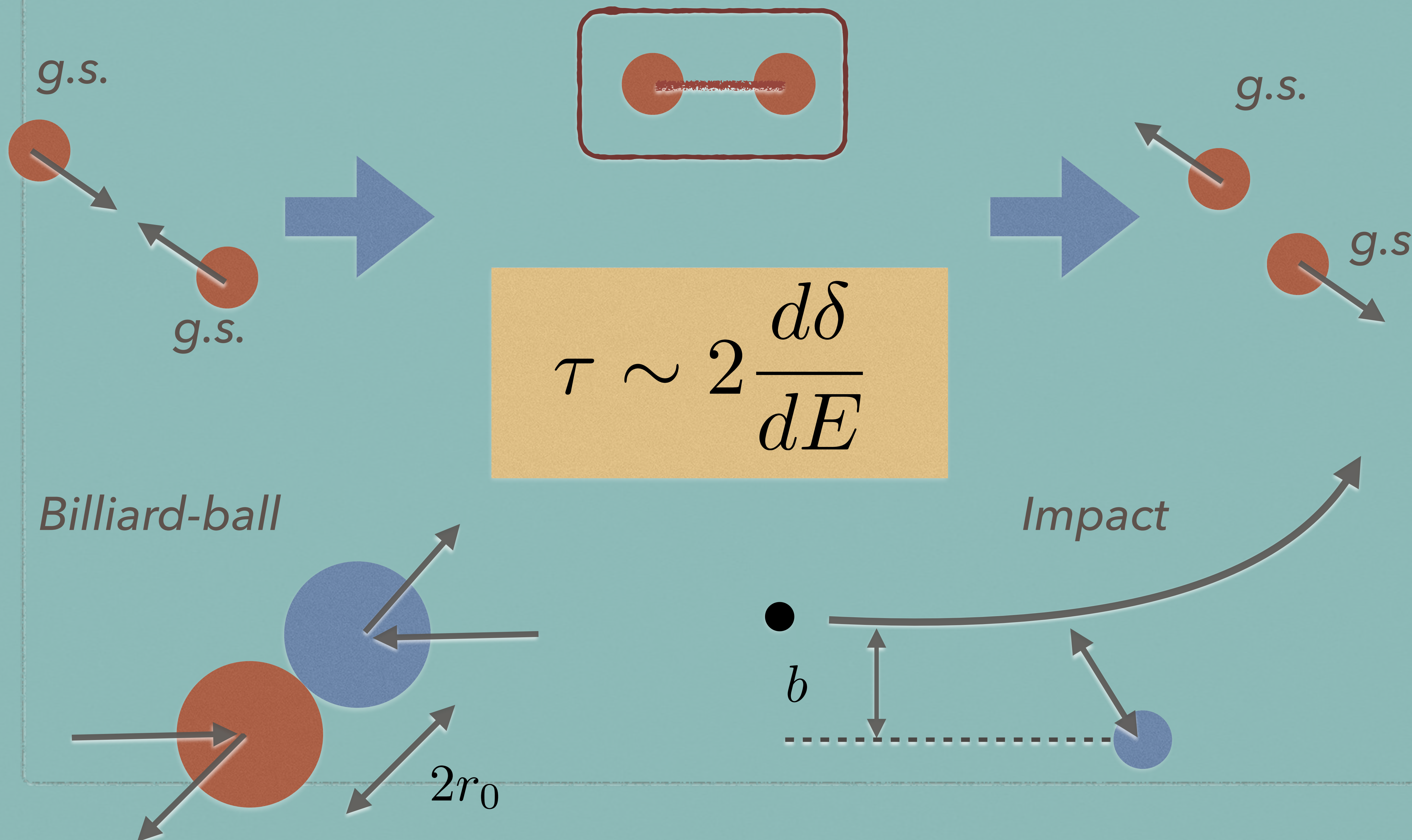

H. Pasi, PML, M. Marczenko,
K. Morita, K. Redlich and C. Sasaki
Phys.Lett. B769 (2017) 509-512



sources of $\pi^+$
other
from $\rho^{+,0}$
from $\omega,\eta$
thermal
0.12
0.1
0.08
0.06
0.04
0.02
0
130
135
140
145
150
155
160
165
170
n (fm$^{-3}$)
$T_f$ (MeV)

# TIME DELAY

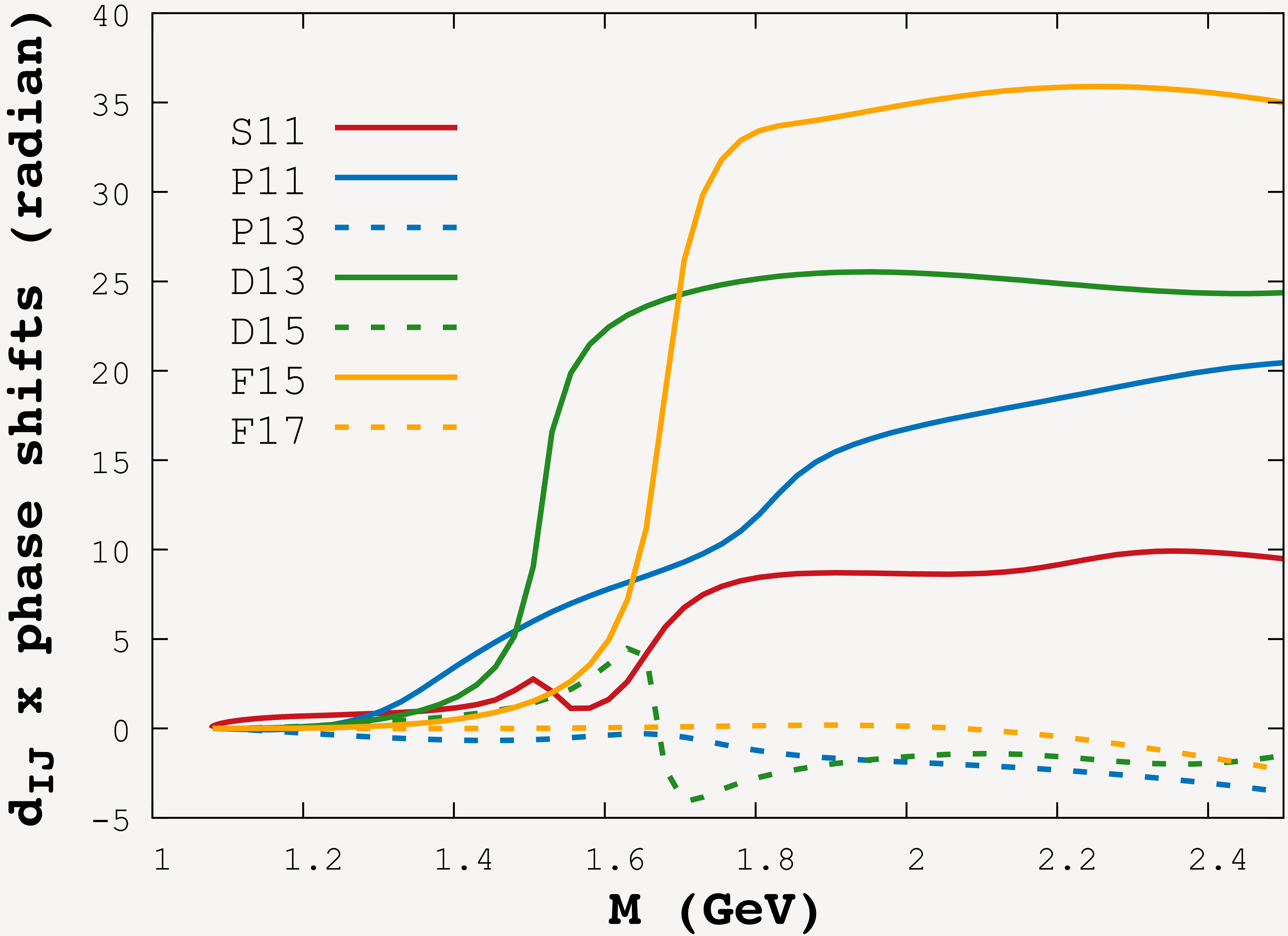
P. Danielewicz and S. Pratt
Phys.Rev. C53 (1996) 249-266



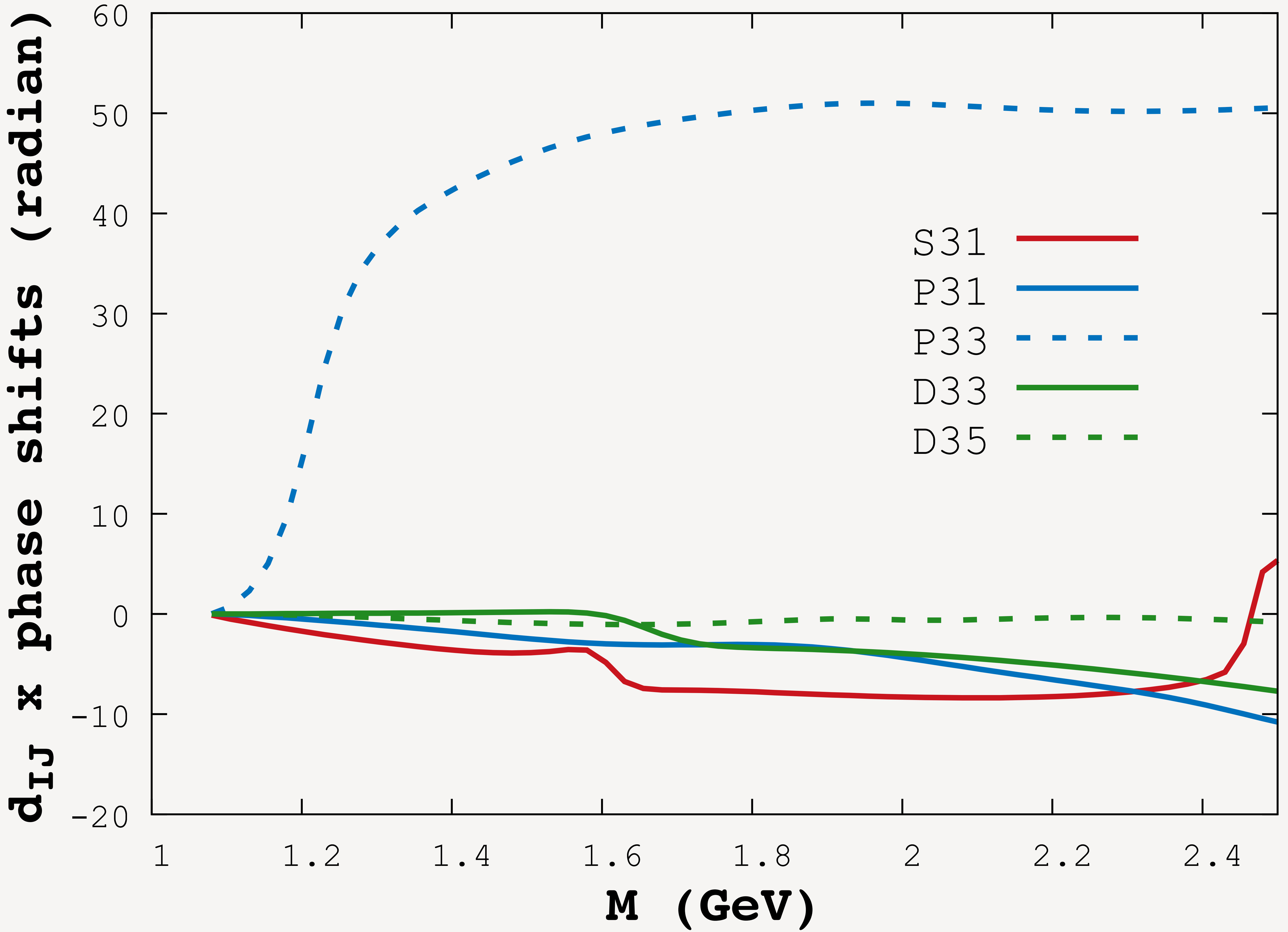

$$\tau \sim 2\frac{d\delta}{dE}$$

# PI-N SYSTEM

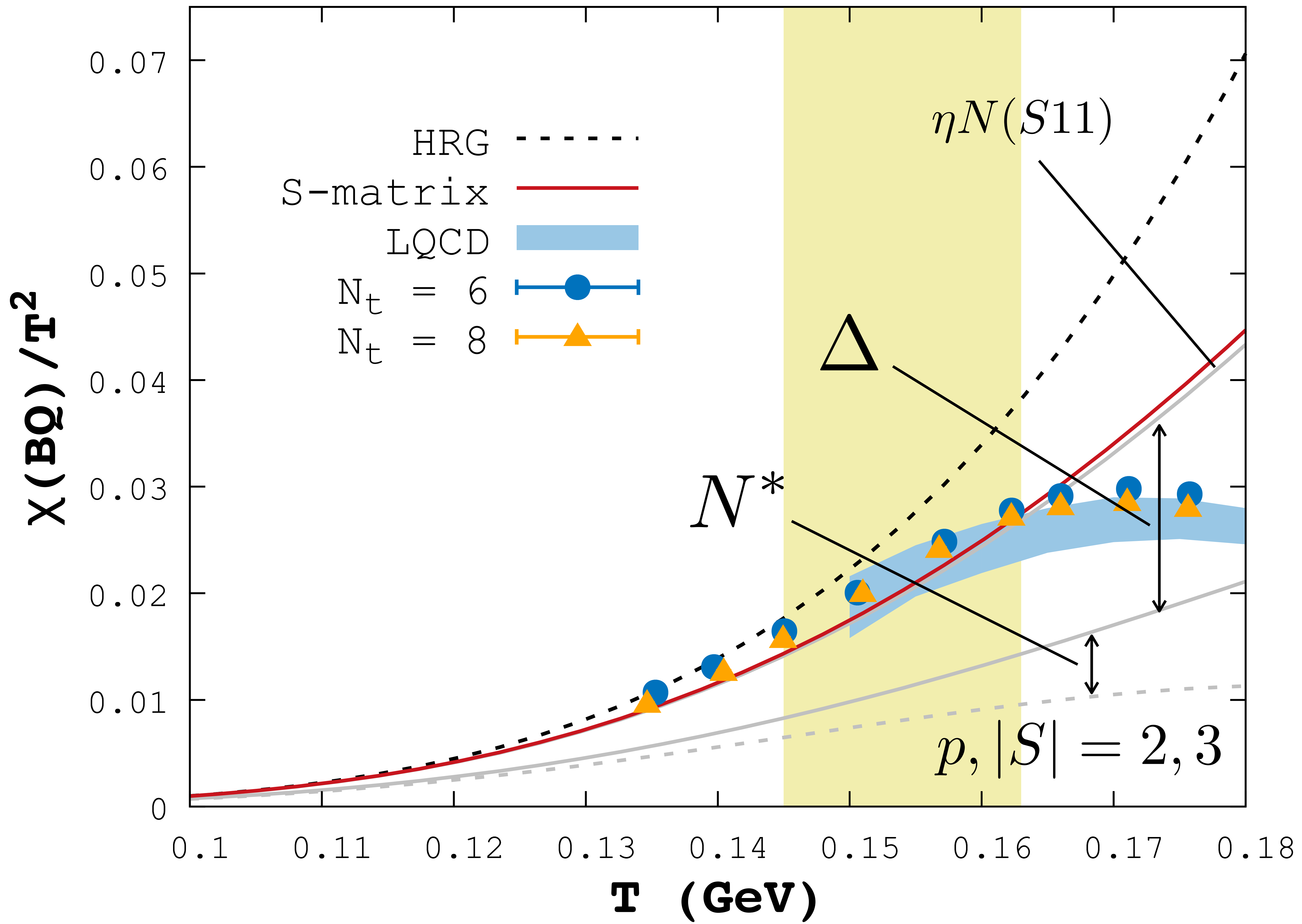
PML, B. Friman, K. Redlich, C. Sasaki, in preparation



40
35
30
25
20
15
10
5
0
-5
1
1.2
1.4
1.6
1.8
2
2.2
2.4
S11
P11
P13
D13
D15
F15
F17
d_IJ x phase shifts (radian)
M (GeV)



d_IJ x phase shifts (radian)
60
50
40
30
20
10
0
-10
-20
1
1.2
1.4
1.6
1.8
2
2.2
2.4
M (GeV)
S31
P31
P33
D33
D35



HRG
S-matrix
LQCD
N_t = 6
N_t = 8
ηN(S11)
Δ
N*
p, |S| = 2, 3
χ(BQ)/T²
T (GeV)
0
0.01
0.02
0.03
0.04
0.05
0.06
0.07
0.1
0.11
0.12
0.13
0.14
0.15
0.16
0.17
0.18

# COUPLED-CHANNEL PROBLEM

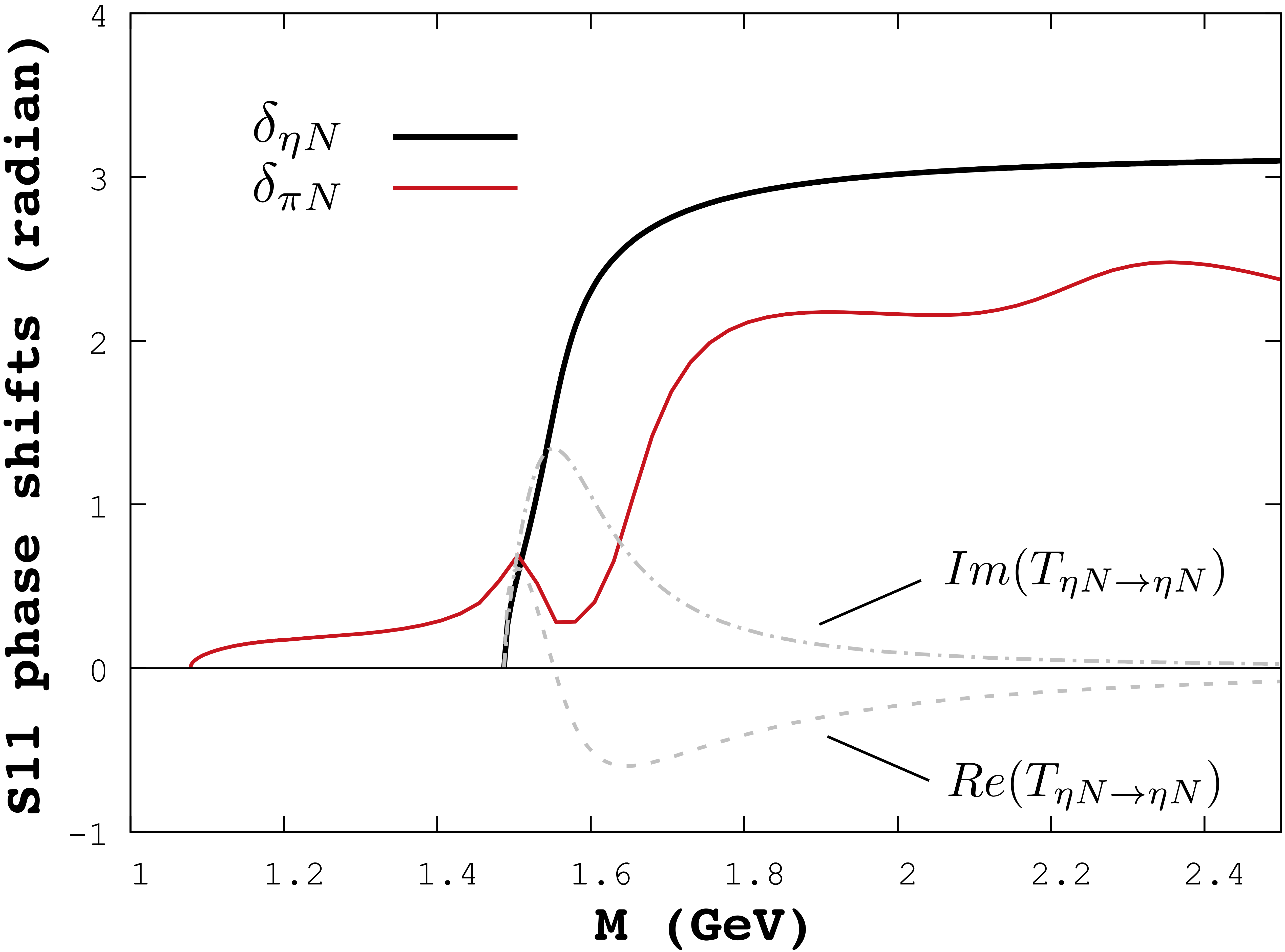
$$S = \begin{pmatrix} \eta\, e^{2\,i\,\delta_I} & i\sqrt{1-\eta^2}\, e^{i\,(\delta_I+\delta_{II})} \\ i\sqrt{1-\eta^2}\, e^{i\,(\delta_I+\delta_{II})} & \eta\, e^{2\,i\,\delta_{II}} \end{pmatrix}$$

$$\begin{aligned} \mathcal{Q}(M) &\equiv \frac{1}{2}\,\mathrm{Im}\,(\mathrm{tr}\ln\,\mathrm{S}) \\ &= \frac{1}{2}\,\mathrm{Im}\,(\ln\,\det\,[\mathrm{S}]) \\ &= \delta_I + \delta_{II}. \end{aligned}$$

$\pi N$ *system*

$$\pi N \to \begin{pmatrix} \pi N \\ \eta N \end{pmatrix} \to \pi N$$

$$\pi \eta \to \begin{pmatrix} \pi N \\ \eta N \end{pmatrix} \to \pi \eta$$



$\delta_{\eta N}$
$\delta_{\pi N}$
$Im(T_{\eta N\to\eta N})$
$Re(T_{\eta N\to\eta N})$
S11 phase shifts (radian)
M (GeV)
4
3
2
1
0
-1
1
1.2
1.4
1.6
1.8
2
2.2
2.4

# N-BODY SCATTERING

PML, Eur. Phys. J. C **77** no.8 533 (2017)

# WHY N-BODY?

- EOS for dense system
  -> need higher coefficients of
  quantum cluster / virial expansion
  (three-body forces, etc.)
- Explore the influence of N-body scatterings on heavy ion collision observables:
  pT-spectra, flow etc.
- phenomenology
  -> model S-matrix element instead…

# RECIPE

*Feynman amplitude*

- generalized phase shift

$$\mathcal{Q}_N(M) = \frac{1}{2}\,\mathrm{Im}\left[\ln\left(1 + \int d\phi_N\, i\mathcal{M}\right)\right]$$

$$d\phi_N = \frac{d^3p_1}{(2\pi)^3}\frac{1}{2E_1}\frac{d^3p_2}{(2\pi)^3}\frac{1}{2E_2}\cdots\frac{d^3p_N}{(2\pi)^3}\frac{1}{2E_N}\times$$
$$(2\pi)^4\,\delta^4(P - \sum_i p_i).$$

*phase space approach*

# PHASE SPACE DOMINANCE

$$\mathcal{Q}_N(M) = \frac{1}{2}\,\mathrm{Im}\left[\ln\left(1 + \int d\phi_N\, i\mathcal{M}\right)\right]$$

- structureless scattering

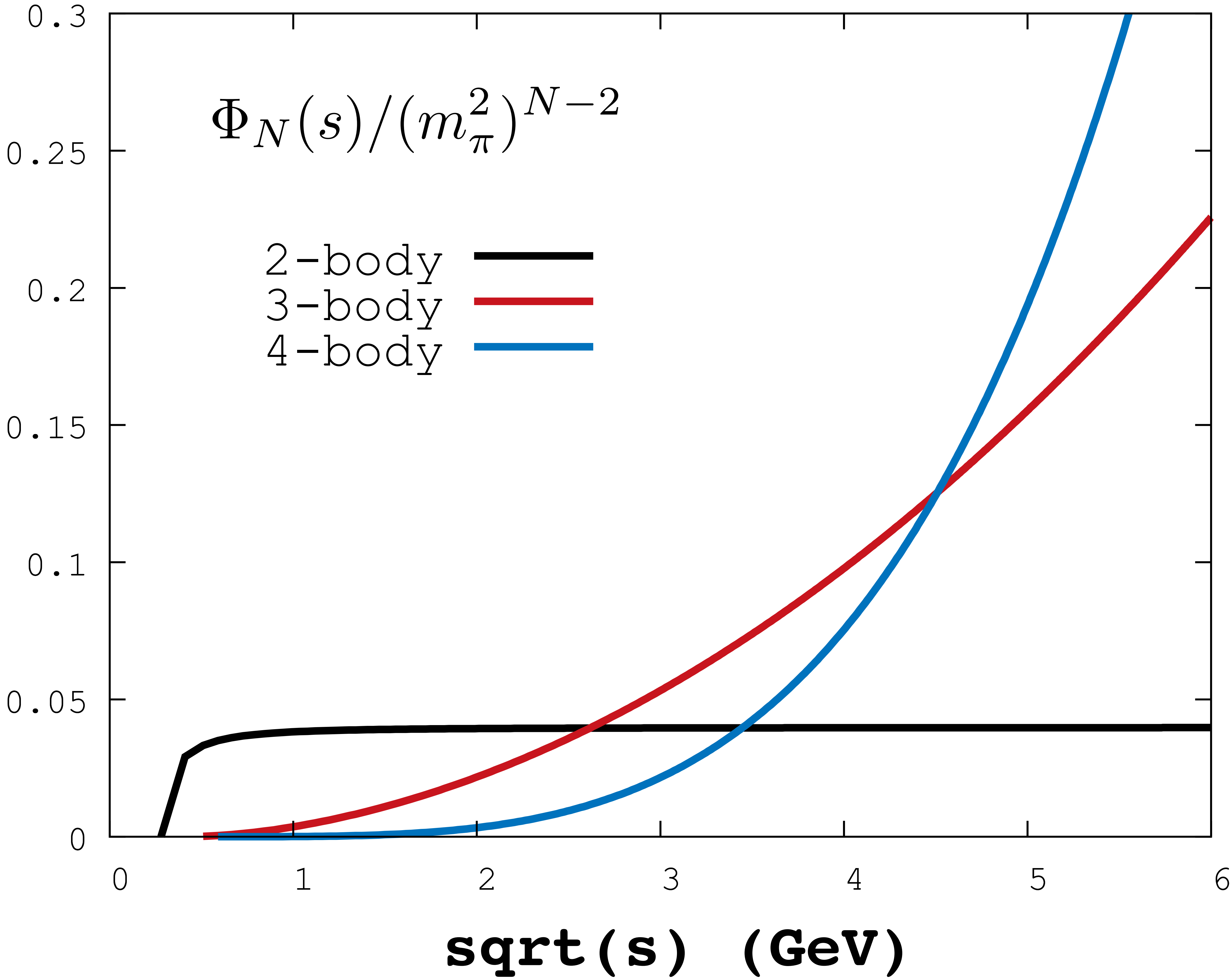
*Dimension:* $\sim E^{2N-4}$

$$i\mathcal{M} = i\,\lambda_N$$

Källén triangle function

$$\phi_N(s) = \frac{1}{16\,\pi^2 s}\int_{s'_-}^{s'_+} ds'\,\sqrt{\lambda(s, s', m_N^2)}\times$$

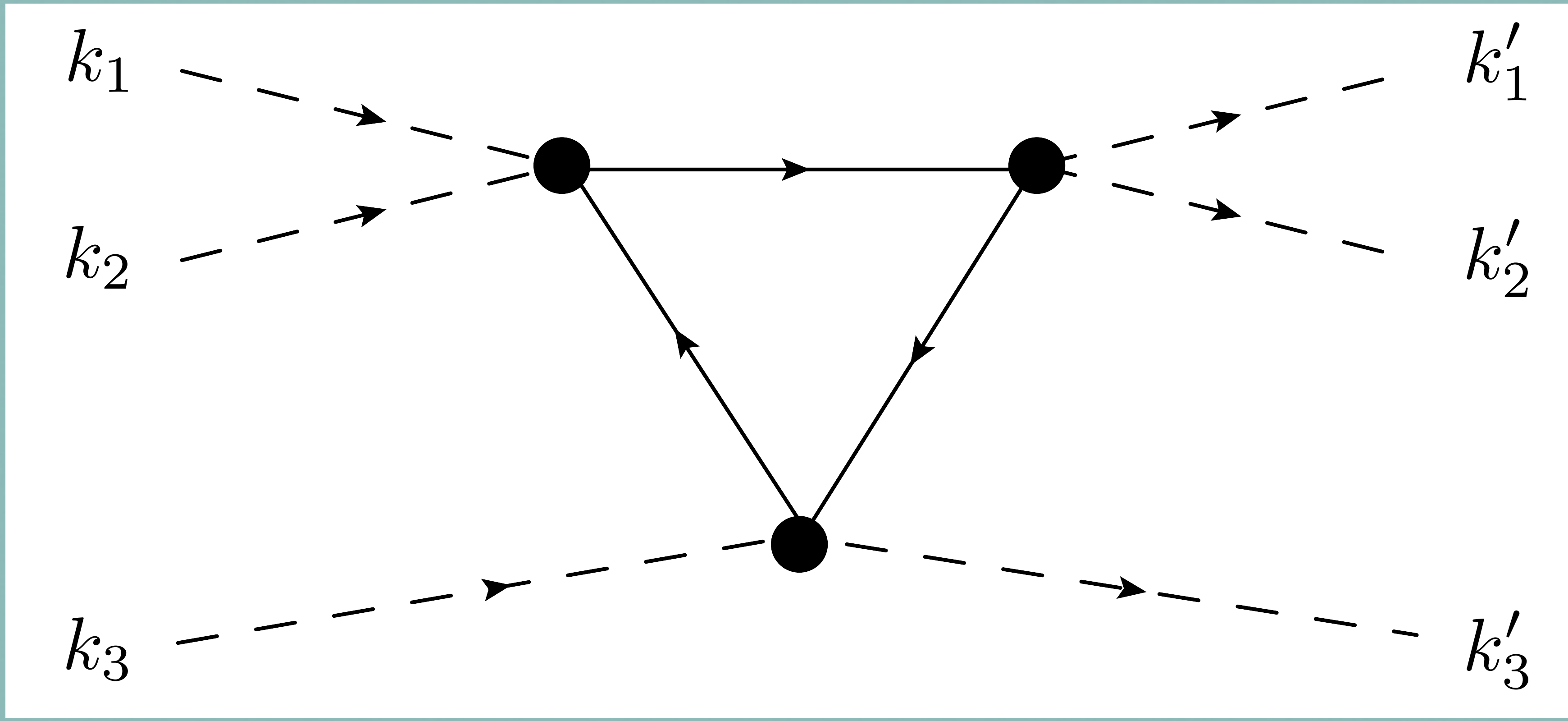
$$\phi_{N-1}(s', m_1^2, m_2^2, ..., m_{N-1}^2)$$



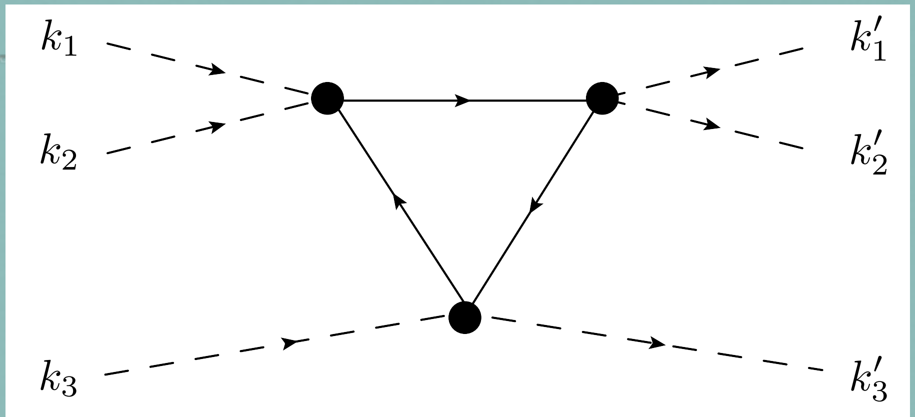
0.3
0.25
0.2
0.15
0.1
0.05
0
0
1
2
3
4
5
6
$\Phi_N(s)/(m_\pi^2)^{N-2}$
2-body
3-body
4-body
sqrt(s) (GeV)

# TRIANGLE DIAGRAM

- 3-body diagram



*Explicit calculation*




$$i\mathcal{M}^{\triangle}(Q_1, Q_2, Q_3) = \int \frac{d^4 l}{(2\pi)^4} \times (-i\,\lambda)^3 \times i\,G(l) \times i\,G(l + Q_1) \times i\,G(l - Q_2)$$

*Feynman's trick + dim reg.*

$$i\mathcal{M}^{\triangle}(Q_1^2, Q_2^2, s = P_I^2) = -i\frac{\lambda^3}{16\pi^2} \int_0^1 dx \int_0^{1-x} dy\, \frac{1}{\Delta(x,y)}$$

$$\Delta(x,y) = m_\pi^2 - x(1-x)\,Q_1^2 - y(1-y)\,Q_2^2$$
$$- 2\,xy\,Q_1 \cdot Q_2 - i\epsilon.$$

- to lowest order $\mathcal{Q}(s) \approx \frac{1}{2}\,\mathrm{Im}\left[\int d\phi_3\, i\mathcal{M}^{\text{triangle}}\right],$

=> only need to deal with on-shell condition

$$k_i' = k_i$$

*analytic result:*

$$i\mathcal{M}^{\triangle,o.s.}(Q_1^2, s) = -i\frac{\lambda^3}{16\,\pi^2}\,\frac{z}{Q_1^2}\,\ln\frac{1-z}{1+z}$$

$$z = \frac{1}{\sqrt{1-\frac{4m_\pi^2}{Q_1^2}}}.$$

$$\mathcal{Q}(s) \approx \frac{1}{2}\,\mathrm{Im}\left[\int d\phi_3\, i\mathcal{M}^{\mathrm{triangle}}\right],$$

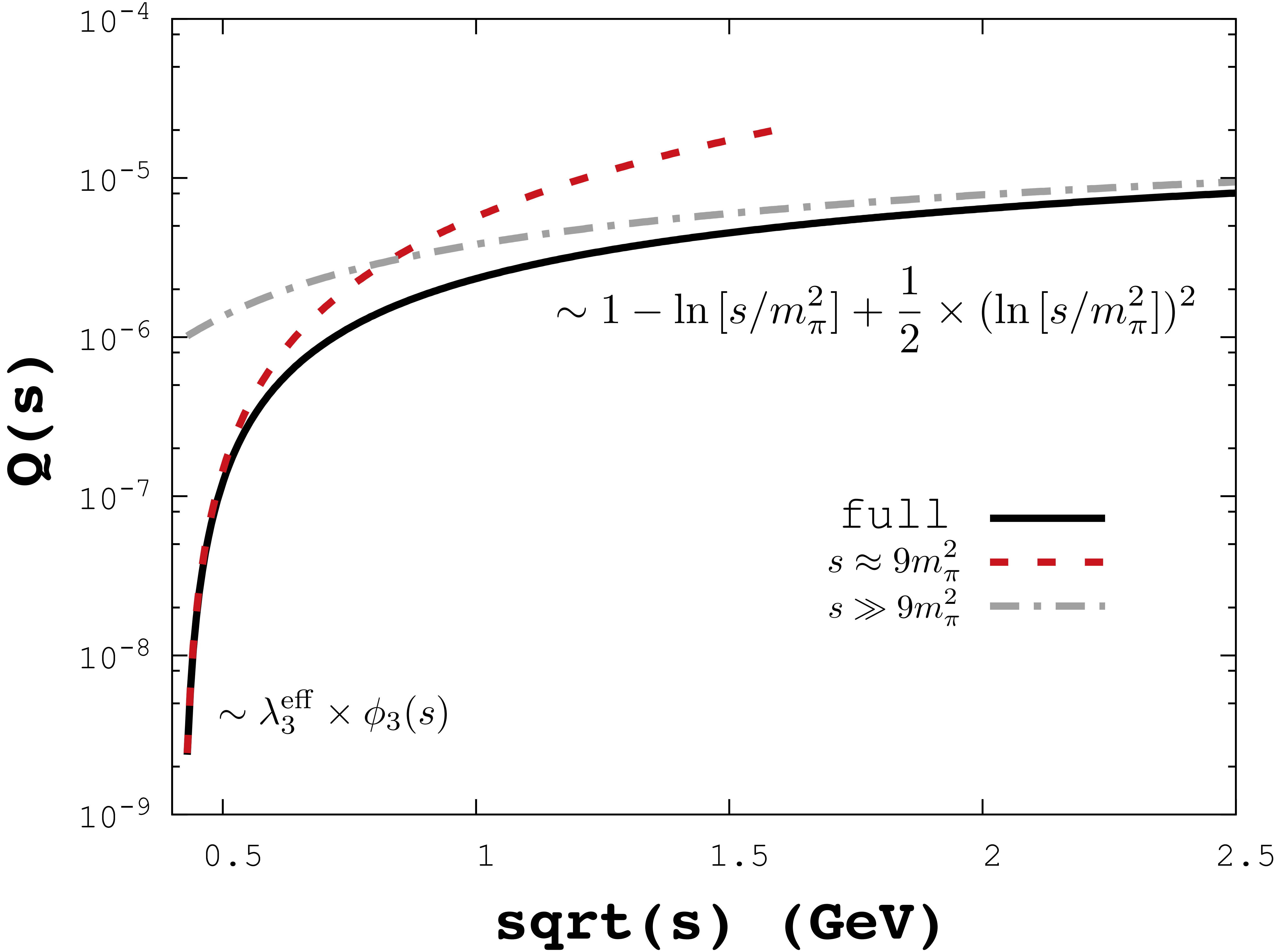
*Limits:*

$$s \to 9m_\pi^2 \qquad \mathcal{Q}(s) \approx \frac{1}{2} \times \lambda_3^{\mathrm{eff}} \times \phi_3(s).$$

$$s \gg 9m_\pi^2 \qquad \mathcal{Q}(s) \approx \frac{\lambda^3}{8192\,\pi^5} \int_{\xi_0}^{1} d\xi\,(\frac{1}{\xi} - 1)\,\left[-z\,\ln\left|\frac{1-z}{1+z}\right|\right]$$

$$\approx \frac{\lambda^3}{4096\,\pi^5} \times \left[1 + \ln\frac{\xi_0}{4} + (\ln\frac{\xi_0}{4})^2\right]$$

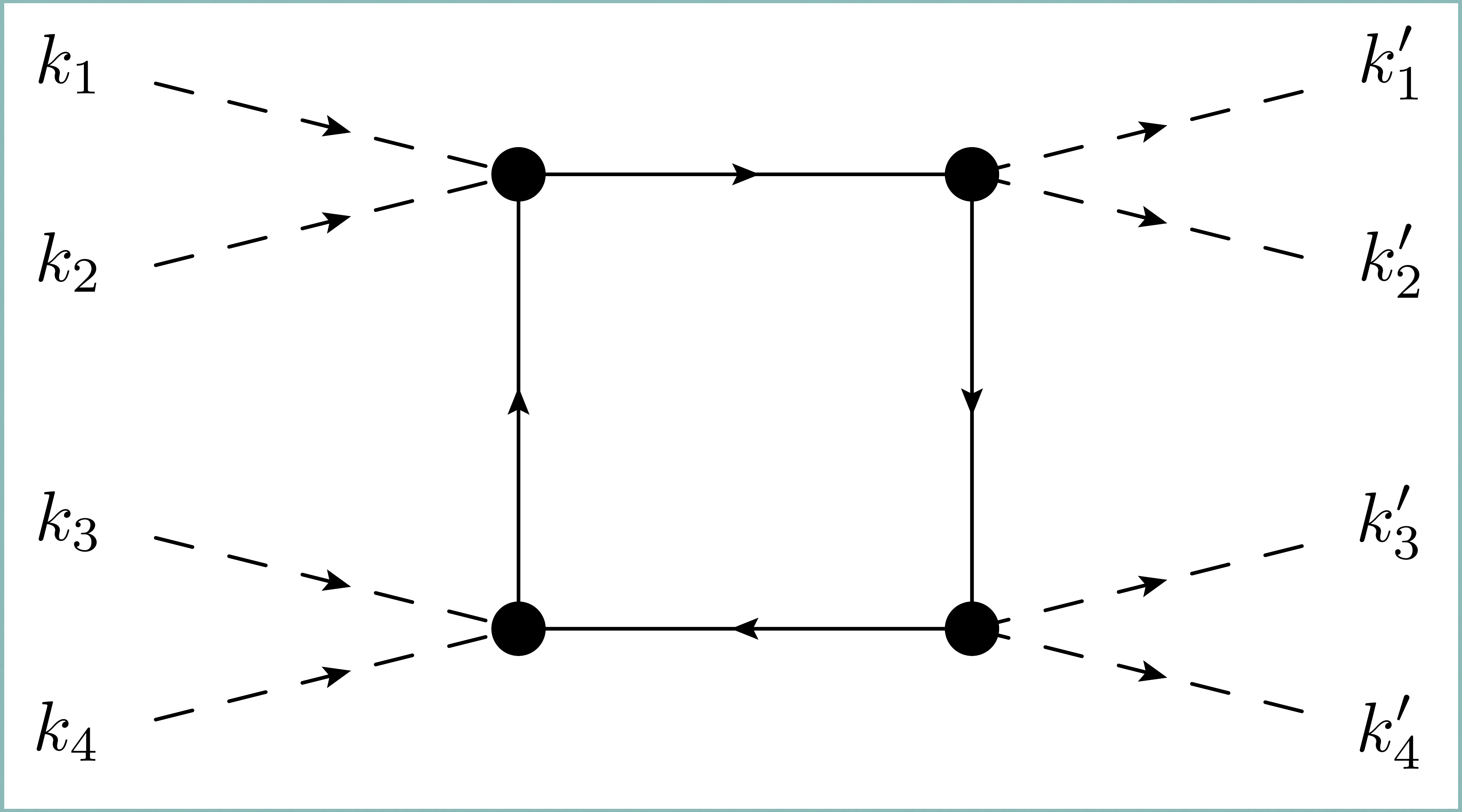
*where* $\quad \xi_0 = \dfrac{4m_\pi^2}{s}.$



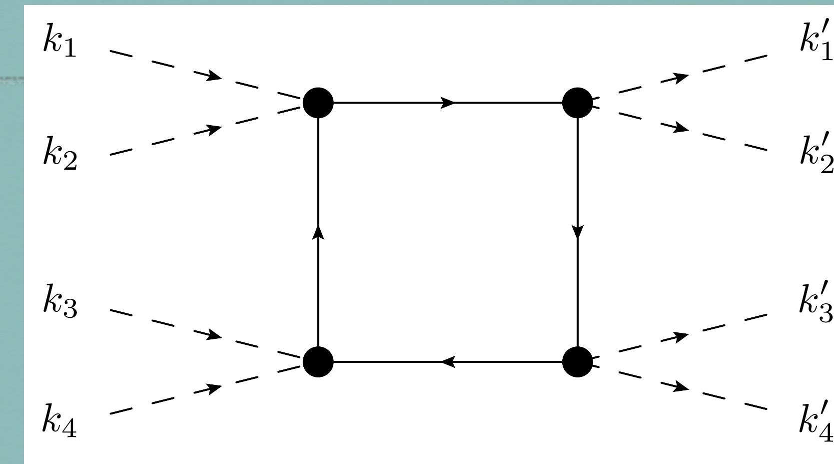
$10^{-4}$
$10^{-5}$
$10^{-6}$
$10^{-7}$
$10^{-8}$
$10^{-9}$
Q(s)
0.5
1
1.5
2
2.5
sqrt(s) (GeV)
$\sim 1 - \ln[s/m_\pi^2] + \frac{1}{2} \times (\ln[s/m_\pi^2])^2$
$\sim \lambda_3^{\mathrm{eff}} \times \phi_3(s)$
full
$s \approx 9m_\pi^2$
$s \gg 9m_\pi^2$

# BOX DIAGRAM

- 4-body diagram



*Explicit calculation*




$$i\mathcal{M}^{\text{box}}(Q_1, Q_2, Q_3, Q_4) = \int \frac{d^4 l}{(2\pi)^4} (-i\,\lambda)^4 \times i\,G(l) \times i\,G(l+Q_1) \times i\,G(l+Q_1-Q_3) \times i\,G(l-Q_2)$$

*Feynman's trick + dim reg.*

$$i\mathcal{M}^{\text{box}}(Q_1, Q_2, Q_3, Q_4) = i\frac{\lambda^4}{16\pi^2} \int_0^1 dx \int_0^{1-x} dy \times \int_0^{1-x-y} dz \times \left(\frac{1}{\Delta(x,y,z)}\right)^2$$

$$\mathcal{Q}(s) \approx \frac{1}{2}\,\mathrm{Im}\,\left[\int d\phi_4\, i\mathcal{M}^{\mathrm{box,o.s.}}\right].$$

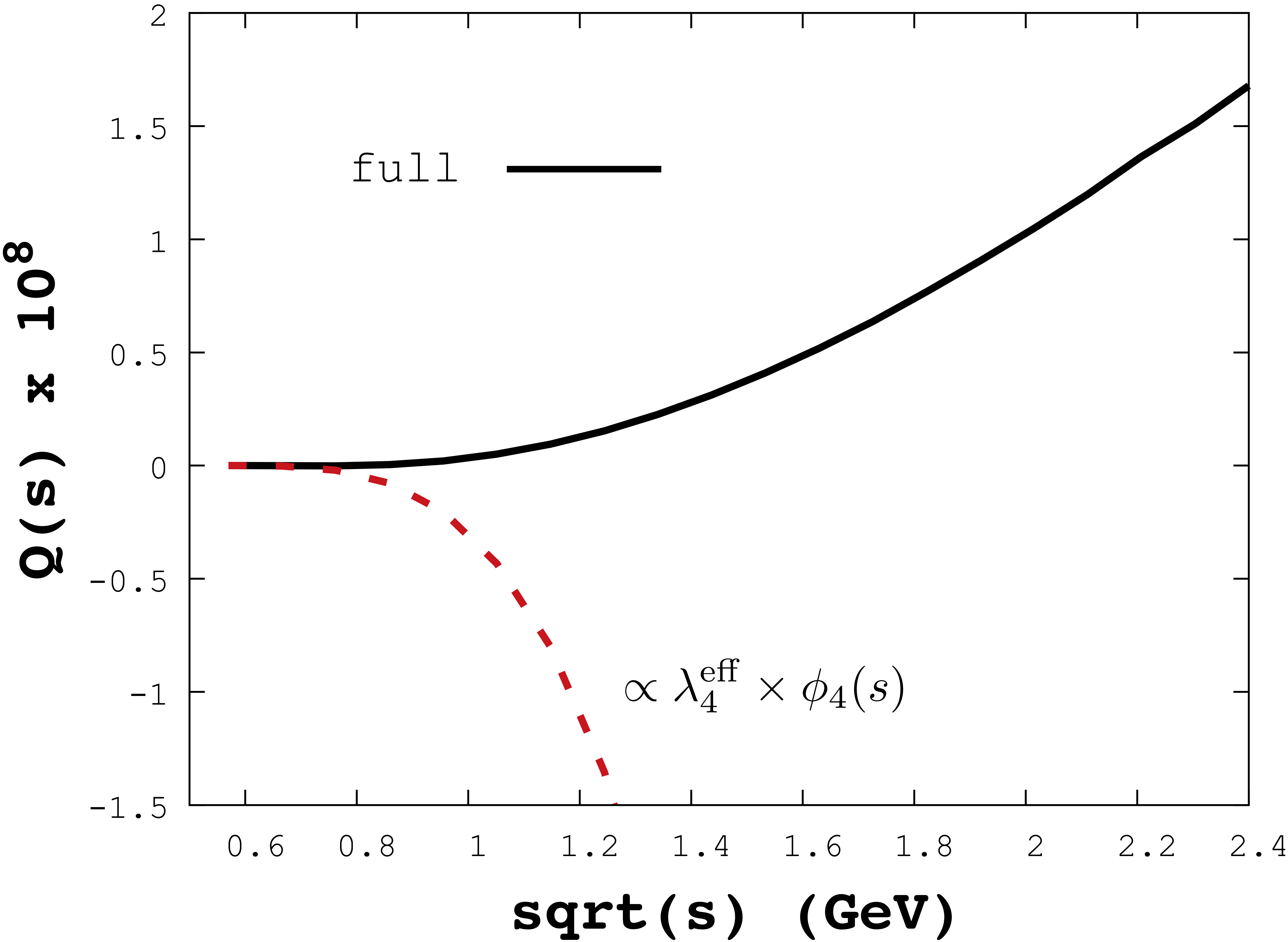
*Limits:* $s \to 16\, m_\pi^2$

$$\mathrm{Im}\,\left(i\mathcal{M}^{\mathrm{box,o.s.}}(q_1^2, q_2^2, s)\right) \approx \lambda_4^{\mathrm{eff}}$$

$$\lambda_4^{\mathrm{eff}} = \frac{\lambda^4}{256\pi^2}\frac{1}{m_\pi^4} \times \left(\frac{\sqrt{3}}{2}\,\ln\left(7 - 4\sqrt{3}\right) + 2\right)$$ *Negative!*

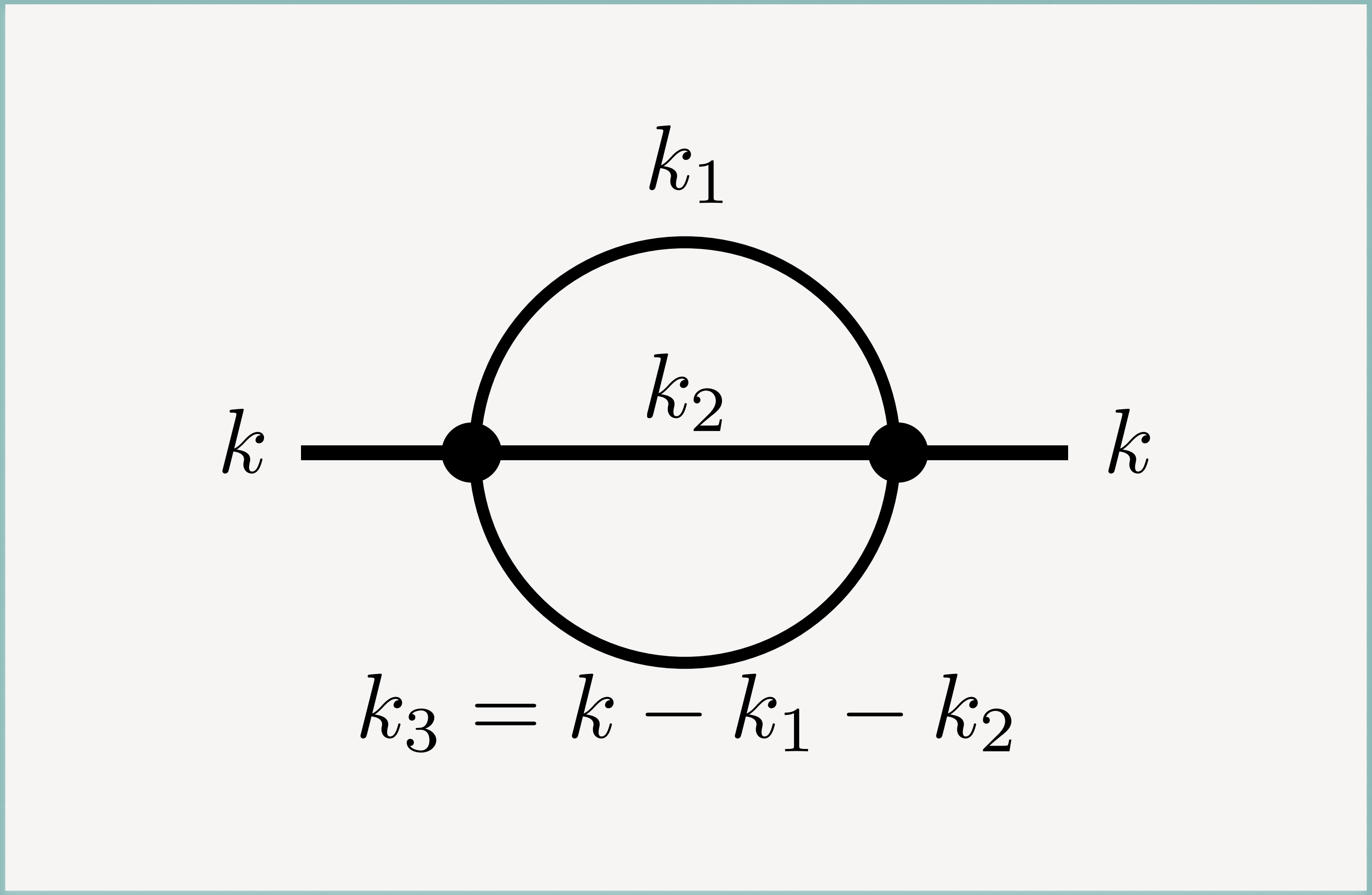
$$\mathcal{Q}(s) \approx \frac{1}{2} \times \lambda_4^{\mathrm{eff}} \times \phi_4(s).$$

$s >> 16\, m_\pi^2$ *???*



Q(s) x 10^8
sqrt(s) (GeV)
full
$\propto \lambda_4^{\rm eff} \times \phi_4(s)$
2
1.5
1
0.5
0
-0.5
-1
-1.5
0.6
0.8
1
1.2
1.4
1.6
1.8
2
2.2
2.4

# SUNSET DIAGRAM




$$\mathrm{Im}\, I \propto \frac{1}{2} \int d\phi_3 \left|\Gamma_{s \to \pi\pi\pi}\right|^2$$

# SUMMARY

- change in density of state / time delay due to interaction

$$2\frac{d\delta}{dE}$$

- S-matrix approach to thermodynamics
- Extend to N-body with phase space expansion

# TO DO LIST…

- Exotics, cusp effects …
- For dense(r) medium

# END OF LECTURE I & II

# LECTURE III

# INTRODUCTION TO FUNCTIONAL METHOD AND SCHWINGER DYSON EQUATIONS

ON THE BOARD…

# EXECUTIVE SUMMARY

$$\mathcal{L} = \frac{1}{2}(\partial\phi)^2 - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4}\phi^4$$

$$Z\left[j(x)\right] = \int D\phi\, e^{i\int(\mathcal{L}+j(x)\phi(x))}$$

$$W\left[j\right] = -i\ln Z\left[j\right]$$

$$\Gamma[\phi] = W - \int j\phi$$

# EXECUTIVE SUMMARY

*Master equation*

$$0 = \int D\phi \, \frac{\delta}{\delta\phi} e^{i\int(\mathcal{L}+j\phi)}$$

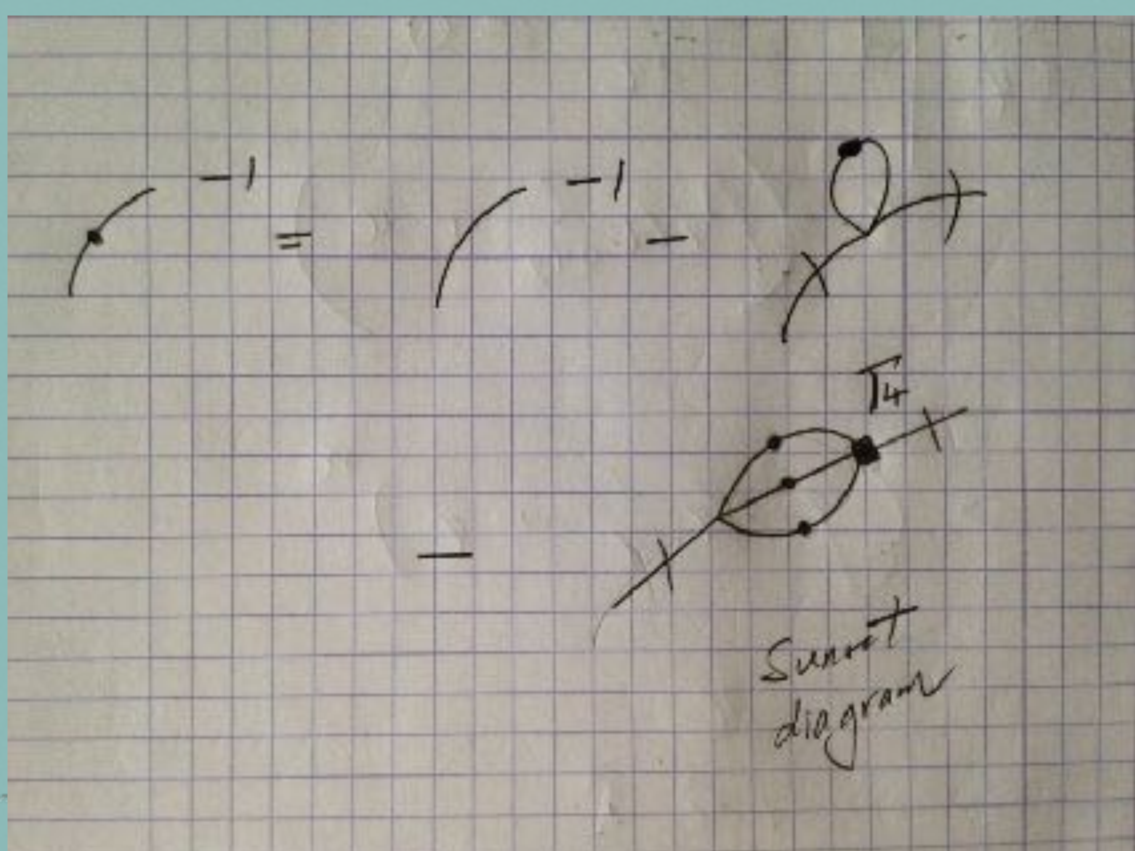
$$\left(\frac{\delta S}{\delta\phi} + j\right) Z[j] = 0 \qquad \textit{with} \qquad \phi \to -i\frac{\delta}{\delta j}$$

$$\left(\frac{\delta S}{\delta\phi} + j\right) I = 0 \qquad \textit{with} \qquad \phi \to -i\frac{\delta}{\delta j} + \frac{\delta W}{\delta j}$$

$$\left(\frac{\delta S}{\delta\phi} + j\right) I = 0 \qquad \textit{with} \qquad \phi \to \int i\,G\,\frac{\delta}{\delta\phi} + \phi$$

# EXECUTIVE SUMMARY

$$G^{-1} = -(\partial^2 + m^2)\delta - \frac{\lambda}{2} iG(x,x)\delta +$$
$$- \frac{\lambda}{6} \int G(x,z)G(x,z)G(x,z)\Gamma_4(z,z,z,y)$$



# THANK YOU

# T-MATRIX REPRESENTATION

$$\frac{1}{4\,i}\,\mathrm{tr}\,\left[S^{-1}\,\overleftrightarrow{\frac{\partial}{\partial E}}\,S\right]_c \longleftrightarrow \frac{\partial \delta_E}{\partial E}$$

$$\frac{1}{4}\,\frac{\partial}{\partial E}\,\mathrm{tr}\,\left[T + T^\dagger\right]_c \longleftrightarrow (1 - 2\sin^2\delta_E) \times \frac{\partial \delta_E}{\partial E}$$

$$\frac{1}{4\,i}\,\mathrm{tr}\,\left(T^\dagger\,\overleftrightarrow{\frac{\partial}{\partial E}}\,T\right)_c \longleftrightarrow 2\sin^2\delta_E \times \frac{\partial \delta_E}{\partial E}.$$

*Landau Lifshitz classification*

# T-MATRIX REPRESENTATION

$$B(M) = A(M) + \delta\rho\,(M).$$

$$B(M) = 2\,\frac{\partial}{\partial M} Q(M)$$

$$A(M) = -2M\,\frac{\sin 2Q(M)}{M^2 - \bar{m}_{\text{res}}^2}$$

*Modern*

# T-MATRIX REPRESENTATION