One-loop divergences in $4D, \mathcal{N} = 2$ harmonic superfield sigma-model

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Alexandra Budekhina

Tomsk State Pedagogical University

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1. Nonlinear sigma-models is an interesting object of study due to their remarkable properties, namely because of the intimate connection with the differential geometry.
2. The divergences of the effective action in four-dimensional N=1 supersymmetric sigma-models are studied in work of Spence in the case of vanishing (anti-)chiral potentials and in the work made by (A.T. Banin , I.L. Buchbinder , N.G. Pletnev, 2006) in the general case in harmonic superspace.
3. Some assumptions about a structure of the possible one-loop divergences in N=2 sigma-models on the base of N=1 divergences were considered in the work of (Spence,1985).
4. The harmonic superspace was originally developed by (A.Galperin, E. Ivanov, S. Kalitsyn, V. Ogievetsky, E. Sokatchev, 1985)
5. There are two types of hypermultiplets in harmonic superspace, the q-hypermultiplet and ω-hypermultiplet (A.Galperin, E. Ivanov, V. Ogievetsky, E. Sokatchev, Harmonic Superspace, 2001).
6. The derivation of the one-loop divergent contributions to the effective action of N=2 supersymmetric sigma model in manifestly covariant and N=2 supersymmetric manner was not held directly in terms of N=2 superfields.

Plan

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Description of the model

The central basis coordinates of the $\mathcal{N}=2$ harmonic superspace (A.Galperin, E. Ivanov, V. Ogievetsky, E. Sokatchev, Harmonic Superspace, 2001)

$$(z,u)=(x^M,\theta_i^\alpha,\bar\theta_{\dot\alpha}^i,u^{\pm i}),\qquad M=0,..,3,\quad \alpha=1,2\quad i=1,2. \tag{1}$$

The analytic harmonic superspace coordinates

$$(\zeta,u)=(x_{\mathcal{A}}^M,\theta_\alpha^+,\bar\theta_{\dot\alpha}^+,u^{\pm i}),\qquad x_{\mathcal{A}}^M=x^M-2i\theta^{(i}\sigma^M\bar\theta^{j)}u_i^+u_j^-,\quad \theta^{+\alpha}=u_i^+\theta^{\alpha i}. \tag{2}$$

The classical action for the model

$$S[\omega]=\int d\zeta^{(-4)}\left(-\frac{1}{2}g_{ab}(\omega)D^{++}\omega^aD^{++}\omega^b+L_a^{++}(\omega)D^{++}\omega^a+L^{(+4)}(\omega)\right), \tag{3}$$

where $a=1,..,n$. The target space metric g_{ab}, and L_a^{++} and $L^{(+4)}$ are the arbitrary analytic functions of the ω^a-superfields.
This action is invariant under reparameterizations transformations

$$\omega^a\to\omega^a+\lambda^a(\omega,u)\,. \tag{4}$$

in the assumption that superfields g_{ab}, L_a^{++} and $L^{(+4)}$ transform under (4) as a tensor of the corresponding rank.

Background-quantum splitting

Due to the manifestly covariant background field formalism we introduce the analytic superfield $\rho^a(s)$ that satisfies the harmonic superspace geodesic equation

$$\frac{d^2\rho^a(s)}{ds^2} + \Gamma^a{}_{bc}(\rho)\frac{d\rho^b(s)}{ds}\frac{d\rho^c(s)}{ds} = 0, \tag{5}$$

with the conditions

$$\rho^a(0) = \Omega^a, \quad \rho^a(1) = \Omega^a + \pi^a, \quad \frac{d}{\rho^a}ds\Big|_{s=0} = \xi^a\,. \tag{6}$$

The solution to the equation (5) reads

$$\rho^a(s) = \Omega^a + \sum_{n=1}^{\infty}\frac{s^n}{n!}\rho^a_{(n)}\,. \tag{7}$$

The decomposition of the classical action (3) under (7)

$$S[\rho] = S[\Omega] + \sum_{n=1}^{\infty}\frac{1}{n!}\frac{d^nS[\rho]}{ds^n}\Big|_{s=0} = S[\Omega] + S_1 + S_2 + \ldots\,. \tag{8}$$

will be manifestly covariant.

The explicit expression for S_2 is written as follows

$$S_2 = \frac{1}{2}\int d\zeta^{-4}\,\xi^a\Big(g_{cd}(\nabla^{++})^c_a(\nabla^{++})^d_b - R^d{}_{abc}\,D^{++}\Omega^c D^{++}\Omega_d + \nabla_{(a}L^{++}_c(\nabla^{++})^c_{b)}$$
$$+\nabla_a\nabla_b L^{++}_c D^{++}\Omega^c + L^{++}_d R^d{}_{abc}D^{++}\Omega^c + \nabla_a\nabla_b L^{(+4)}\Big)\xi^b, \quad (9)$$

where we have introduced the harmonic covariant derivative

$$(\nabla^{++}\xi)^a = (\nabla^{++})^a{}_b\xi^b = D^{++}\xi^a + \Gamma^a{}_{bc}(\Omega)D^{++}\Omega^c\xi^b, \quad (10)$$

and ∇_a is a covariant derivative along the curve $\rho^a(s)$ in the target space.

The one-loop quantum correction $\Gamma^{(1)}$ to the classical action (3)

$$\Gamma^{(1)}[\Omega] = \frac{i}{2}\,\mathrm{Tr}_{\,(4,0)}\ln\Big((\nabla^{++})^2 + (\nabla L^{++})\nabla^{++} + X^{(+4)}\Big), \tag{11}$$

where the operator $(\nabla L^{++})\nabla^{++}$ means $\nabla_{(a}L_c^{++}(\nabla^{++})^c{}_{b)}$. The expression $X^{(+4)}$ is written as follows

$$\begin{aligned} X^{(+4)}_{ab} \quad &= \quad -R^d{}_{abc}D^{++}\Omega^c D^{++}\Omega_d + \nabla_a\nabla_b L_c^{++}D^{++}\Omega^c \\ &\quad + L_d^{++}R^d{}_{abc}D^{++}\Omega^c + \nabla_a\nabla_b L^{(+4)}. \end{aligned} \tag{12}$$

We introduce the new covariant derivative $\mathcal{D}^{++} = D^{++} + \mathcal{V}^{++} = \nabla^{++} + \tilde{\Gamma}^{++}$ in terms of new analytic connection $\mathcal{V}^{++} = \Gamma^{++} + \tilde{\Gamma}^{++}$. Here $(\Gamma^{++})^a{}_b = \Gamma^a{}_{bc}(\Omega)D^{++}\Omega^c$ and $(\tilde{\Gamma}^{++})^a{}_b = g^{ac}\nabla_c L_b^{++}$. Then (11) reads

$$\Gamma^{(1)} = \frac{i}{2}\,\mathrm{Tr}_{\,(4,0)}\ln\Big((\mathcal{D}^{++})^2 + \tilde{X}^{(+4)}\Big), \tag{13}$$

$$\begin{aligned} \tilde{X}^{(+4)}_{ab} \quad &= \quad -R^d{}_{abc}D^{++}\Omega^c D^{++}\Omega_d + L_d^{++}R^d{}_{abc}D^{++}\Omega^c \\ &\quad -{}^cL_a^{++}{}_cL_b^{++} - (\nabla^{++})^c_a{}_c L_b^{++} + \nabla_a\nabla_b L_c^{++}D^{++}\Omega^c + \nabla_a\nabla_b L^{(+4)}. \end{aligned} \tag{14}$$

The zero curvature condition

$$\left[(\mathcal{D}^{++})^a_c, (\mathcal{D}^{--})^c_b\right] = \delta^a_b D_0. \tag{15}$$

Assuming $\mathcal{D}^{--} = D^{--} + \mathcal{V}^{--}$ we obtain

$$\mathcal{V}^{--} = \sum_{n=1}^{\infty} (-1)^n \int du_1 \dots du_n \frac{\mathcal{V}_1^{++}\mathcal{V}_2^{++} \dots \mathcal{V}_n^{++}}{(u^+u_1^+)\dots(u_n^+u^+)}. \tag{16}$$

Algebra of covariant derivatives like in $\mathcal{N}=2$ SYM theory (A.Galperin, E. Ivanov, V. Ogievetsky, E. Sokatchev, Harmonic Superspace, 2001)

$$\begin{aligned}
&\{D^+_\alpha, \mathcal{D}^-_\beta\} = 2\varepsilon_{\alpha\beta}\bar{\mathcal{W}}, \qquad \{\bar{D}^+_{\dot\alpha}, \bar{\mathcal{D}}^-_{\dot\beta}\} = 2\varepsilon_{\dot\alpha\dot\beta}\mathcal{W}, \\
&\{\bar{D}^+_{\dot\alpha}, \mathcal{D}^-_\alpha\} = -\{D^+_\alpha, \bar{\mathcal{D}}^-_{\dot\alpha}\} = 2i\mathcal{D}_{\alpha\dot\alpha}, \\
&\left[D^+_\alpha, \mathcal{D}_{\beta\dot\beta}\right] = \bar{D}^+_{\dot\beta}\varepsilon_{\alpha\beta}\bar{\mathcal{W}}, \qquad \left[\mathcal{D}^-_\alpha, \mathcal{D}_{\beta\dot\beta}\right] = \bar{\mathcal{D}}^-_{\dot\beta}\varepsilon_{\alpha\beta}\bar{\mathcal{W}}, \\
&\left[\bar{D}^+_{\dot\alpha}, \nabla_{\beta\dot\beta}\right] = D^+_\beta\varepsilon_{\dot\alpha\dot\beta}\mathcal{W}, \qquad \left[\bar{\mathcal{D}}^-_{\dot\alpha}, \mathcal{D}_{\beta\dot\beta}\right] = \mathcal{D}^-_\beta\varepsilon_{\dot\alpha\dot\beta}\mathcal{W} \\
&[\mathcal{D}^{++}, \bar{\mathcal{D}}^-_{\dot\alpha}] = \bar{D}^+_{\dot\alpha}, \qquad [\mathcal{D}^{--}, D^+_\alpha] = \mathcal{D}^-_\alpha\ .
\end{aligned} \tag{17}$$

Here we have denoted

$$\begin{aligned}
&\mathcal{D}^-_\alpha = D^-_\alpha - D^+_\alpha\mathcal{V}^{--}, \qquad \mathcal{D}_{\alpha\dot\alpha} = \partial_{\alpha\dot\alpha} - \tfrac{i}{2}D^+_\alpha\bar{D}^+_{\dot\alpha}\mathcal{V}^{--}, \\
&\bar{\mathcal{W}} = (D^+)^2\mathcal{V}^{--}, \quad \mathcal{W} = (\bar{D}^+)^2\mathcal{V}^{--}.
\end{aligned} \tag{18}$$

To calculate the effective action, we represent it in the form

$$\Gamma^{(1)}[\Omega] = i \,\mathsf{Tr}_{(2,2)} \ln \mathcal{D}^{++} + \tfrac{i}{2} \,\mathsf{Tr}_{(4,0)} \ln \Big(\mathbf{1} + G^{(0,0)} \tilde{X}^{+4} \Big), \tag{19}$$

where the Green function $G^{(0,0)}$ satisfies the equation

$$(\mathcal{D}_1^{++})^2 G^{(0,0)}(1,2) = \delta_{\mathcal{A}}^{(4,0)}(1,2). \tag{20}$$

Explicit solution of this equation has the form (A.Galperin, E. Ivanov, V. Ogievetsky, E. Sokatchev, Harmonic Superspace, 2001)

$$G^{(0,0)}(1,2) = -\frac{1}{\widehat{\Box}_1} (D_1^+)^4 (D_2^+)^4 \delta^{12}(z_1 - z_2) \frac{(u_1^- u_2^-)}{(u_1^+ u_2^+)^3}, \tag{21}$$

where $\delta^{12}(z_1 - z_2)$ is a full $\mathcal{N} = 2$ superspace delta-function and the analytic covariant d'Alembertian

$$\begin{aligned} \widehat{\Box} &= \tfrac{1}{2} (D^+)^4 (\mathcal{D}^{--})^2 \\ &= \mathcal{D}_M \mathcal{D}^M - \tfrac{1}{4} (D^+)^2 \mathcal{W} \mathcal{D}^{--} - \tfrac{1}{2} D_\alpha^+ \mathcal{W} \mathcal{D}^{-\alpha} - \tfrac{1}{2} \bar{D}_{\dot\alpha}^+ \bar{\mathcal{W}} \bar{\mathcal{D}}^{-\dot\alpha} \\ &\quad + \tfrac{1}{8} \mathcal{D}_\alpha^- D^{+\alpha} \mathcal{W} - \tfrac{1}{2} \mathcal{W} \bar{\mathcal{W}}. \end{aligned} \tag{22}$$

We use the proper-time representation for the operator $\widehat{\Box}^{\,-1}$ in the Green function (21)

$$\frac{1}{\widehat{\Box}_1} = \int_0^\infty d(is)(is\mu^2)^{\frac{\varepsilon}{2}} e^{-is\widehat{\Box}_1} \tag{23}$$

The divergent contribution of the effective action (19)

$$\begin{aligned}\Gamma^{(1)}_{\rm div} \quad &= \quad \frac{1}{2(4\pi)^2\varepsilon}\,\mathrm{tr}\int d^8z\,\mathcal{W}^2 \\ &\quad -\frac{1}{4(4\pi)^2\varepsilon}\int d^{12}z\,du_1du_2\,\frac{(u_1^-u_2^-)^2}{(u_1^+u_2^+)^2}\,\tilde{X}^{(+4)ab}(1)\,\tilde{X}^{(+4)}_{ba}(2).\end{aligned} \tag{24}$$

Special cases

Assumption $L_a^{++} = 0$ and $L^{(+4)} = 0$. The divergent contribution

$$\begin{aligned}\Gamma^{(1)}_{R,\,\mathrm{div}}[\Omega] &= \frac{1}{2(4\pi)^2\varepsilon}\,\mathrm{tr}\int d^8z\, W^2 \\ &\quad -\frac{1}{4(4\pi)^2\varepsilon}\int d^{12}z\, du_1 du_2\, \frac{(u_1^- u_2^-)^2}{(u_1^+ u_2^+)^2} R^{cab}{}_d(1) R^e{}_{bak}(2) \\ &\quad \times D^{++}\Omega^d(1) D^{++}\Omega_c(1) D^{++}\Omega^k(2) D^{++}\Omega_e(2). \end{aligned} \tag{25}$$

The superfield connection $\mathcal{V}^{++}$ coincides with the Levi-Civita analytic connection $(\Gamma^{++})^a_b = \Gamma^a{}_{bc}(\Omega) D^{++}\Omega^c$.
Assumption the background metric, $g_{ab}(\Omega) = h_{ab}$, does not depend on the superfield Ω and superspace point z. The divergent contribution

$$\begin{aligned}\Gamma^{(1)}_{L,\,\mathrm{div}} &= \frac{1}{2(4\pi)^2\varepsilon}\,\mathrm{tr}\int d^8z\, \tilde{W}^2 \\ &\quad -\frac{1}{4(4\pi)^2\varepsilon}\int d^{12}z\, du_1 du_2\, \frac{(u_1^- u_2^-)^2}{(u_1^+ u_2^+)^2}\, L^{(+4)}_{ab}(1)\, L^{(+4)\,ba}(2), \end{aligned} \tag{26}$$

where

$$\begin{aligned}\nabla^{++}_{ab} &= h_{ab} D^{++} + \tilde{\Gamma}^{++}_{ab}, \\ L^{(+4)}_{ab} &= \partial_a\partial_b L^{(+4)} + \partial_a\partial_b L^{++}_c D^{++}\Omega^c - D^{++}\tilde{\Gamma}^{++}_{ba} + (\tilde{\Gamma}^{++})^2_{ab}. \end{aligned} \tag{27}$$

The component structure of divergent contribution in bosonic sector

$$\Gamma^{(1)}_{\rm div}[\omega] = -\frac{1}{128\pi^2\varepsilon}\int d^4x\, R^{cab}{}_d R^e{}_{bak}\,(\nabla_{\alpha\dot\alpha}\omega)^d\,(\nabla^{\alpha\dot\alpha}\,\omega)_c(\nabla_{\beta\dot\beta}\omega)^k(\nabla^{\beta\dot\beta}\omega)_e + \dots, \qquad (28)$$

1. The manifestly covariant approach for studying the quantum structure of the general $\mathrm{N} = 2$ supersymmetric sigma-model in four dimensions was developed.
2. 4D,$\mathrm{N} = 2$ supersymmetric sigma-model (3), is formulated in $\mathrm{N} = 2$ harmonic superspace in terms of analytic omega-hypermultiplet superfields.
3. The one-loop effective action for such a model is constructed in the framework of the manifestly covariant and manifestly $\mathrm{N} = 2$ supersymmetric background-quantum splitting in $\mathrm{N} = 2$ harmonic superspace .
4. One loop divergent contributions to the effective action was constructed for arbitrary background hypermultiplet Ω with the use of proper-time technique.
5. The one-loop divergences in two special cases were calculated.

Thank you for your attention!