Multipoint Conformal Blocks and their Vertex Integrable Systems

with I. Burić, S. Lacroix, L. Quintavalle & V. Schomerus

based on:

– **[2112.10827]**,
– **[2108.00023]**, *JHEP* 11 (2021) 182,
– **[2009.11882]**, *JHEP* 10 (2021) 139,
– **[1910.10427]**, *Phys.Rev.Lett.* 126 (2021) 2, 021602.

Motivation

Correlation functions in conformal field theory

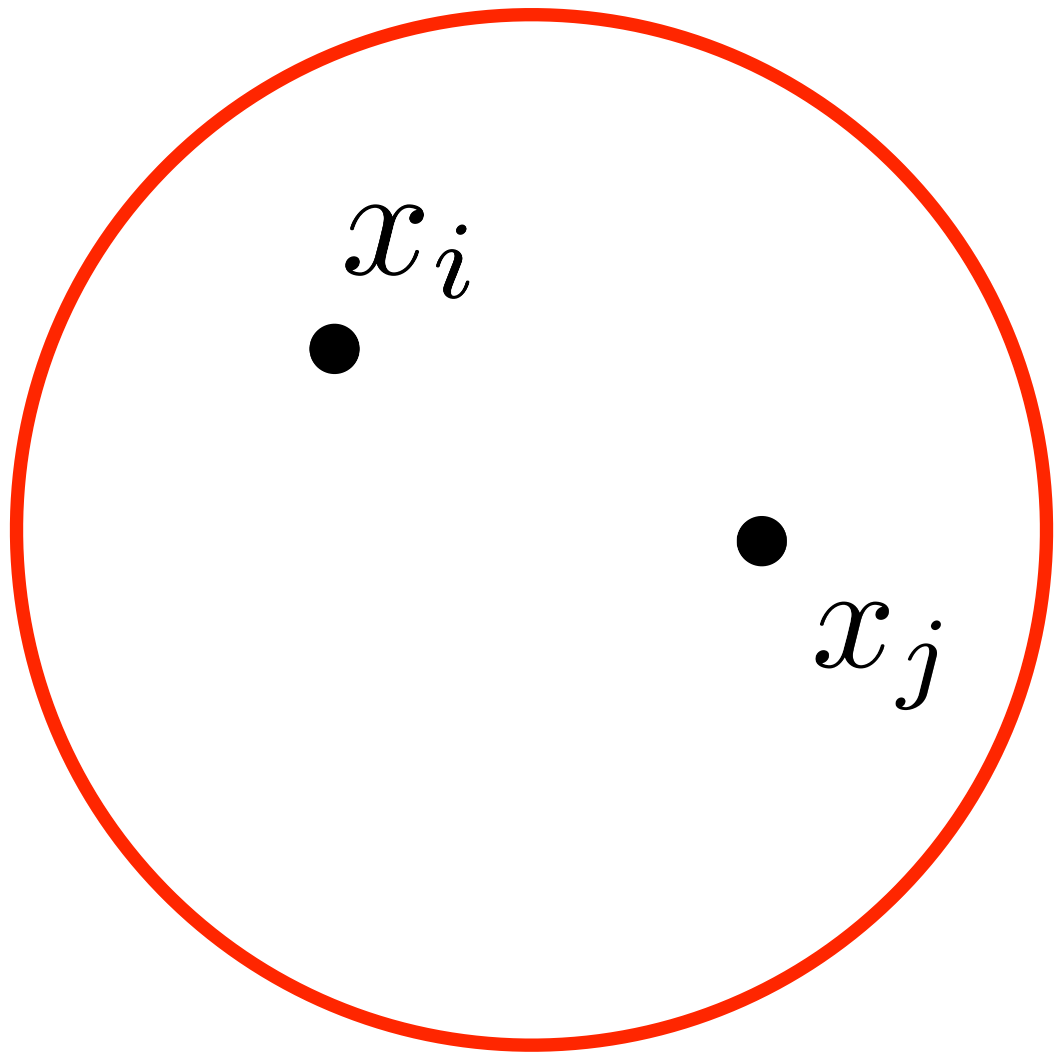
- Ingredients of a CFT in $\mathbb{R}^d$:

1. **Primary** fields: $x \to x + \xi(x), \quad \mathrm{d}s^2 \to (1 - \omega_\xi(x))\mathrm{d}s^2, \quad {\xi_A}^B \in \mathfrak{so}(1, d+1)$

$$\mathcal{O}_i(x_i) \to \mathcal{O}_i(x_i) + {\xi_A}^B \underline{L_i(x_i, \partial_{x_i})^A{}_B}\, \mathcal{O}_i(x_i)$$

2. **Operator product expansion:**

$$\boxed{\mathcal{O}_i(x_i)\mathcal{O}_j(x_j) = \sum_{[\mathcal{O}_k] \in [\mathcal{O}_i] \otimes [\mathcal{O}_j]} \sum_{n=1}^{\text{multiplicity}} C_{ijk}^{(n)} f_{ijk}^{(n)}(x_i - x_j, \partial_{x_j}) \mathcal{O}_k(x_j)}$$



Motivation

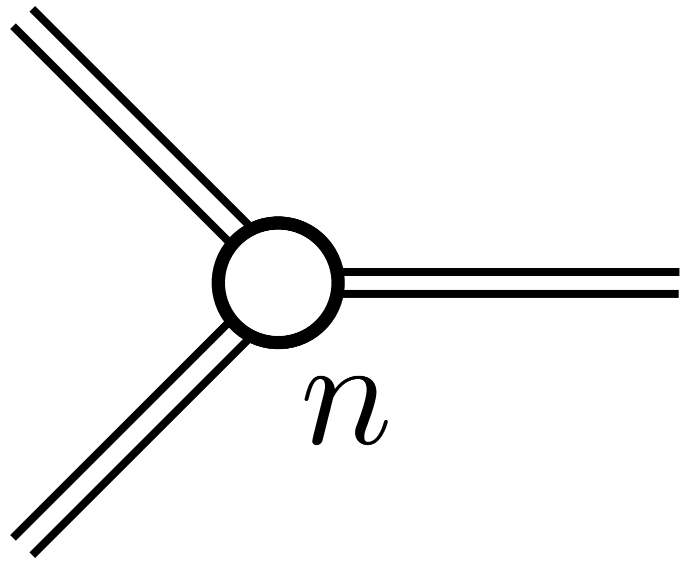
Correlation functions in conformal field theory

- **OPE** determines **all** correlation functions

$$\langle \mathcal{O}_1 \rangle = \begin{cases} 1 & \text{if} \quad \mathcal{O}_1 = \mathbf{1} \\ 0 & \text{otherwise} \end{cases}$$

$$\langle \widehat{\mathcal{O}_1 \mathcal{O}_2} \rangle = \begin{cases} \overset{=1}{C_{\mathcal{O}_1\mathcal{O}_1\mathbf{1}}} f_{\mathcal{O}_1\mathcal{O}_1\mathbf{1}}(x_1 - x_2) & \text{if} \quad \mathcal{O}_1 = \mathcal{O}_2 \\ 0 & \text{otherwise} \end{cases}$$

$$\langle \widehat{\mathcal{O}_1 \mathcal{O}_2} \mathcal{O}_3 \rangle = \sum_n C_{123}^{(n)} \underline{f_{123}^{(n)}(x_1 - x_2, \partial_{x_2}) f_{33\mathbf{1}}(x_2 - x_3)} = \sum_n C_{123}^{(n)}$$



Motivation

Correlation functions in conformal field theory

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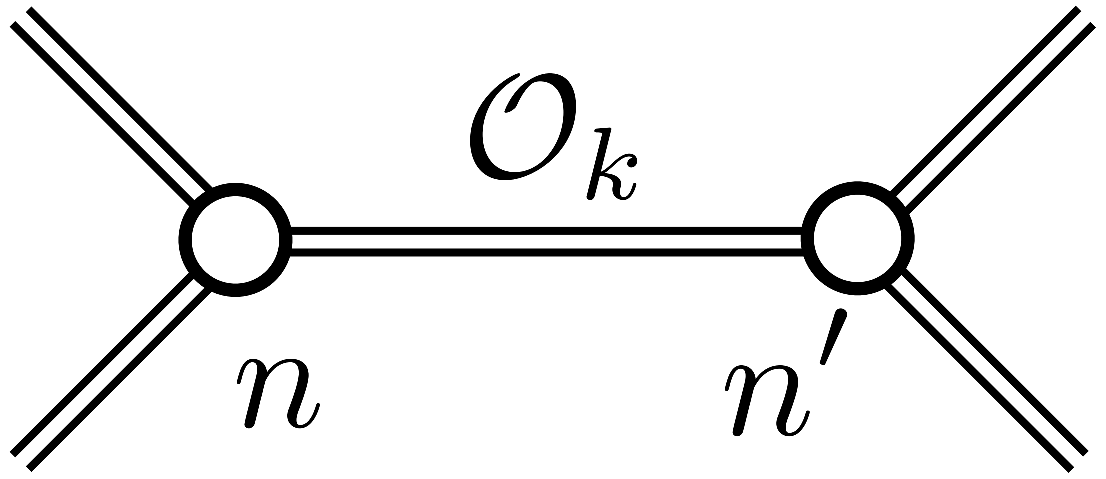
$$\langle \mathcal{O}_1 \mathcal{O}_2 \mathcal{O}_3 \mathcal{O}_4 \rangle = \sum_{\mathcal{O}_k, n, n'} C^{(n)}_{12k} C^{(n')}_{k34}$$



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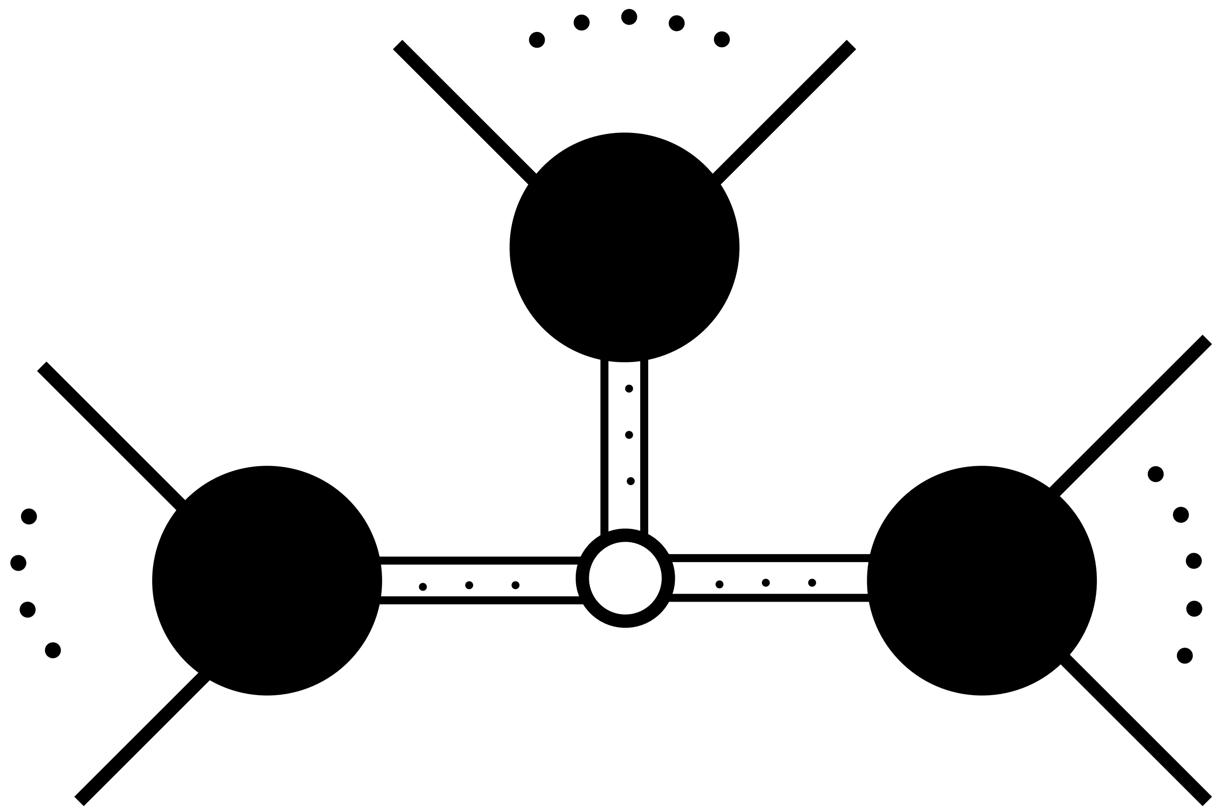
$$\langle \mathcal{O}_1 \dots \mathcal{O}_N \rangle = \sum \underbrace{C \dots C}_{\times (N-2)}$$



Motivation

Correlation functions in conformal field theory

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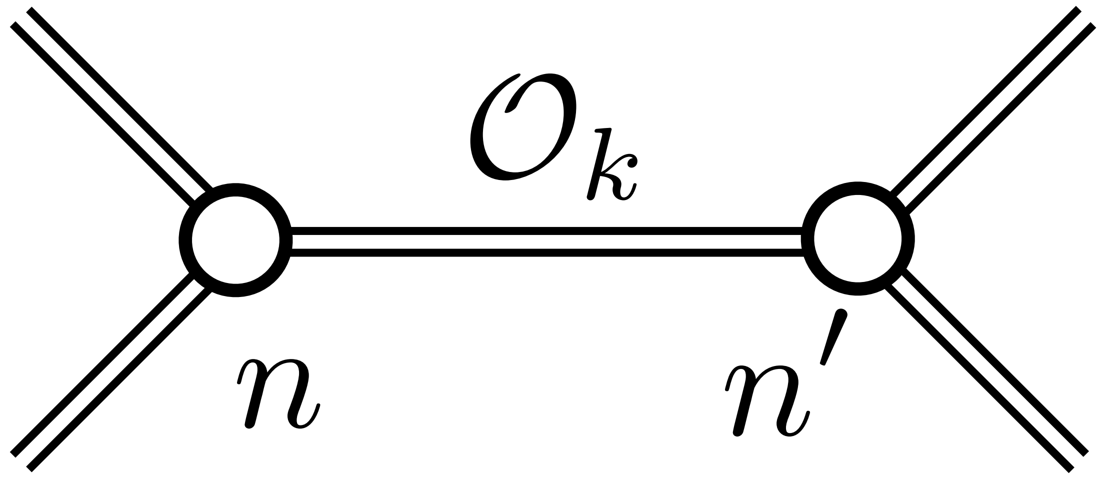

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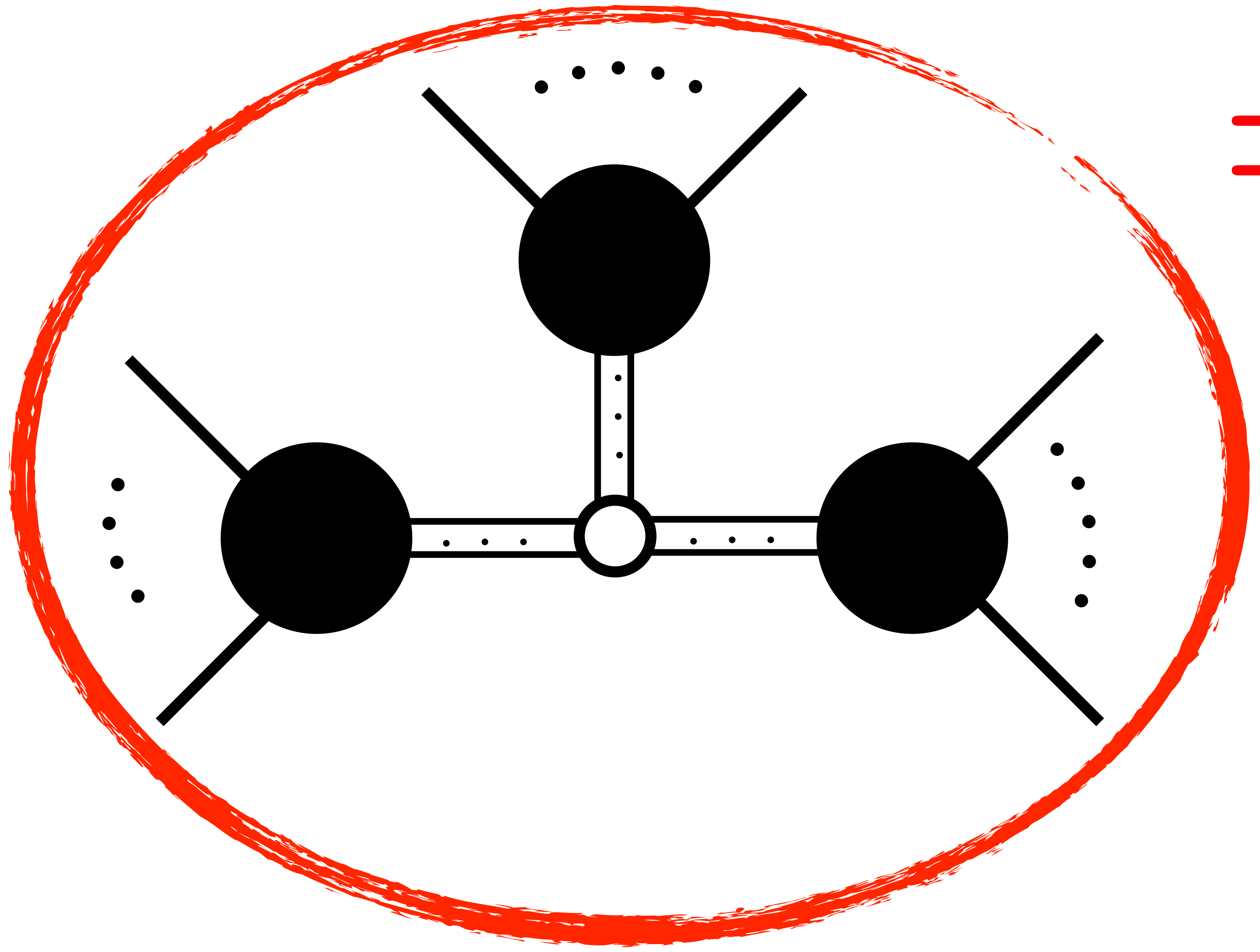
$$\langle \mathcal{O}_1 \ldots \mathcal{O}_N\rangle = \sum \underbrace{C \ldots C}_{\times(N-2)}$$



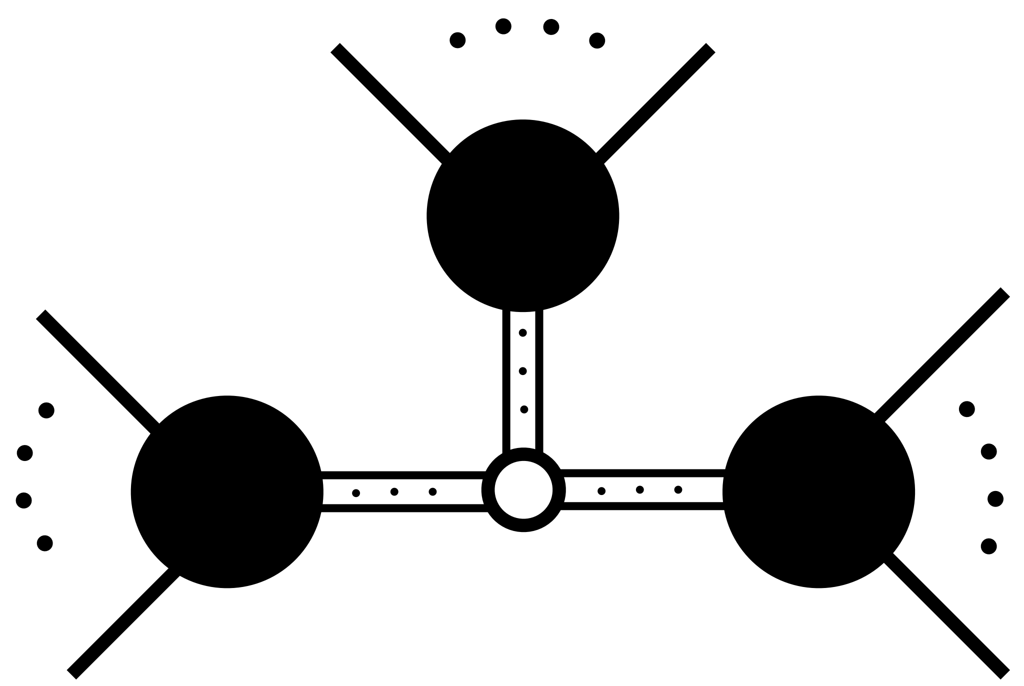
$=?$

Integrability based approach

N-point Gaudin models

[1910.10427]
[2009.11882]

basis of N-point blocks = eigenbasis of Gaudin Hamiltonians



$$H_I \psi = E_I \psi \qquad [H_I, H_J] = 0$$

Integrability based approach

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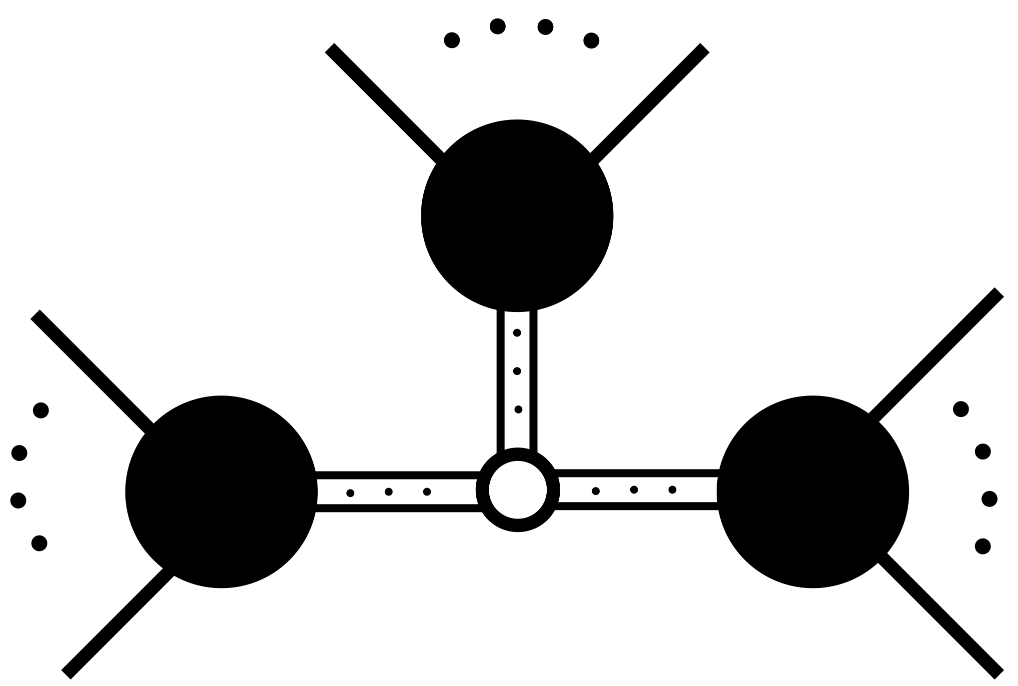


$$H_I \psi = E_I \psi \qquad [H_I, H_J] = 0$$

$$\sum_{i_1,\ldots,i_p=1}^{N} c^{(I)}_{i_1\ldots i_p} \operatorname{tr} L_{i_1}(x_{i_1}, \partial_{x_{i_1}}) \ldots L_{i_1}(x_{i_p}, \partial_{x_{i_p}})$$

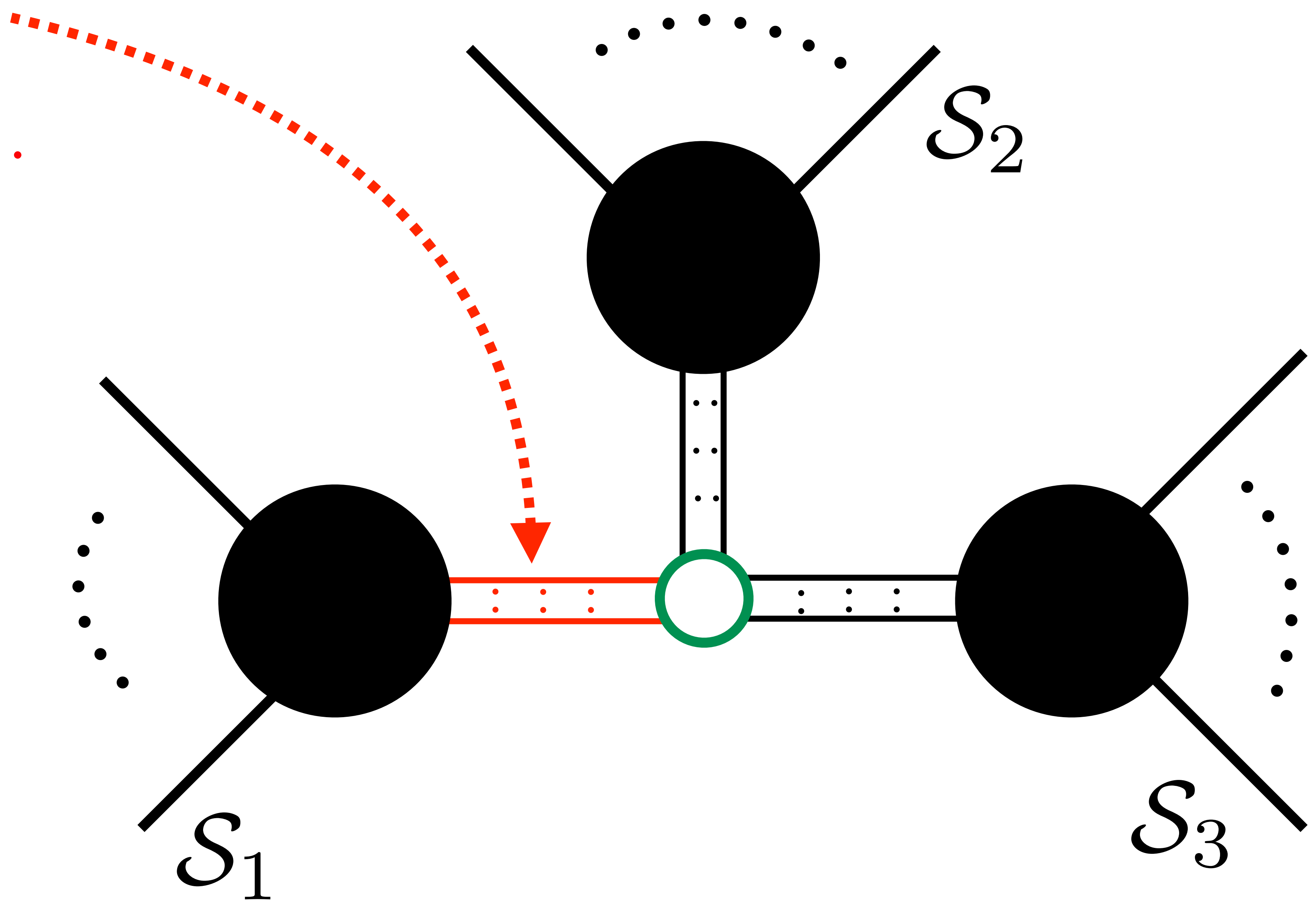
Integrability based approach
N-point Gaudin models

[1910.10427]
[2009.11882]

basis of N-point blocks $=$ eigenbasis of Gaudin Hamiltonians

$$\left\{ \mathrm{tr} \left(\sum_{i \in \mathcal{S}_1} L_i \right)^p \right\}_{p=2,4,\ldots}$$

$\Rightarrow$ **quantum numbers of** $\mathcal{O}_k$



Integrability based approach

N-point Gaudin models

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basis of N-point blocks = eigenbasis of Gaudin Hamiltonians

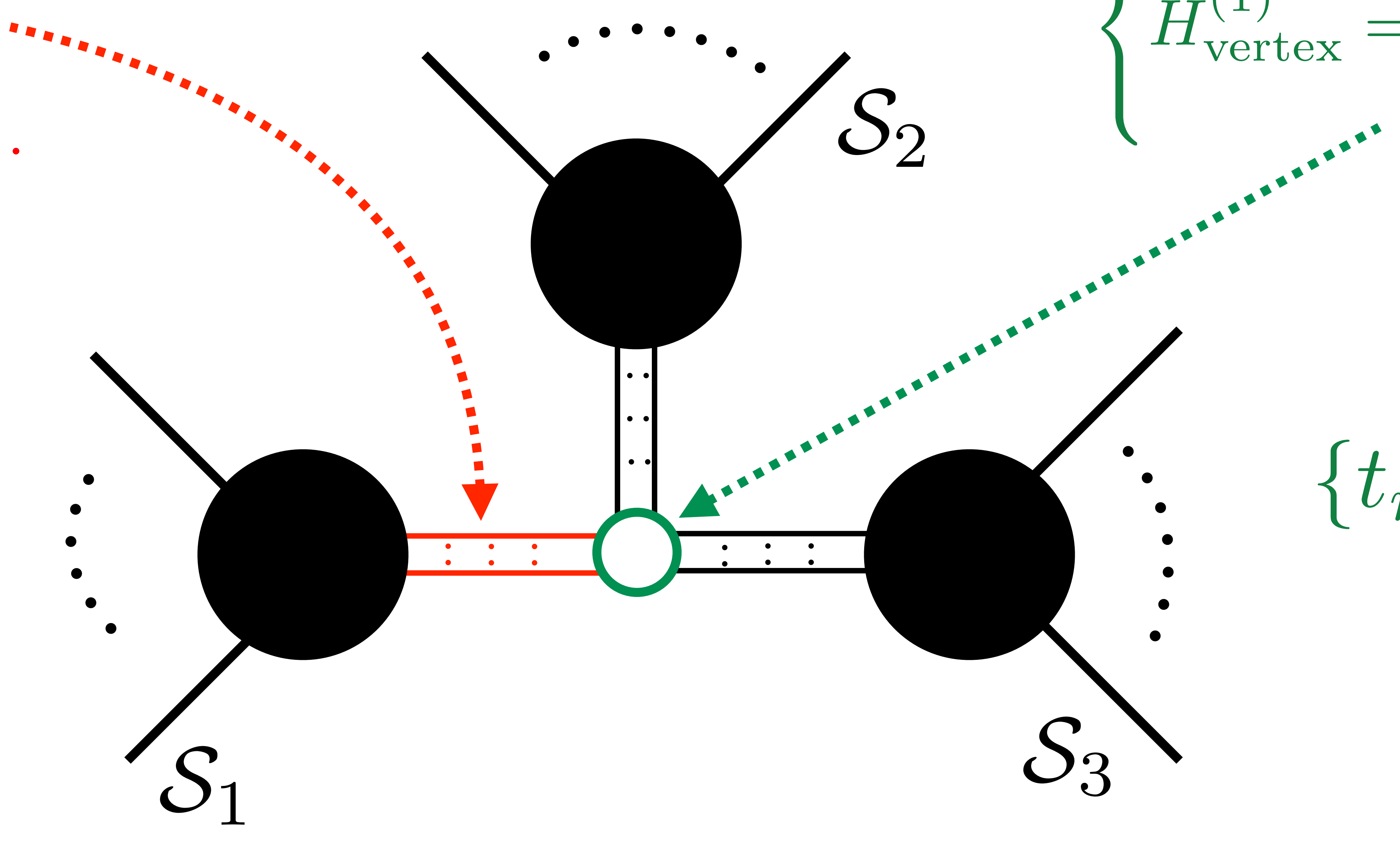
$$\left\{ \operatorname{tr}\left(\sum_{i\in\mathcal{S}_1} L_i \right)^p \right\}_{p=2,4,\ldots}$$

$\Rightarrow$ **quantum numbers of $\mathcal{O}_k$**

$$\left\{ H^{(1)}_{\text{vertex}} = \operatorname{tr}\left(\sum_{i\in\mathcal{S}_1} L_i \right)^3 \left(\sum_{i\in\mathcal{S}_2} L_i \right), \ldots \right\}$$

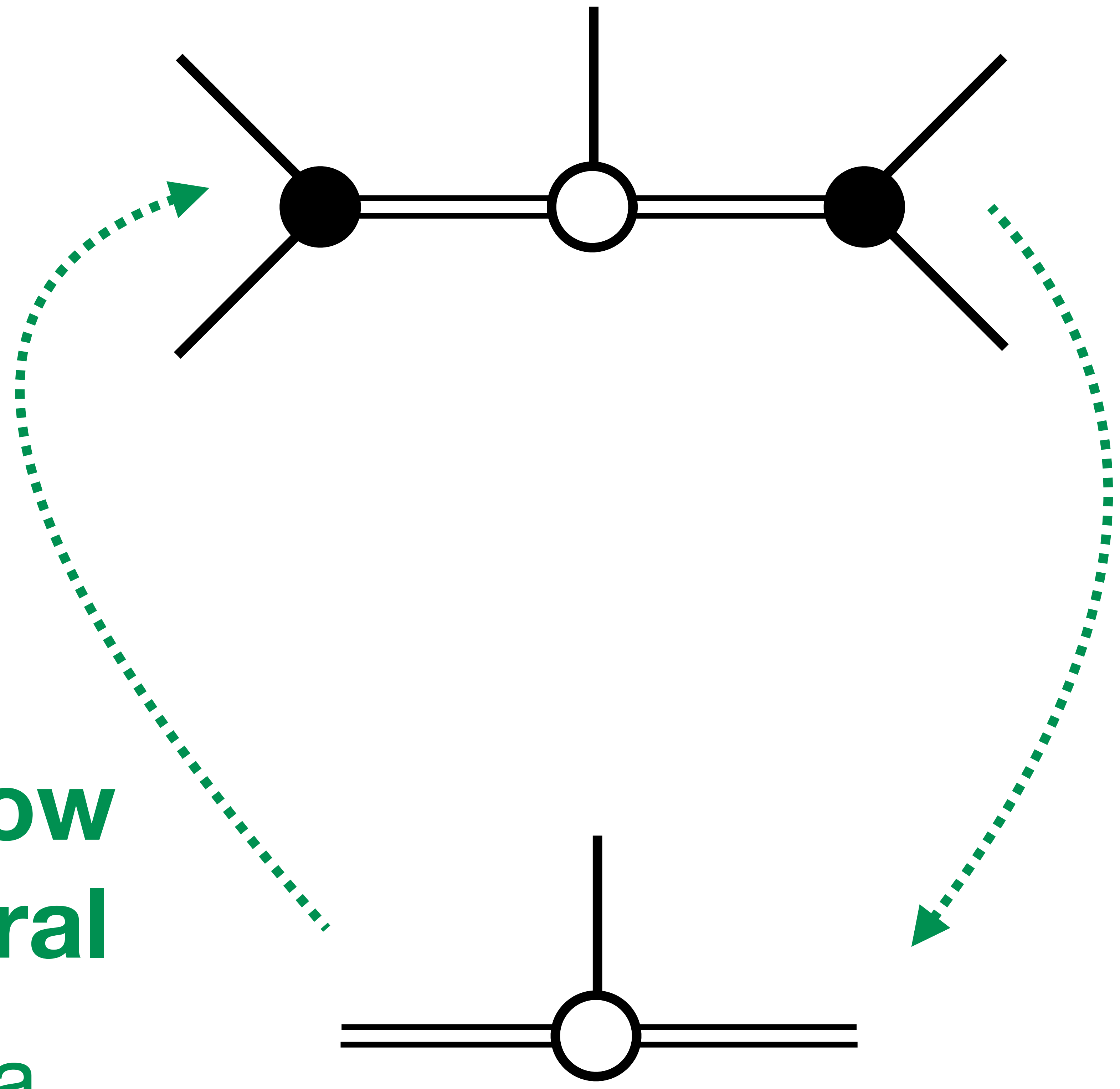
$\Rightarrow$ **basis of**

$$\{t_n : [\mathcal{O}_k] \hookrightarrow [\mathcal{O}_i] \otimes [\mathcal{O}_j]\}$$



Integrability based approach

From N-point blocks to vertex systems and back



shadow integral

[Ferrara, Gatta, Grillo, Parisi 1972]

OPE limits

$x_i \to x_j$

Scalar 5-point Gaudin

Hamiltonian reduction

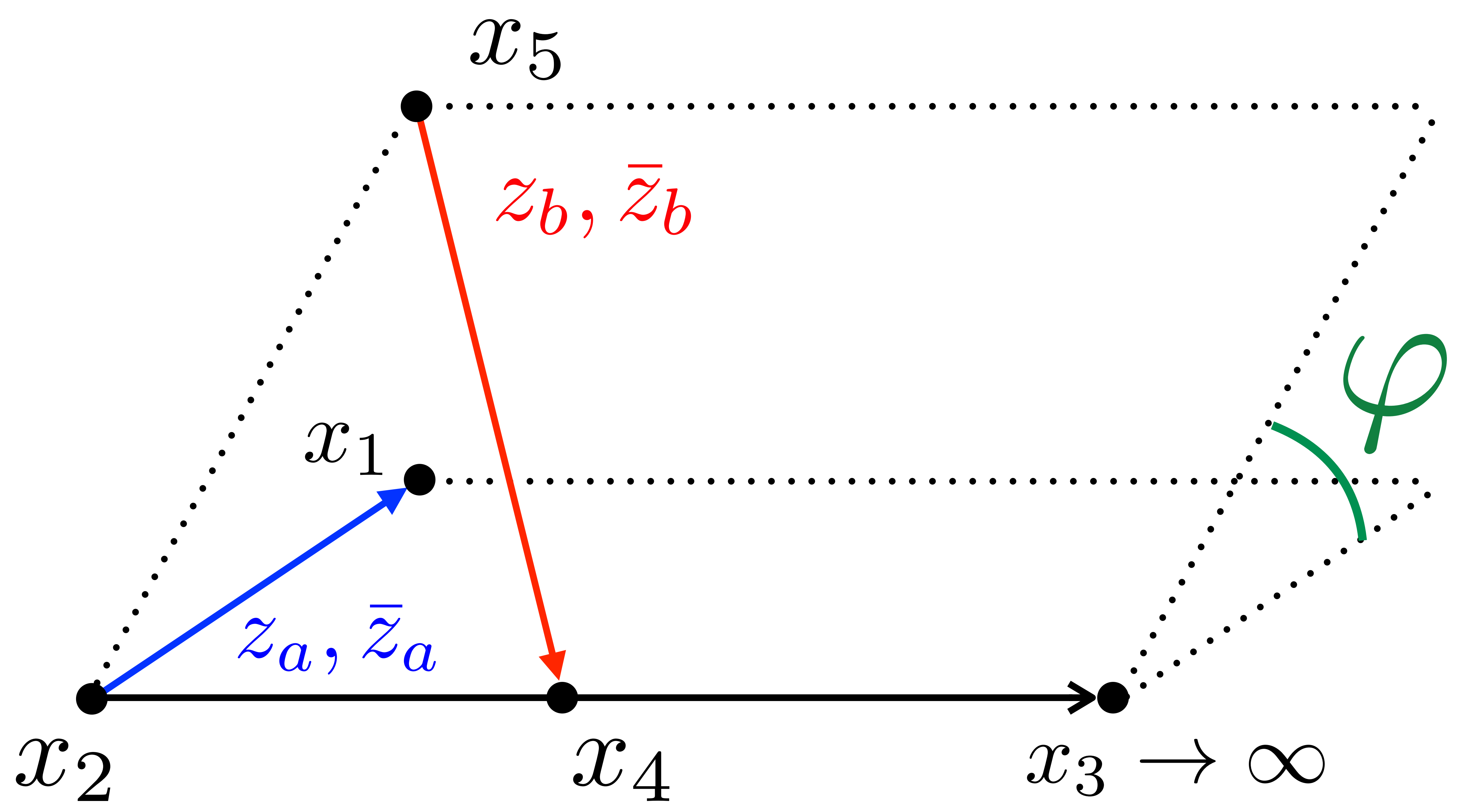
Spinning 3-point Gaudin

= **Vertex system**

Example

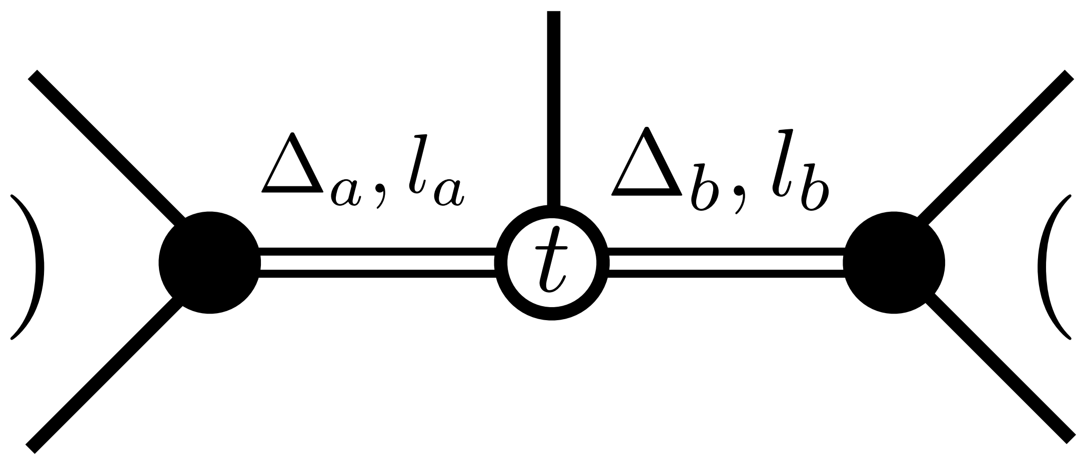
Scalar five-point function

- Use conformal symmetry to fix frame:



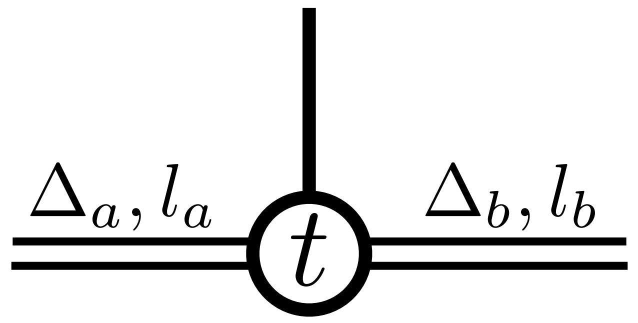
$$\mathcal{X} = \cos^2 \frac{\varphi}{2}$$

$$H_{\text{vertex}}(z, \bar{z}, \mathcal{X}, \partial_z, \partial_{\bar{z}}, \partial_{\mathcal{X}})$$



$$(z, \bar{z}, \mathcal{X}) \overset{z, \bar{z} \to 0}{\sim}$$

$$z_a^{\frac{\Delta_a + l_a}{2}} \bar{z}_a^{\frac{\Delta_a - l_a}{2}} \Big[H_{\text{reduced}}(\mathcal{X}, \partial_{\mathcal{X}})$$



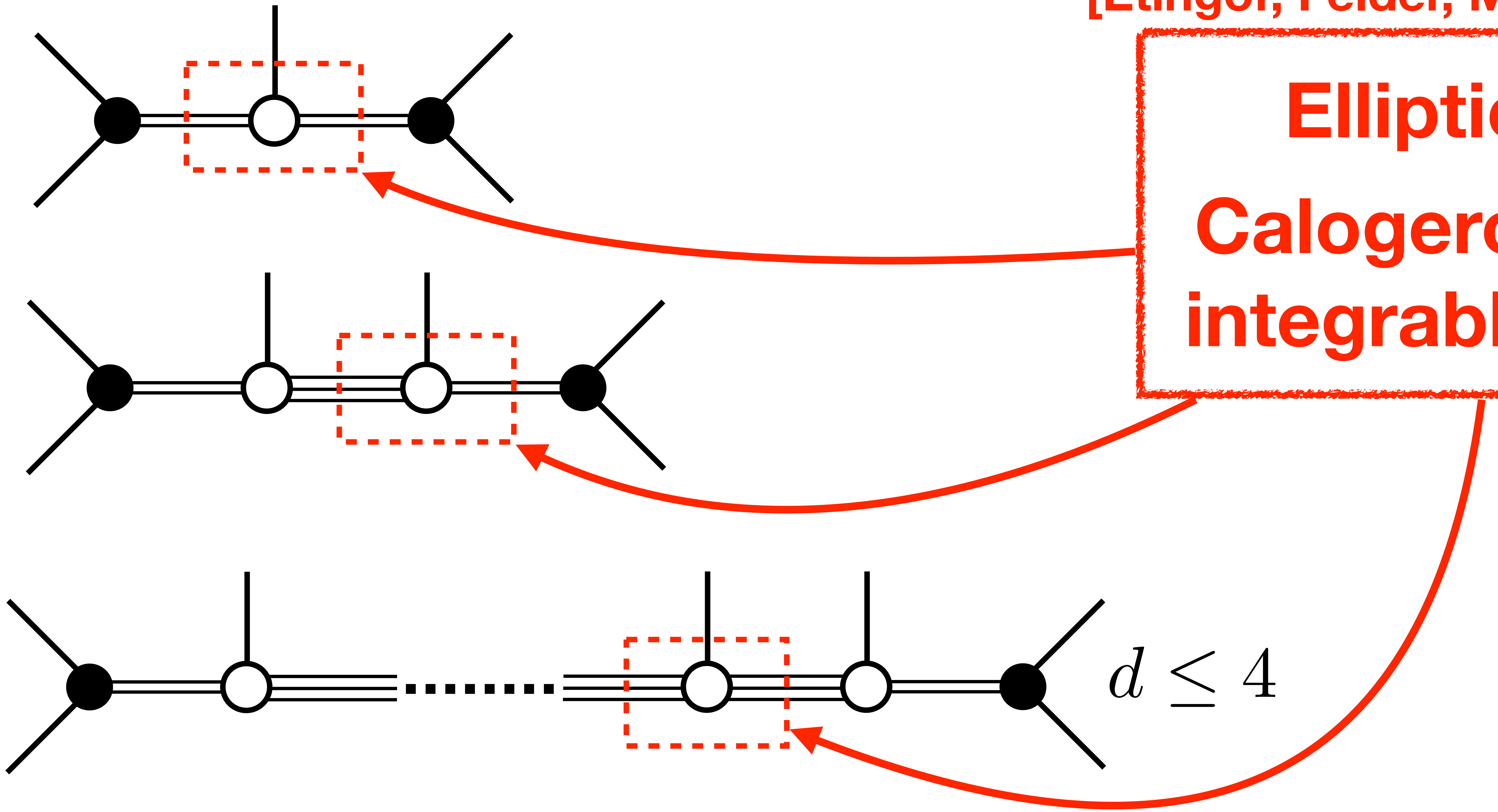
$$(\mathcal{X}) \Big] z_b^{\frac{\Delta_b + l_b}{2}} \bar{z}_b^{\frac{\Delta_b - l_b}{2}}$$

Main result
Comb channel vertex systems

[2108.00023]

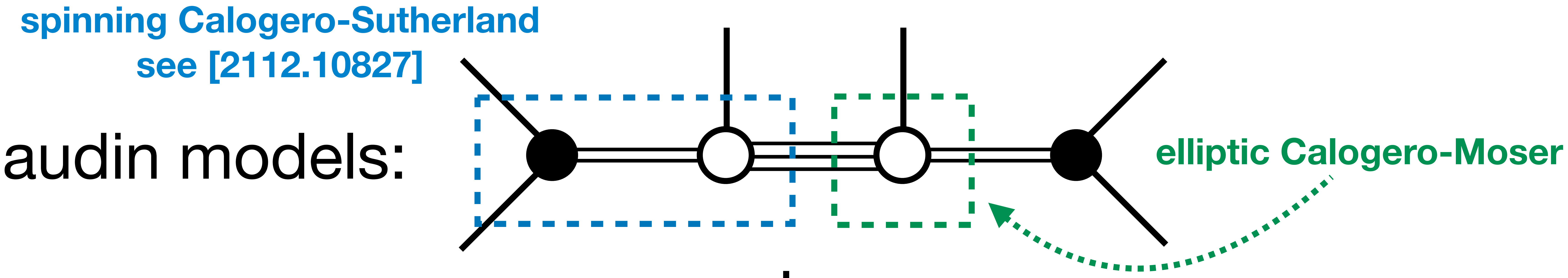
[Etingof, Felder, Ma, Veselov 2010]

Elliptic $\mathbb{Z}/4\mathbb{Z}$ Calogero-Moser integrable model

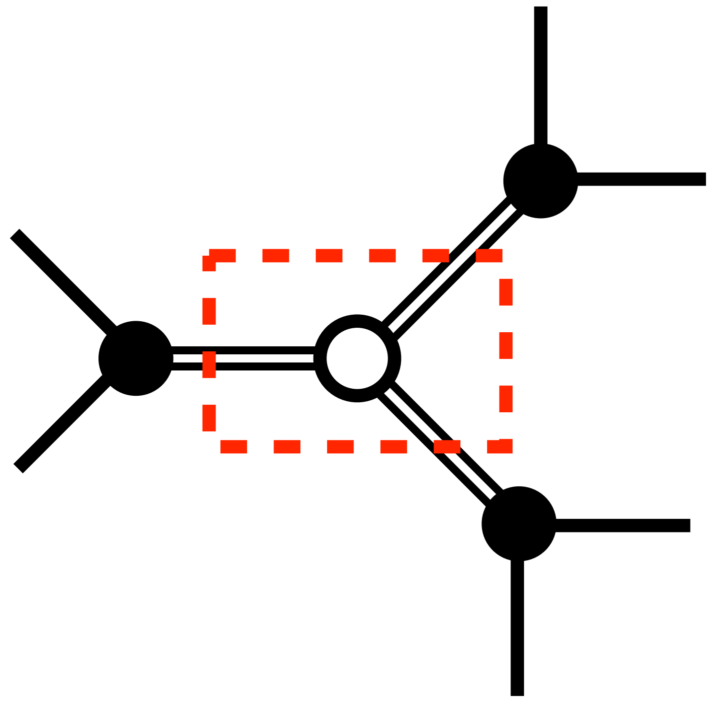


Ongoing research

- **Solving** N-point Gaudin models:

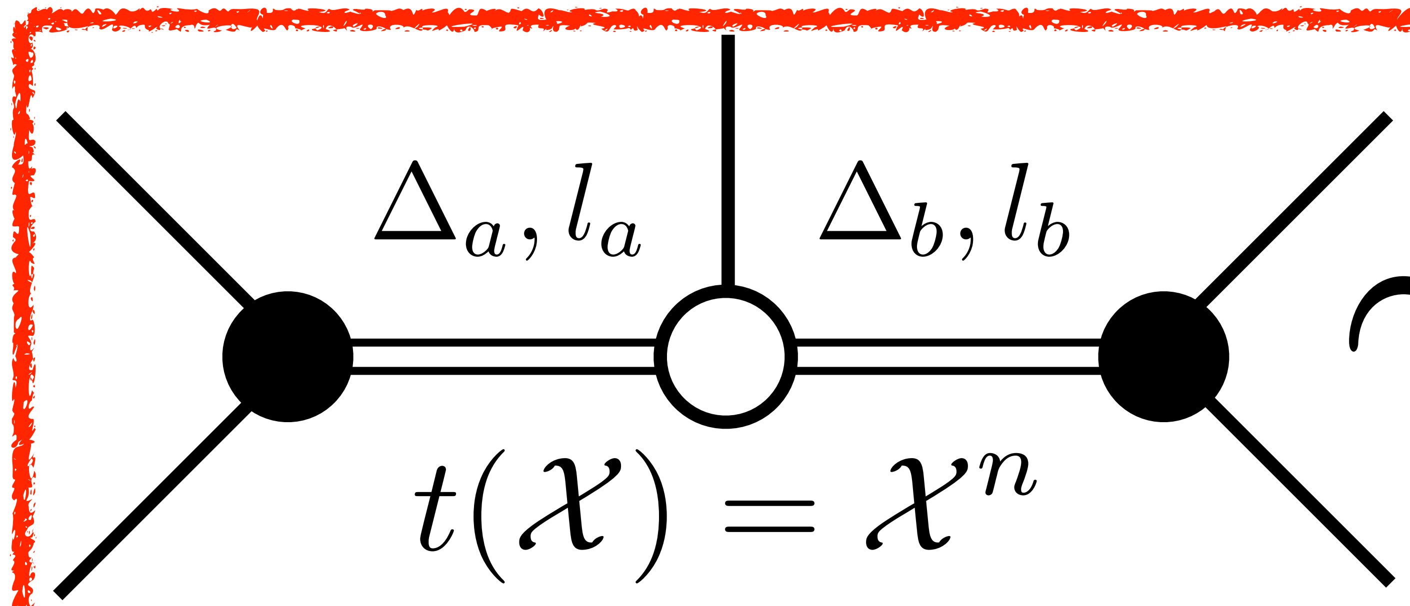


- **OPE channels of different topology**:
  using quivers & cluster algebras



$$= C_2, A_3$$

- N-point blocks in **lightcone limits**: $\bar{z}_{a,b} \to 0, \quad z_{a,b} \to 1, \quad l_{a,b} \to \infty$



$$\sim \quad \mathcal{X}^n \prod_{s=a,b} \frac{\bar{z}_s^{\frac{\Delta_s - l_s}{2}}}{(1-z_s)^{\frac{n}{2} + \frac{\Delta_s - l_s - \Delta_3}{4}}} K_{n + \frac{\Delta_s - l_s - \Delta_3}{2}}\left(2 l_s \sqrt{1 - z_s}\right)$$

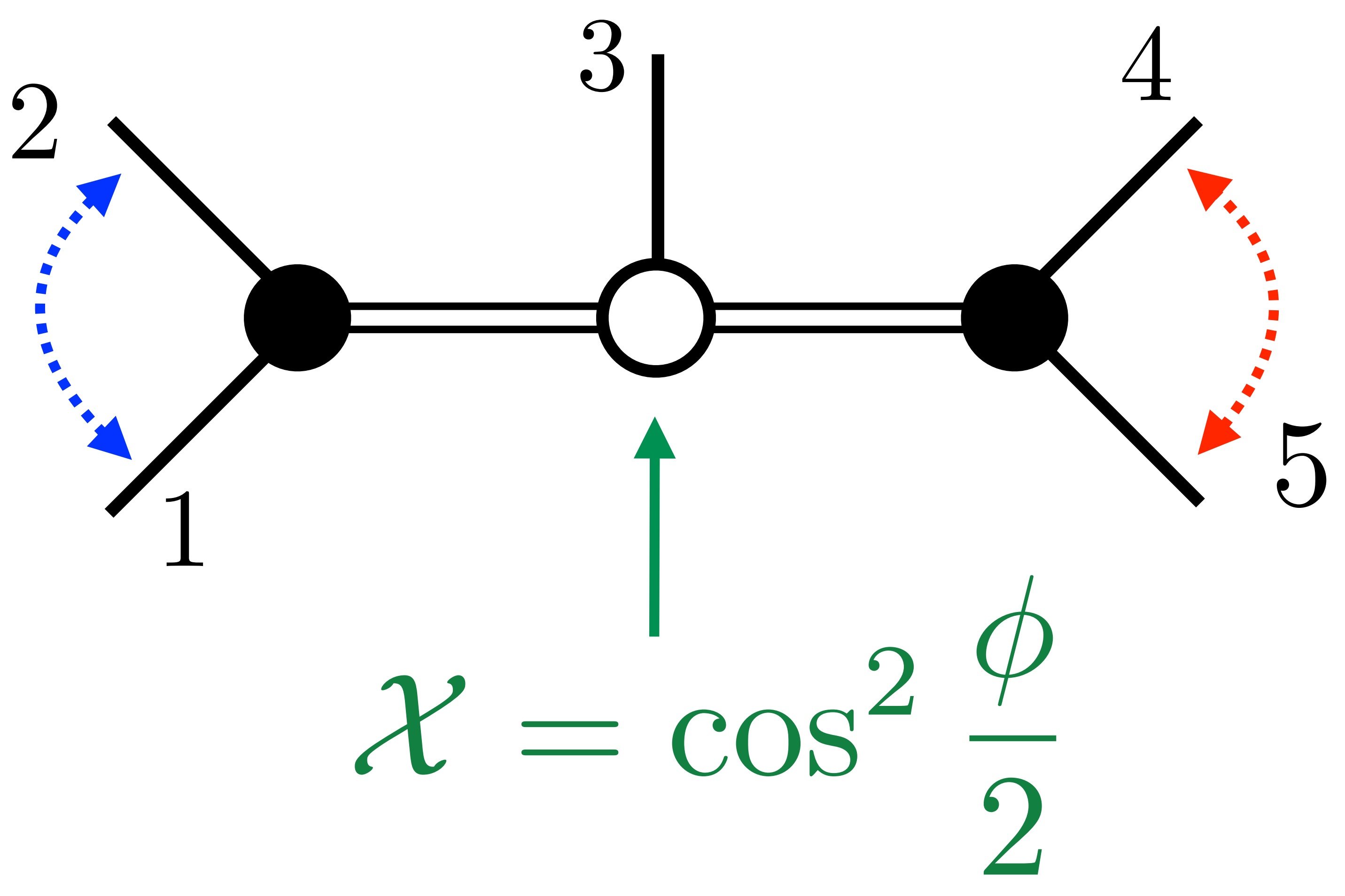
Appendix 1
OPE limit

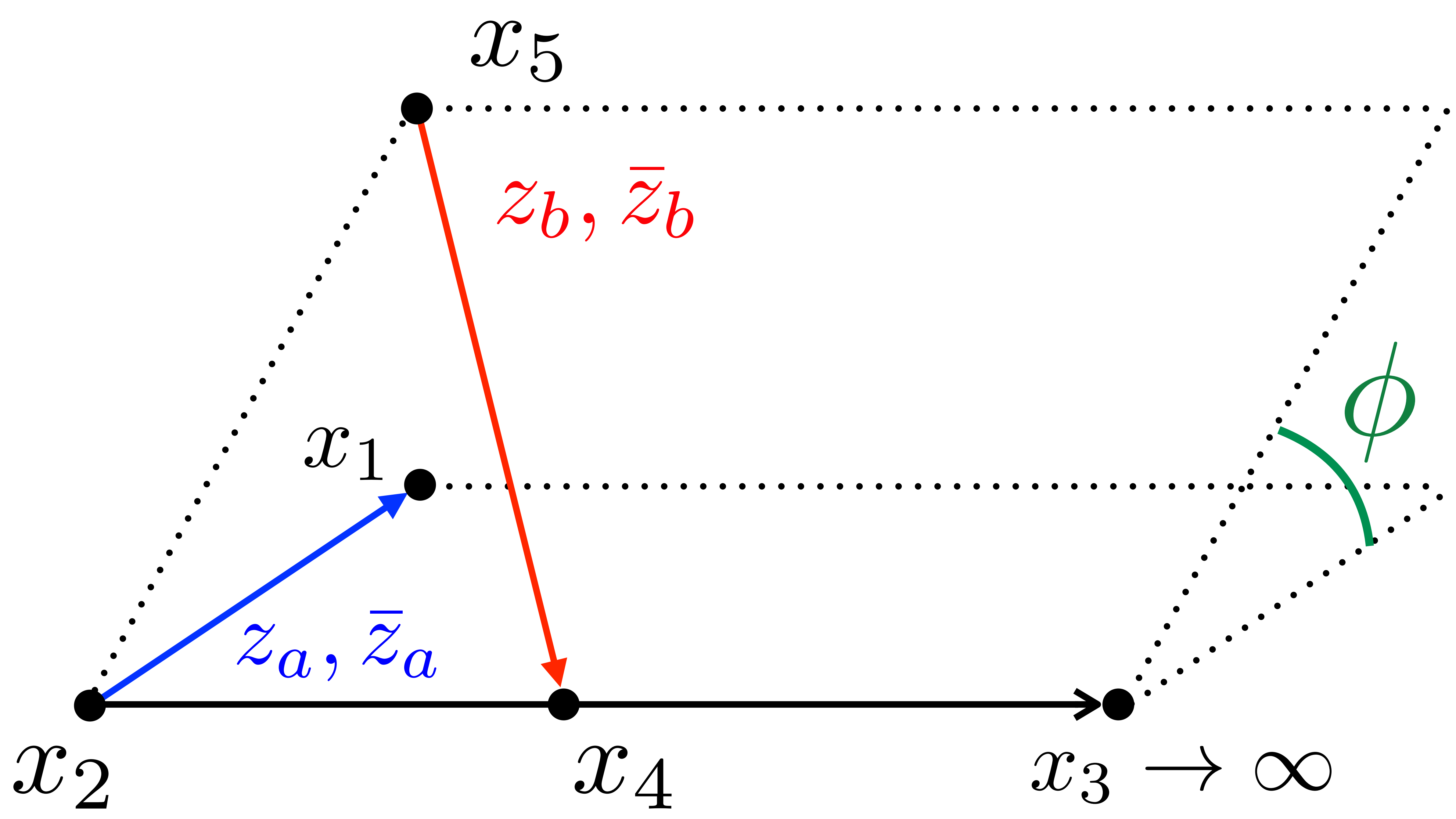
- Most singular contribution to OPE:

$$\phi_1(x_1)\phi_2(x_2) \overset{x_1 \to x_2}{\sim}$$

$$(x_1 - x_2)^{2q_a} C_{\phi_1\phi_2\mathcal{O}} \mathcal{D}(x_1 - x_2, \partial_{x_2}) \mathcal{O}_{\Delta_a, l_a}(x_2) + (\text{less singular})$$

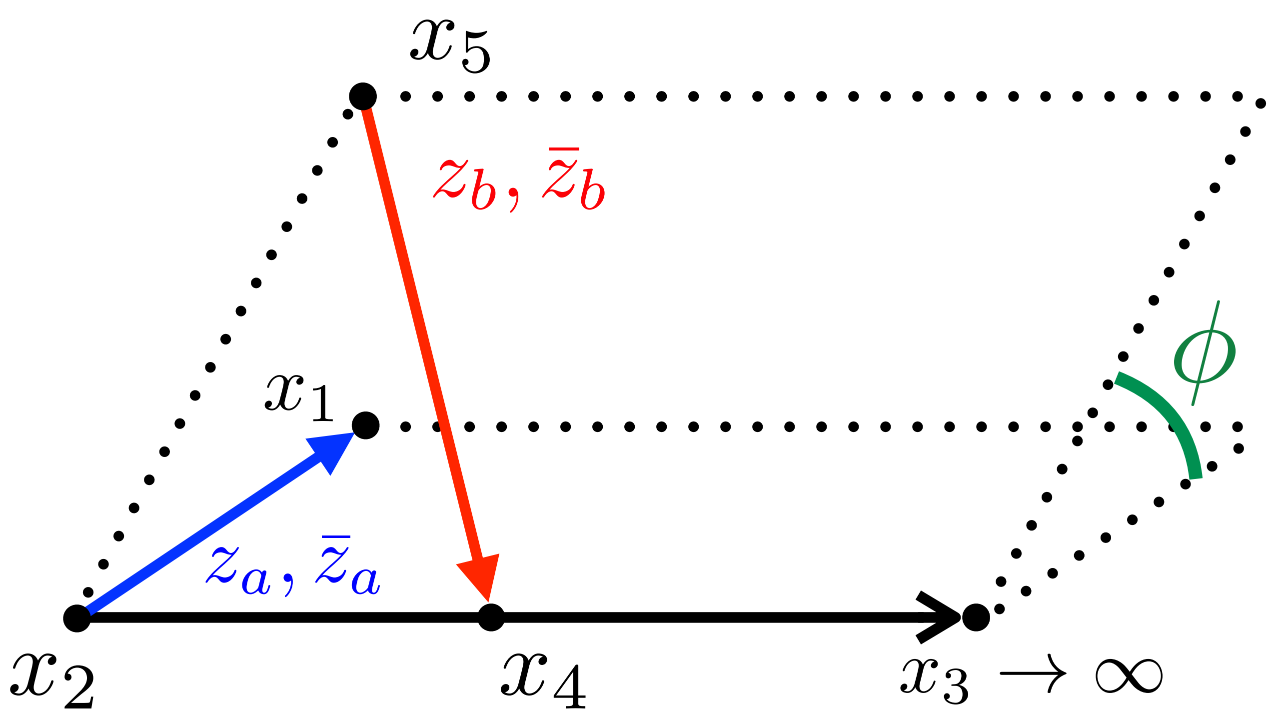
- In a N-point function:





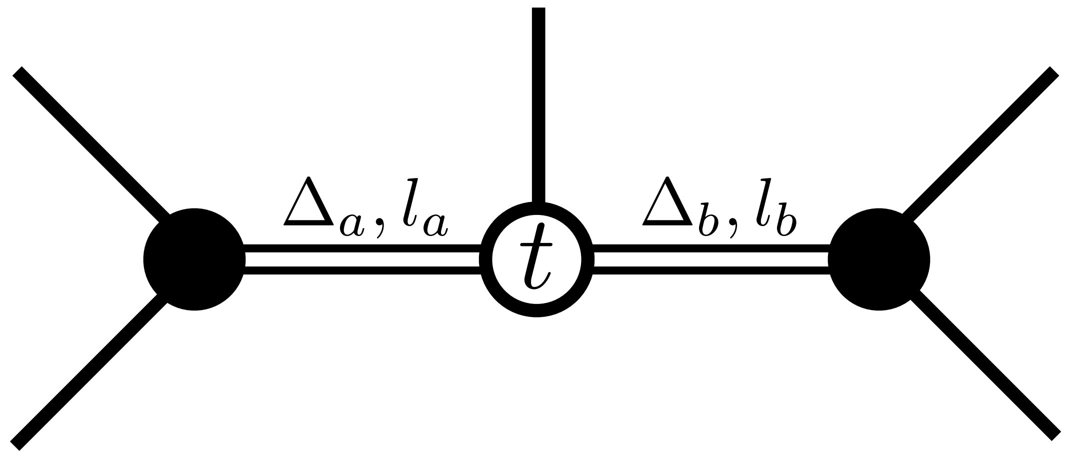
Appendix 1

OPE limit

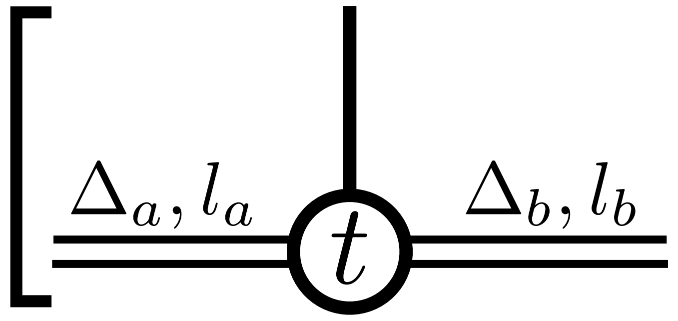


$$\mathcal{X} = \cos^2 \frac{\phi}{2}$$

Ansatz:



$$(z, \bar{z}, \mathcal{X}) \overset{!}{\underset{z,\bar{z}\to 0}{\sim}} z_a^{q_a}\, \bar{z}_a^{\bar{q}_a} \Big[$$



$$(\mathcal{X}) \Big]\, z_b^{q_b}\, \bar{z}_b^{\bar{q}_b}$$

$$\mathrm{tr}\,(L_1 + L_2)^{2,4} \sim \mathrm{Cas}^{2,4}(\Delta_a, l_a)$$

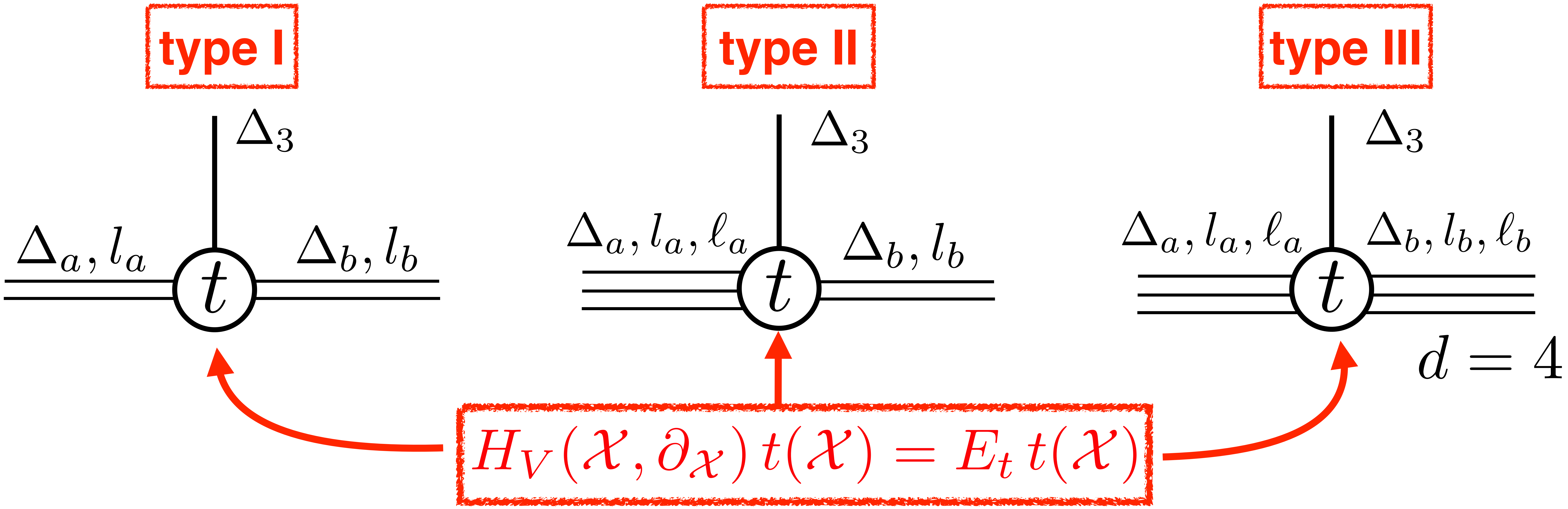
$$\Longrightarrow q_a, \bar{q}_a = \frac{\Delta_a \pm l_a}{2}$$

$$\mathrm{tr}\,(L_4 + L_5)^{2,4} \sim \mathrm{Cas}^{2,4}(\Delta_b, l_b)$$

$$\Longrightarrow q_b, \bar{q}_b = \frac{\Delta_b \pm l_b}{2}$$

Appendix 2

From vertex system to Calogero-Moser model



$=$ **OPE limits** & **shadow integrands** of N-point blocks

$=$ **Gaudin/Hitchin systems** on the three-punctured sphere

Appendix 2
From vertex system to Calogero-Moser model

- **Conjugation by** $\Theta = \Theta_0\, \mathcal{X}^{\frac{l_1+l_2-2(\ell_1+\ell_2)+\Delta_3+(1-d)/2}{4}} (1-\mathcal{X})^{\frac{l_1+l_2-2(\ell_1+\ell_2)-\Delta_3+(1+d)/2}{4}}$

- **Parameters**: $a_\nu := -11 + \sum_{1\le i<j\le 4} m_{i,\nu} m_{j,\nu}\,,$

$$b_\nu := \frac{1}{2}\left(-a_\nu - 6 + \sum_{1\le i<j<k\le 4} m_{i,\nu} m_{j,\nu} m_{k,\nu}\right)$$

$$c_\nu := \prod_{i=1}^{4} m_{i,\nu}$$

$$\sum_{i=1}^{4} m_{i,\nu} = 6$$

$$m_{i,2} = m_{i,3}$$

$$m_{1,2} = k+1$$

$$m_{2,2} = -k$$

$$m_{3,2} = k+4$$

$$m_{4,2} = -k+2$$

Appendix 2

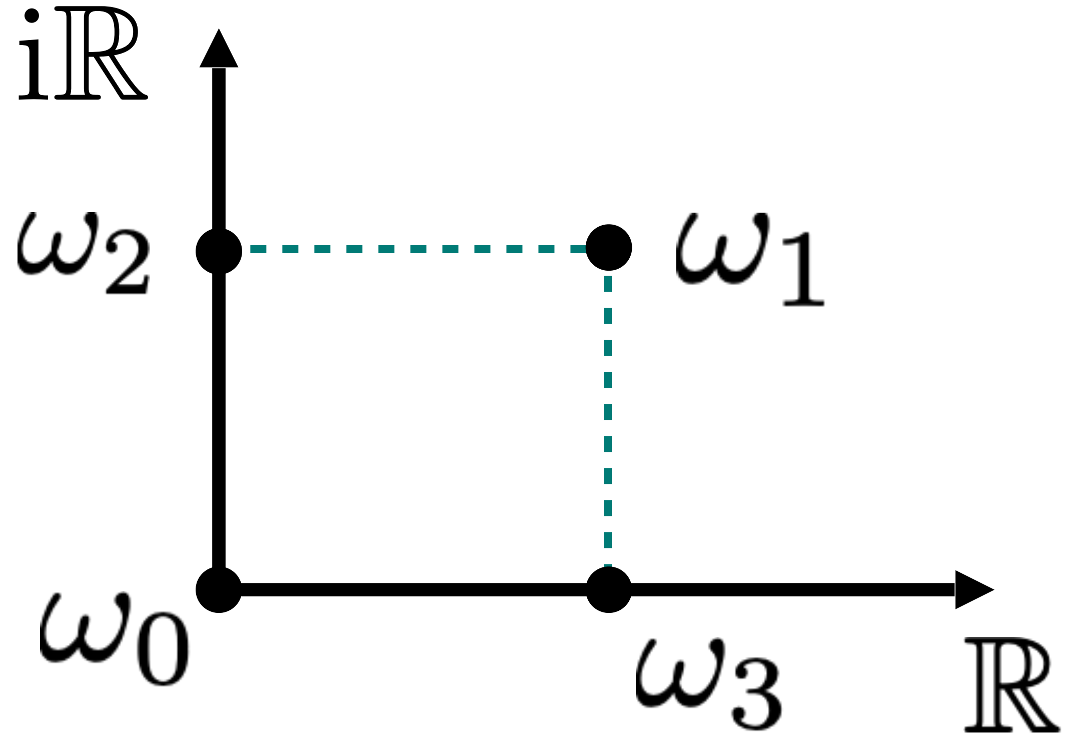
From vertex system to Calogero-Moser model

[c.f. Etingof, Felder, Ma, Veselov 2010 – eq. (4.3)]

$$\Theta^{-1} H_V \Theta + \text{const} =$$

$$\partial_\zeta^4 + \sum_{\nu=0}^{3} \left[a_\nu \wp(\zeta - \omega_\nu)\partial_\zeta^2 + b_\nu \wp'(\zeta - \omega_\nu)\partial_\zeta + c_\nu \wp^2(\zeta - \omega_\nu) \right]$$

$$+\wp(\omega_3)(a_0 - a_1)k(k+1)\left[\wp(\zeta - \omega_2) - \wp(\zeta - \omega_3)\right]$$



$$\mathcal{X} = \left(1 - \frac{\wp^2(\zeta)}{\wp^2(\omega_3)}\right)^{-1}$$

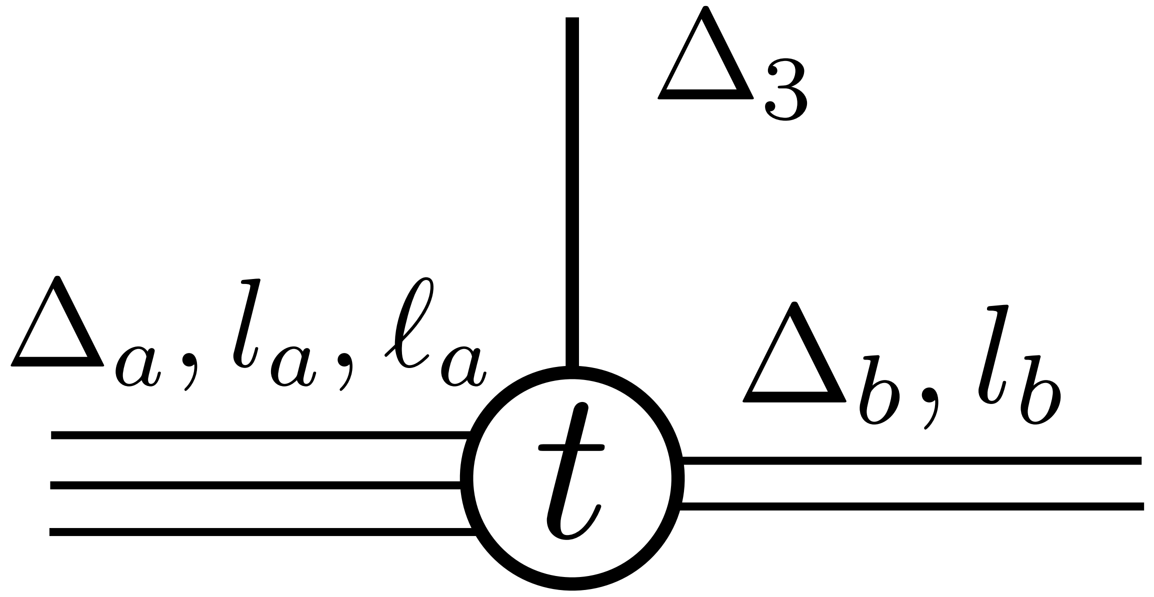
$$a_\nu, b_\nu, c_\nu = \text{polynomial}(m_{i,\nu}) \qquad \sum_{i=1}^{4} m_{i,\nu} = 6$$

$$m_{i,2} = m_{i,3} \propto k$$

$$\mathbb{Z}/4\mathbb{Z} : \zeta \mapsto \mathrm{i}^n \zeta$$

Appendix 2

Calogero-Moser multiplicities for type I,II vertices



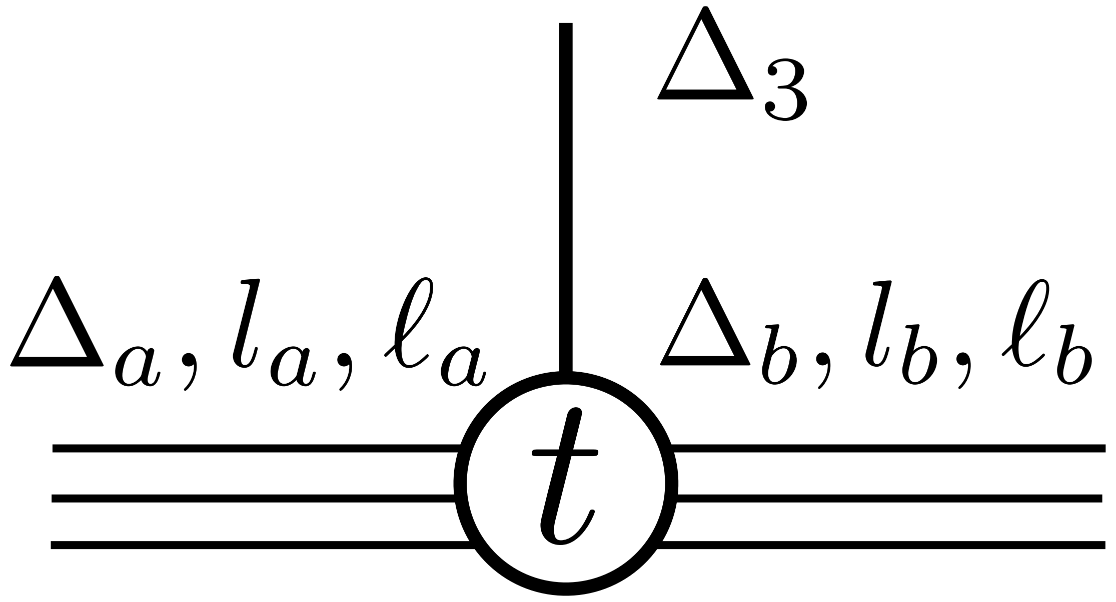
$$
\begin{aligned}
m_{1,0} &= 3\,\frac{5-d}{2} - (l_a + l_b) - \Delta_3 - 2\ell_a\,,\\
m_{2,0} &= \frac{d-1}{2} - (l_a + l_b) - \Delta_3 + 2\ell_a\,,\\
m_{3,0} &= \frac{d-1}{2} + (l_a + l_b) + \Delta_3 + 2(\Delta_a - \Delta_b)\,,\\
m_{4,0} &= \frac{d-1}{2} + (l_a + l_b) + \Delta_3 - 2(\Delta_a - \Delta_b)\,,\\
m_{1,1} &= -5\,\frac{d-3}{2} - (l_a + l_b) + \Delta_3 - 2\ell_a\,,\\
m_{2,1} &= -\frac{d+1}{2} - (l_a + l_b) + \Delta_3 + 2\ell_a\,,\\
m_{3,1} &= -\frac{d+1}{2} + (l_a + l_b) - \Delta_3 + 2(\Delta_a + \Delta_b)\,,\\
m_{4,1} &= \frac{7d-1}{2} + (l_a + l_b) - \Delta_3 - 2(\Delta_a + \Delta_b)\,,
\end{aligned}
$$

$$k = |l_a - l_b| - \frac{1}{2}$$

Appendix 2

Calogero-Moser multiplicities for type III vertices

$$d = 4$$



$$l = \frac{j+\bar{j}}{2}, \quad \ell = \frac{j-\bar{j}}{2}$$

$$\begin{aligned}
m_{1,0} &= \frac{3}{2} - (l_a + l_b) - \Delta_3 - 2(\ell_a - \ell_b)\,, \\
m_{2,0} &= \frac{3}{2} - (l_a + l_b) - \Delta_3 + 2(\ell_a - \ell_b)\,, \\
m_{3,0} &= \frac{3}{2} + (l_a + l_b) + \Delta_3 + 2(\Delta_a - \Delta_b)\,, \\
m_{4,0} &= \frac{3}{2} + (l_a + l_b) + \Delta_3 - 2(\Delta_a - \Delta_b)\,, \\
m_{1,1} &= -\frac{5}{2} - (l_a + l_b) + \Delta_3 - 2(\ell_a + \ell_b)\,, \\
m_{2,1} &= -\frac{5}{2} - (l_a + l_b) + \Delta_3 + 2(\ell_a + \ell_b)\,, \\
m_{3,1} &= -\frac{5}{2} + (l_a + l_b) - \Delta_3 + 2(\Delta_a + \Delta_b)\,, \\
m_{4,1} &= \frac{27}{2} + (l_a + l_b) - \Delta_3 - 2(\Delta_a + \Delta_b)\,.
\end{aligned}$$

$$k = |l_a - l_b| - \frac{1}{2}$$

Appendix 3

Scalar N-point kinematics in embedding space

→ **Embedding space:** $x \in \mathbb{R}^d \to X \in \mathbb{R}^{1,d+1}, \quad X^2 = 0, \quad X \sim \lambda X$

→ **Scalar primary:** $\phi_i(\lambda_i X_i) = \lambda_i^{-\Delta_i} \phi_i(X_i), \quad (L_i)^A{}_B = X_i^A \dfrac{\partial}{\partial X_i^B} - [A \leftrightarrow B]$

→ **N-point function:** $\langle \phi_1 \ldots \phi_N \rangle = \text{homogeneous}(X_i \cdot X_j) =$

$$\prod_{i<j} (X_i \cdot X_j)^{a_{ij}} \, F\left(\left\{ u_I = \frac{\prod X_i \cdot X_j}{\prod X_i \cdot X_j} \right\}_{I=1}^{n_{\text{CR}}} \right)$$

→ **Counting cross-ratios:** $n_{\text{CR}} = \begin{cases} N(N-1)/2 - N & \text{if} \quad d+2 \geq N \\ Nd - (d+2)(d+1)/2 & \text{if} \quad d+2 \leq N \end{cases}$

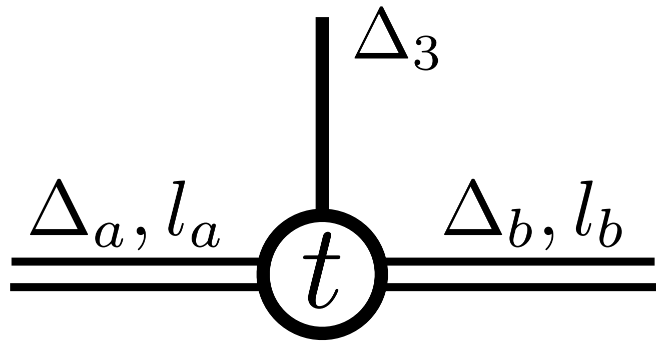
Appendix 3

Spinning three-point kinematics in embedding space

- **Spinning embedding space:** $X, Z \in \mathbb{R}^{1,d+1} \qquad X^2 = X \cdot Z = Z^2 = 0$

- **Spinning primary:** $\mathcal{O}_{\Delta,l}(\lambda X, \alpha Z + \beta X) = \lambda^{-\Delta} \alpha^l \mathcal{O}_{\Delta,l}(X, Z)$

- **Spinning three-point function:**



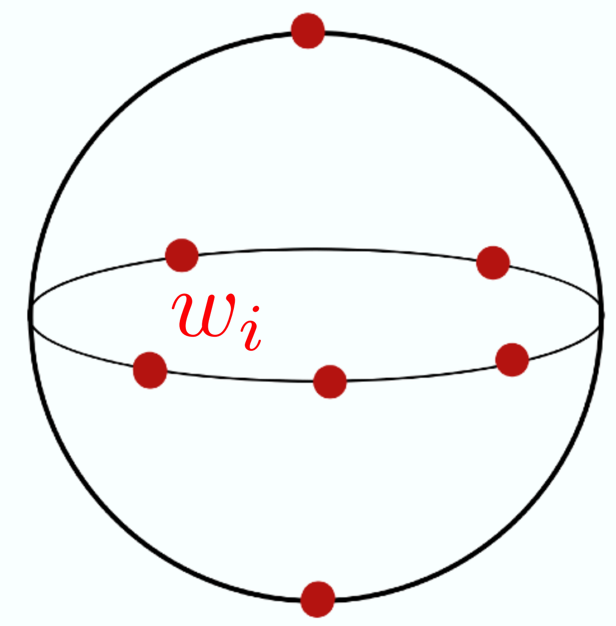
$$\langle \mathcal{O}_{\Delta_a,l_a}(X_a, Z_a)\phi_3(X_3)\mathcal{O}_{\Delta_b,l_b}(X_b, Z_b)\rangle =$$
$$(X_1 \cdot X_2)^{a_{12}}(X_2 \cdot X_3)^{a_{23}}(X_3 \cdot X_1)^{a_{31}}[X_3(X_a \wedge Z_a)X_b]^{b_1}[X_3(X_b \wedge Z_b)X_a]^{b_2} t(\mathcal{X})$$

$$\mathcal{X} = \frac{[X_3(X_a \wedge Z_a)X_b][X_3(X_b \wedge Z_b)X_a]}{[(X_a \wedge Z_a) \cdot (X_b \wedge Z_b)]\,(X_3 \cdot X_a)(X_3 \cdot X_b)}$$

Appendix 4: N-point Gaudin Models

Lax matrix and Hamiltonians

$w \in$



- Lax matrix: $\mathcal{L}^A{}_B(w) = \sum_{i=1}^{N} \frac{L_i(x_i, \partial_{x_i})^A{}_B}{w - w_i} \in \bigotimes_{i=1}^{N} [\mathcal{O}_i] \otimes \mathfrak{so}(1, d+1)$

- Invariants: $\mathcal{H}_p(w) = \operatorname{tr} \mathcal{L}(w)^p + \mathrm{O}(\mathcal{L}^{p-1})$ $\qquad [\mathcal{H}_p(w), \mathcal{H}_q(w')] = 0$

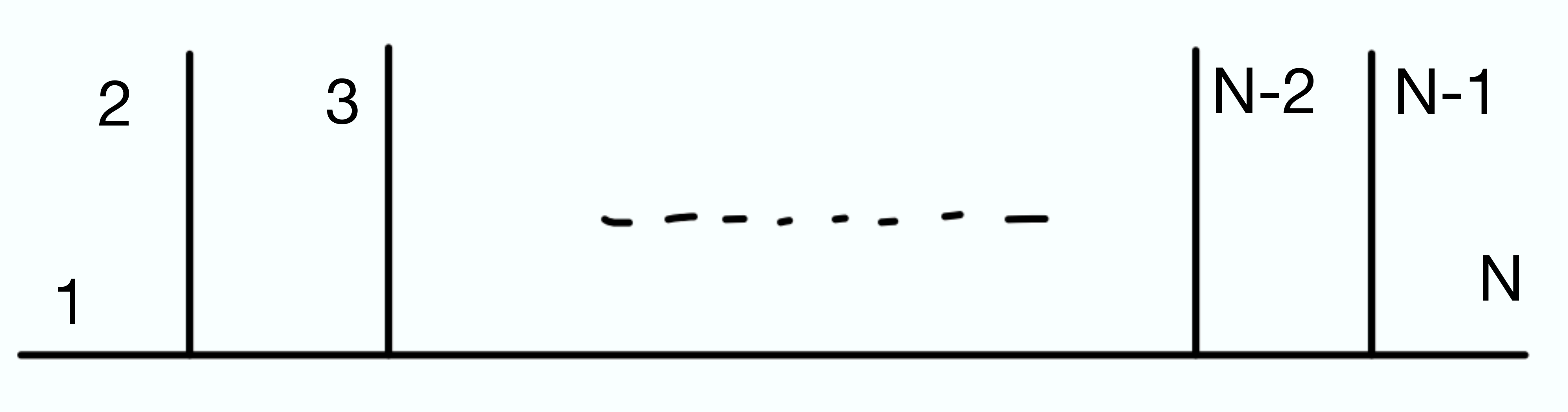
- Obtain Gaudin Hamiltonians by **partial fraction decomposition**, e.g.

$$\mathcal{H}_2(w) = \sum_{i=1}^{N} \left\{ \frac{\operatorname{tr} L_i^2}{(w - w_i)^2} + \frac{\sum_{j \neq i} \frac{\operatorname{tr} L_i L_j}{w_i - w_j}}{w - w_i} \right\}$$

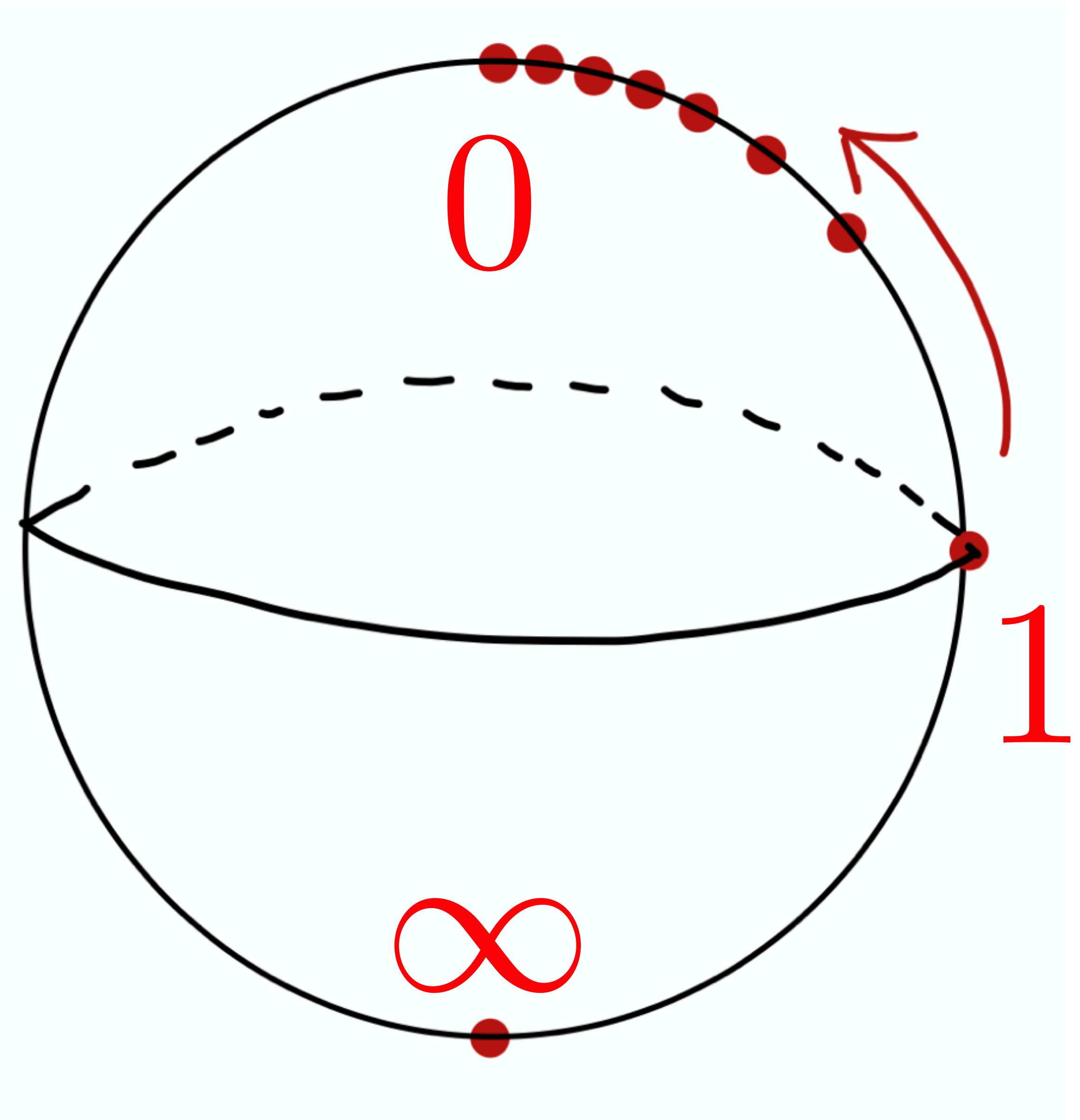
Appendix 4: N-point Gaudin Models

OPE channel limits

- take limits of **colliding punctures:** $w_i - w_j = \epsilon \to 0 \quad \epsilon^{p\lambda}\mathcal{H}_p\left(\epsilon^{\lambda} w\right) \to \text{finite}$
- Example: **comb channel:**



$\Longleftrightarrow$



$$w_1 = 0, w_2 = \epsilon^{N-3}, ..., w_{N-2} = \epsilon, w_{N-1} = 1, w_N = \infty$$

$$r = 2, 3, ..., N-2 \qquad \mathcal{H}_p^{[r]}(w) = \lim_{\epsilon \to 0} \epsilon^{p(N-r-2)} \mathcal{H}_p(\epsilon^{N-r-2} w)$$