Identification of discrete Painlevé equations

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Introduction

- Painlevé equations are nonlinear differential equations of the second order whose only movable singularities are poles.
- There are six families of such equations.
- The discrete Painlevé equations are nonlinear recurrence relations that reproduce one of the Painlevé differential equations in the continuous limit.
- In 2001, H. Sakai suggested a classification of discrete Painlevé equations based on rational surfaces associated with affine root systems.
- Each discrete system in Sakai method is characterized by a pair of affine root systems, for example $(A_2^{(1)}/E_6^{(1)})$.

Problem of identification

- How to identify some discrete system as discrete Painlevé equation?

$$\begin{cases} x_{n+1} = \left(t - \frac{1}{2}y_n\right) y_n - x_n, \\ y_{n+1} = \dfrac{(y_n^2 - 2ty_n + 2x_n)^2}{y_n\left(2(n+1) - y_n^2 + 2ty_n - 2x_n\right)}, \end{cases} \qquad \begin{cases} f_{n+1} + f_n = g_n - t - \dfrac{a_2}{g_n}, \\ g_{n-1} + g_n = f_n + t + \dfrac{a_1}{f_n}. \end{cases}$$

- The first system arises in random matrix theory, the second system is the standard equation of $(A_2^{(1)}/E_6^{(1)})$-type.
- We will show how to obtain explicit change of coordinates matching the two equations.

Space of initial conditions

- First, we need to find the type of surface.
- Consider the space of initial conditions. Naturally, we think that it is $\mathbb{C}^2$.
- But for Painlevé equations, we want to consider poles as initial conditions. This is why we make the compactification:

$$\mathbb{C}^2 \quad \to \quad \mathbb{P}^1 \times \mathbb{P}^1$$

- Here, we move from the complex plane $\mathbb{C}$ to the projective line $\mathbb{P}^1$.
- This allows us to consider two charts for each variable: x, $X = 1/x$ and y, $Y = 1/y$.
- However, such compactification leads to some problems.

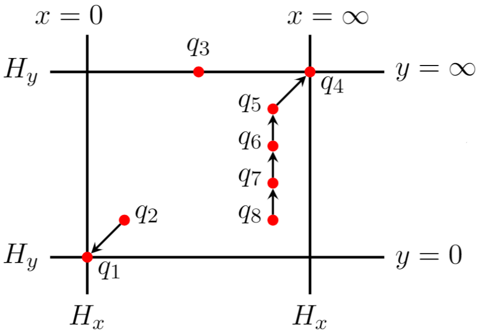
Base points

- After the compactification procedure, infinitely many solutions can pass through some points of $\mathbb{P}^1 \times \mathbb{P}^1$. Such points are called base points.
- In such points indeterminacies appear, i.e., both the numerator and the denominator of the map vanish.
- For example, in our system in point $(x = 0, y = 0)$:

$$y_{n+1} = \frac{(y_n^2 - 2ty_n + 2x_n)^2}{y_n\,(2(n+1) - y_n^2 + 2ty_n - 2x_n)} = \frac{0}{0}.$$

Base points

- It turns out that every Painlevé equation has exactly eight base points. Their configuration defines the type of the surface. In our case:



- To resolve such indeterminacies, we need to perform a blow-up procedure.

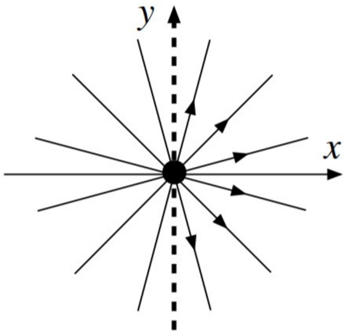
Blow-up procedure

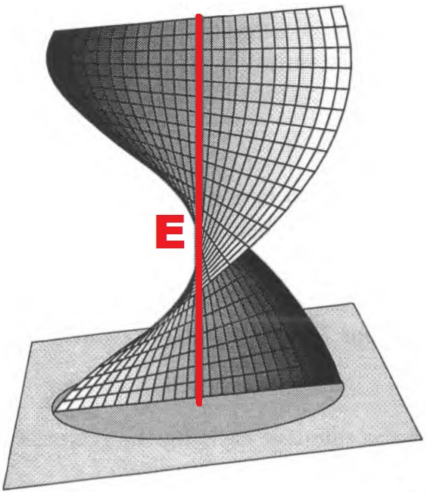
- The blow-up procedure in point (a, b) is given by:

$$\begin{cases} x = a + u_i = a + U_i V_i, \\ y = b + u_i v_i = b + V_i \end{cases}$$

- We adding two additional charts in point, which is equivalent to the adding Riemann sphere (that's why we call it a "blow-up").
- Such procedure allows us to get rid of the indeterminacy.
- However, sometimes after blowing-up, we can find new base points in (u, v)-chart.

Blow-up procedure

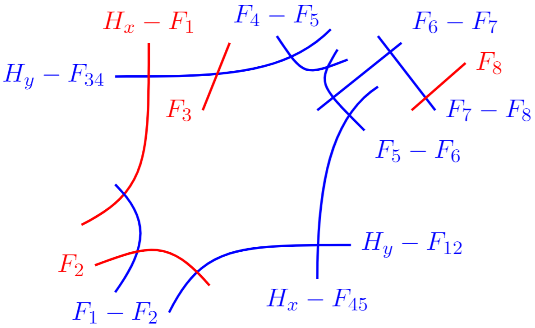




Sakai surface

- We see that the point (a, b) of blowing-up becomes a line E that we call exceptional divisor.
- From this point, we will use the algebro-geometric language of divisors.
- After eight blowing-ups, our $\mathbb{P}^1 \times \mathbb{P}^1$ initial space becomes Sakai surface $\mathcal{X}$.
- Divisors forms a basis on Sakai surface called Picard lattice: $\mathrm{Pic}(\mathcal{X}) = \mathrm{Span}\{H_x, H_y, E_1, \dots E_8\}$.
- If we know the change of basis matching Picard lattice of our equation with the Picard lattice of the standard equation, it is easy to obtain an explicit change of coordinates.

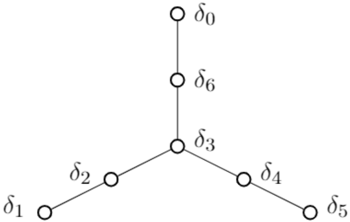
- After eight blowing-ups, we have the following Sakai surface:



- Blue lines are -2 curves, if you look closely at them ...

Sakai surface

- ... you will see that they form an $E_6^{(1)}$ affine root system:



$$
\begin{aligned}
\delta_0 &= \mathcal{F}_7 - \mathcal{F}_8, & \delta_4 &= \mathcal{H}_x - \mathcal{F}_4 - \mathcal{F}_5,\\
\delta_1 &= \mathcal{H}_y - \mathcal{F}_3 - \mathcal{F}_4, & \delta_5 &= \mathcal{H}_y - \mathcal{F}_1 - \mathcal{F}_2,\\
\delta_2 &= \mathcal{F}_4 - \mathcal{F}_5, & \delta_6 &= \mathcal{F}_6 - \mathcal{F}_7.\\
\delta_3 &= \mathcal{F}_5 - \mathcal{F}_6, & &
\end{aligned}
$$

- This is the type of our surface!
- Comparing our choice of $E_6^{(1)}$ roots with the standard one, we can obtain a preliminary change of basis on the Picard lattice.

Symmetries of surface

- For differential Painlevé equations, we have the special class of the so-called Backlund transformations that transform the solutions of equation to the solution of equation from the same Painlevé family. For example, P-II:

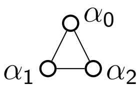
$$y'' = 2y^3 + ty + \beta - \frac{1}{2},$$

  has two Backlunds $s(\widetilde{q} = q + \beta/p, \widetilde{p} = p, \widetilde{t} = t)$ and $r(\widetilde{q} = -q,$ $\widetilde{p} = -p + 2q^2 + t, \widetilde{t} = t)$.
- It turns out that the Backlund transformations preserve the type of Sakai surface.
- Discrete Painlevé equation is nothing but a some combination of Backlund transformations ($r \circ s$ is d-P $(A_1^{(1)}/E_7^{(1)})$).

Group of symmetries

- How to find all Backlund transformations?
- All symmetries of the $E_6^{(1)}$ Sakai surface are described by $\widetilde{W}(A_2^{(1)})$ group:

$$\widetilde{W}(A_2^{(1)}) = \mathrm{Aut}(A_2^{(1)}) \ltimes W(A_2^{(1)}) = \mathbb{D}_3 \ltimes W \left(\quad \right)$$



- We have three Backlunds w_0, w_1, w_2 from the $W(A_2^{(1)})$ group and two π_1, r from the automorphisms of $W(A_2^{(1)})$.

Dynamic of the equation

- To finally obtain the change of variables, we need to compare the dynamics of our discrete system with the dynamics of the standard discrete $(A_2^{(1)}/E_6^{(1)})$ Painlevé equation:

$$\psi = r \circ w_1 \circ w_0, \quad \varphi_{st} = r \circ w_0 \circ w_2$$

- We see that they are indeed equivalent up to small transformation of basis:

$$\psi_f = r \circ \varphi_{st} \circ r^{-1}$$

- Acting on our basis by r^{-1}, we can obtain the final change of basis on the Picard lattice.

Change of variables

- After some calculations, we can obtain the explicit change of coordinates that matches our discrete system and standard d-P$(A_2^{(1)}/E_6^{(1)})$ Painlevé equation:

$$
\begin{aligned}
\mathcal{H}_x &= 2\mathcal{H}_f + \mathcal{H}_g - \mathcal{E}_{3567}, \\
\mathcal{H}_y &= \mathcal{H}_f + \mathcal{H}_g - \mathcal{E}_{56}, \\
\mathcal{F}_1 &= \mathcal{H}_f + \mathcal{H}_g - \mathcal{E}_{567}, \\
\mathcal{F}_2 &= \mathcal{E}_8, \quad \mathcal{F}_3 = \mathcal{E}_4, \qquad\qquad \to x(f,g) = f(g-f-c), \\
\mathcal{F}_4 &= \mathcal{H}_f + \mathcal{H}_g - \mathcal{E}_{356}, \qquad \to y(f,g) = \sqrt{2}(g-f-c). \\
\mathcal{F}_5 &= \mathcal{H}_f - \mathcal{E}_6, \\
\mathcal{F}_6 &= \mathcal{H}_f - \mathcal{E}_5, \\
\mathcal{F}_7 &= \mathcal{E}_1, \quad \mathcal{F}_8 = \mathcal{E}_2.
\end{aligned}
$$


Merci