Particle creation in a toy $O(N)$ model

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Particle creation in a free theory

- Particle creation in external classical fields is usually considered in a **tree-level approximation**
- In this approximation, number of created particles is estimated using the following relation:

$$\mathcal{N}_n^{\text{free}} = \langle in | (a_n^{\text{out}})^\dagger a_n^{\text{out}} | in \rangle$$

- Creation-annihilation operators and mode functions in the past and future infinity are related by a **Bogoliubov transformations**:

$$f_n^{\text{out}} = \sum_k \left[\alpha_{nk}^* f_k^{\text{in}} - \beta_{nk} (f_k^{\text{in}})^* \right],$$

$$a_n^{\text{out}} = \sum_k \left[\alpha_{nk} a_k^{\text{in}} + \beta_{nk}^* (a_k^{\text{in}})^\dagger \right],$$

- Hence, for the initial vacuum state, we get the following identity:

$$\mathcal{N}_n^{\text{free}} = \sum_k |\beta_{kn}|^2$$

Interactions are important!

- Nevertheless, in an interacting theory, quantum averages receive **substantial loop corrections** due to the change of the initial quantum state:

$$n_{kl}(t) = \langle in|U^\dagger(t,t_0)(a_k^{\text{in}})^\dagger a_l^{\text{in}} U(t,t_0)|in\rangle$$
$$\kappa_{kl}(t) = \langle in|U^\dagger(t,t_0) a_k^{\text{in}} a_l^{\text{in}} U(t,t_0)|in\rangle$$

- Therefore, the expression for the created particle number should be corrected:

$$\begin{aligned}\mathcal{N}_n &= \langle in|U^\dagger(t,t_0)(a_n^{\text{out}})^\dagger a_n^{\text{out}} U(t,t_0)|in\rangle \\ &= \mathcal{N}_n^{\text{free}} + \sum_{k,l} \left(\alpha_{kn}\alpha_{ln}^* + \beta_{kn}\beta_{ln}^*\right) n_{kl} + \sum_{k,l} \alpha_{kn}\beta_{ln}\kappa_{kl} + \sum_{k,l} \alpha_{kn}^*\beta_{ln}^*\kappa_{kl}^*\end{aligned}$$

- Moreover, the exact quantum averages and particle number cannot be calculated perturbatively due to the **secular growth of loop corrections**
- For example, one faces such a secular growth in the nonlinear dynamical Casimir effect or theory of interacting light fields in de Sitter space
- Today, I consider a simplified model of these systems — a nonstaionary $O(N)$ model in $(0+1)$ dimensions

The model and its quantization

- Consider the system of $N \gg 1$ quantum mechanical oscillators with a time-dependent frequency and $O(N)$-symmetric quartic coupling:

$$S = \int dt \left[\frac{1}{2} \dot{\phi}_i \dot{\phi}_i - \frac{\omega^2(t)}{2} \phi_i \phi_i - \frac{\lambda}{4N} (\phi_i \phi_i)^2 \right]$$

- The quantized field is decomposed as usual:

$$\phi_i(t) = a_i f(t) + a_i^\dagger f^*(t), \qquad [a_i, a_j^\dagger] = \delta_{ij}$$

- Due to the nonstationarity of the model, mode function contains a **reflected wave**:

$$f(t) = \begin{cases} \frac{1}{\sqrt{2\omega_-}} e^{-i\omega_- t}, & \text{as} \quad t \to -\infty, \\ \frac{\alpha}{\sqrt{2\omega_+}} e^{-i\omega_+ t} + \frac{\beta}{\sqrt{2\omega_+}} e^{i\omega_+ t}, & \text{as} \quad t \to +\infty, \end{cases}$$

where $|\alpha|^2 - |\beta|^2 = 1$ and $\omega(t) \to \omega_\pm$ as $t \to \pm\infty$

Bogoliubov coefficients α and β

- For simplicity, I assume that variations of the frequency are small: $\omega(t) = \omega + \delta\omega(t)$ with $\omega = \text{const}$ and $|\delta\omega| \ll \omega$ for all t
- In the **nonresonant case**, Bogoliubov coefficient β is small:

$$|\beta|^2 \approx \frac{|\beta|^2}{|\alpha|^2} \approx \left| \int_{-\infty}^{\infty} \delta\omega(t) e^{-2i\omega t} dt \right|^2 \ll 1$$

- In the **resonant case**, e.g., $\omega(t) = \omega\left[1 + 2\gamma\cos(2\omega t)\right]$, $\gamma \ll 1$, it exponentially grows with the duration of oscillations t_R:

$$\alpha = \cosh\left(\omega\gamma t_R\right), \quad \beta = -i\sinh\left(\omega\gamma t_R\right)$$

- Note that in quantum mechanics, we can calculate the average number of excitations $\mathcal{N}$, which is an **analog of the created particle number**
- If we neglect interactions and assume that the initial state is close to the vacuum, this number is as follows:

$$\mathcal{N}_{\text{free}} = N|\beta|^2$$

Effective Hamiltonian in the RWA

- Let us substitute the modes into the interacting Hamiltonian ($\lambda\phi^4$), neglect rapidly oscillating terms and transform it to the normal-ordered form:

$$H_{\text{int}} \approx \frac{\lambda}{16N\omega_+^2}\left(|\alpha|^4 + 4|\alpha|^2|\beta|^2 + |\beta|^4\right)\left(a_i^\dagger a_i^\dagger a_j a_j + 2a_i^\dagger a_j^\dagger a_i a_j\right) + \\ + \frac{3\lambda\alpha\beta}{4N\omega_+^2}\left(|\alpha|^2 + |\beta|^2\right) a_i^\dagger a_i a_j a_j + \frac{3\lambda\alpha^2\beta^2}{8N\omega_+^2} a_i a_i a_j a_j + h.c.$$

- In other words, let us work in the limit $\lambda \to 0$, $t \to \infty$, $\lambda t = \text{const}$
- This approximation is somewhat similar to the **rotating-wave approximation** (RWA) from quantum optics
- Note that the **quadratic terms** that appear after the normal ordering lead to the **renormalization** of the frequency and Bogoliubov coefficients:

$$\omega_+ \to \omega_+ + \frac{\lambda(N+2)}{4N\omega_+^2}\left(|\alpha|^2 + |\beta|^2\right), \quad \alpha \to \alpha + \frac{\lambda(N+2)}{8N\omega_+^3}|\beta|^2\alpha$$

Quantum averages for small β

- Using the effective Hamiltonian and assuming small deviations from the stationarity (small β), we readily estimate the evolution operator and calculate the **correction to the initial quantum state**:

$$|\Psi(t)\rangle \approx |0\rangle + 9\frac{\beta^*|\beta|^2}{N}\left[\exp\left(\frac{-i\lambda t}{2\omega_+^2}\right) + \frac{i\lambda t}{2\omega_+^2} - 1\right] a_i^\dagger a_i^\dagger |0\rangle + \\ + \frac{3}{4}\frac{(\beta^*)^2}{N}\left[\exp\left(\frac{-i\lambda t}{2\omega_+^2}\right) - 1\right] a_i^\dagger a_i^\dagger a_j^\dagger a_j^\dagger |0\rangle + \mathcal{O}\left(\frac{1}{N^2}\right) + \mathcal{O}\left(\beta^4\right)$$

- We can also straightforwardly calculate the resummed **quantum averages**:

$$n_{ij}(t) \approx 72|\beta|^4 \frac{\delta_{ij}}{N} \sin^2\left(\frac{\lambda t}{4\omega_+^2}\right) + \mathcal{O}\left(\frac{1}{N^2}\right) + \mathcal{O}\left(\beta^6\right),$$

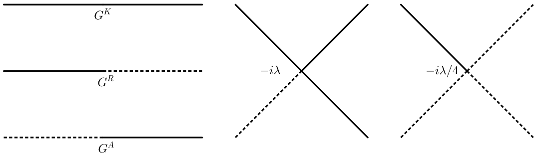
$$\kappa_{ij}(t) \approx 18\beta^*|\beta|^2 \frac{\delta_{ij}}{N}\left[\exp\left(\frac{-i\lambda t}{2\omega_+^2}\right) + \frac{i\lambda t}{2\omega_+^2} - 1\right] + \mathcal{O}\left(\frac{1}{N^2}\right) + \mathcal{O}\left(\beta^5\right)$$

- And **average number of excitations**:

$$\mathcal{N} = \mathcal{N}_{\text{free}} + 36|\beta|^4\left[3 + \cos\left(\frac{\lambda t}{2\omega_+^2}\right) - 4\cos\left(\frac{\lambda t}{4\omega_+^2}\right)\right] + \mathcal{O}\left(\frac{1}{N}\right) + \mathcal{O}\left(\beta^5\right)$$

Diagram calculations

- These results can be generalized to arbitrary β using the **Schwinger — Keldysh diagram technique**:



- The tree-level propagators are defined as follows:

$$iG^K_{0,ij}(t_1,t_2) = \langle \phi_{i,\text{cl}}(t_1)\phi_{j,\text{cl}}(t_2)\rangle_0 = \frac{1}{2}\langle 0\big|\left\{\phi_i(t_1),\phi_j(t_2)\right\}\big|0\rangle,$$
$$iG^R_{0,ij}(t_1,t_2) = \langle \phi_{i,\text{cl}}(t_1)\phi_{j,\text{q}}(t_2)\rangle_0 = \theta(t_1-t_2)\langle 0\big|\left[\phi_i(t_1),\phi_j(t_2)\right]\big|0\rangle,$$
$$iG^A_{0,ij}(t_1,t_2) = \langle \phi_{i,\text{q}}(t_1)\phi_{j,\text{cl}}(t_2)\rangle_0 = \theta(t_2-t_1)\langle 0\big|\left[\phi_j(t_2),\phi_i(t_1)\right]\big|0\rangle,$$

- The **exact Keldysh propagator** contains the quantum averages (no sum):

$$iG^K_{nk}(t,t) \approx \left[\left(\frac{1}{2}\delta_{nk} + n_{nk}(t)\right) f_n(t) f^*_k(t) + \kappa_{nk}(t) f_n(t) f_k(t) + h.c.\right]$$

$\mathcal{O}(1)$ corrections to lines

- In the leading order in $1/N$, loop corrections to the propagators are summed with the following **Schwinger — Dyson equations**:

$$\tilde{G}^R_{ij}(t_1,t_2) = G^R_{0,ij}(t_1,t_2) - \frac{i\lambda}{N}\int_{t_0}^{\infty} dt G^R_{0,ik}(t_1,t)\tilde{G}^K_{kk}(t,t)\tilde{G}^R_{kj}(t,t_2),$$

$$\tilde{G}^K_{ij}(t_1,t_2) = G^K_{0,ij}(t_1,t_2) - \frac{i\lambda}{N}\int_{t_0}^{\infty} dt\Big[G^R_{0,ik}(t_1,t)\tilde{G}^K_{kk}(t,t)\tilde{G}^K_{kj}(t,t_2)+ + G^K_{0,ik}(t_1,t)\tilde{G}^K_{kk}(t,t)\tilde{G}^A_{kj}(t,t_2)\Big].$$

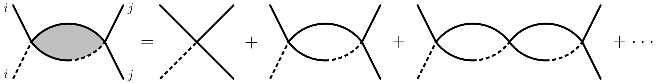
- These corrections simply **renormalize the frequency**, i.e., they are not related to the change in the quantum state:

$$\omega^2(t) \to \omega^2(t) + \lambda\tilde{G}^K(t,t) \approx \omega_+^2 + \frac{\lambda}{2\omega_+}\left(|\alpha|^2+|\beta|^2\right) + \mathcal{O}(\lambda^2) + \mathcal{O}\left(\frac{1}{N}\right)$$

$\mathcal{O}(1)$ corrections to vertices

- In the same order in $1/N$, loop corrections to the vertices are summed with a similar **Schwinger — Dyson equation**:

$$\tilde{B}(t_1,t_2) = 2G^R_{0,kl}(t_1,t_2)G^K_{0,kl}(t_1,t_2) - \frac{2i\lambda}{N}\int_{t_0}^{\infty} dt_3 G^R_{0,kl}(t_1,t_3)G^K_{0,kl}(t_1,t_3)\tilde{B}(t_3,t_2)$$



- This equation has the following approximate solution ($f_i \equiv f(t_i)$):

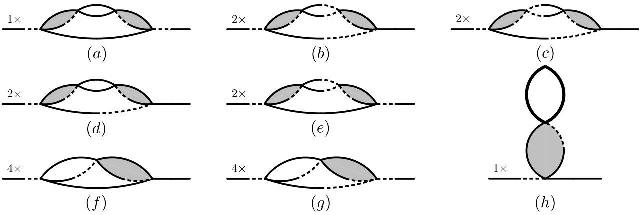
$$\tilde{B}(t_1,t_2) = \frac{\theta(t_{12})}{(2\omega_+)^2}\Bigg[\left(\cos\frac{\lambda R t_{12}}{4\omega_+^2} - i\frac{1+6|\beta|^2+6|\beta|^4}{R}\sin\frac{\lambda R t_{12}}{4\omega_+^2}\right)(f_1^*)^2 f_2^2$$
$$-\left(\cos\frac{\lambda R t_{12}}{4\omega_+^2} + i\frac{1+6|\beta|^2+6|\beta|^4}{R}\sin\frac{\lambda R t_{12}}{4\omega_+^2}\right)f_1^2 (f_2^*)^2$$
$$+\frac{6(\alpha^*)^2(\beta^*)^2}{R}\sin\frac{\lambda R t_{12}}{4\omega_+^2} f_1^2 f_2^2 + \frac{6\alpha^2\beta^2}{R}\sin\frac{\lambda R t_{12}}{4\omega_+^2}(f_1^*)^2(f_2^*)^2\Bigg]$$

$\mathcal{O}(1/N)$ corrections to lines

Substituting the exact vertices into the expression for the Keldysh propagator, we obtain the $\mathcal{O}(1/N)$ corrections to the propagator and quantum averages:

$$n_{ij} = 72\frac{\delta_{ij}}{N}\frac{|\alpha|^4|\beta|^4}{R^2}\sin^2\left(\frac{\lambda t}{4\omega_+^2}R\right) + \mathcal{O}\left(\frac{1}{N}\right),$$

$$\kappa_{ij} = 36\frac{\delta_{ij}}{N}\frac{\alpha^*\beta^*|\alpha|^2|\beta|^2\left(|\alpha|^2+|\beta|^2\right)}{R^2}\left[\frac{1+6|\beta|^2+6|\beta|^4}{R^2}\cos\left(\frac{\lambda t}{2\omega_+^2}R\right) - \frac{i}{R}\sin\left(\frac{\lambda t}{2\omega_+^2}\right) - \right.$$
$$\left. - \frac{2}{R^2}\cos\left(\frac{\lambda t}{4\omega_+^2}R\right) + \frac{2i}{R}\sin\left(\frac{\lambda t}{4\omega_+^2}R\right) + \frac{1-6|\beta|^2-6|\beta|^4}{R^2}\right] + \mathcal{O}\left(\frac{1}{N}\right).$$



Average number of excitations for arbitrary β

- Using the resummed corrections to the quantum averages, we estimate the average number of excitations at large evolution times and arbitrary β:

$$\mathcal{N} \approx \mathcal{N}_{\text{free}} + 36 \frac{|\alpha|^4 |\beta|^4 \left(|\alpha|^2 + |\beta|^2\right)}{R^4} \left[3 + \cos\left(\frac{\lambda t}{2\omega_+^2} R\right) - 4\cos\left(\frac{\lambda t}{4\omega_+^2} R\right)\right]$$

- In a strongly nonstationary case (e.g., for resonant oscillations), tree-level and loop contributions **are proportional to the same power of** β:

$$\mathcal{N}(t) \approx N|\beta|^2 + \frac{1}{2}|\beta|^2 \left[3 + \cos\left(\frac{\lambda t}{\omega_+^2}|\beta|^2\sqrt{3}\right) - 4\cos\left(\frac{\lambda t}{2\omega_+^2}|\beta|^2\sqrt{3}\right)\right] \approx$$
$$\approx N|\beta|^2 + \frac{3}{2}|\beta|^2.$$

- Thus, loop corrections act as additional **"phantom" degrees of freedom** that modify the average number and energy of excitations: $N \to N + \frac{3}{2}$

Conclusion

- I considered a **toy model of interacting nonstationary quantum theory** — a system of N coupled harmonic oscillators with time-dependent frequency and $O(N)$ symmetric quartic coupling
- Using two different methods, I **resummed loop corrections** to the average number of excitations in the limit $\lambda \to 0$, $t \to \infty$, $\lambda t = \text{const}$ and $N \gg 1$
- At large deviations from the stationarity, loop contribution can be interpreted as additional **"phantom" degrees of freedom**, $N \to N + \frac{3}{2}$, that modify the average number and energy of excitations at large evolution times
- In fact, this method and conclusion can be also **extended to the nonlinear dynamical Casimir effect**, at least for weak deviations from the stationarity[1]
- I also note that a deformation of the considered model can be used as a convenient testing ground for the study of quantum chaos — in particular, of the relation between the out-of-time ordered correlation functions and classical Lyapunov exponents

[1] See arXiv:2108.07747