Introduction to twistors and supertwistors

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XVII International DIAS-BLTP Winter School
"Supersymmetry and Integrability"
31 January - 4 February, 2022, Dubna, Russia

These lectures are devoted to brief discussion of the main positions of the twistor theory.

Twistor formalism discovered by R. Penrose in 1967 is effective in the study of dynamical systems possessing conformal symmetries. But their use is not limited to such theories. Twistors are used

- in gravity;
- in the construction of new supersymmetric models;
- in the higher spin (HS) theory;
- in the calculation of scattering amplitudes;
- and in other areas of theoretical physics.

Twistors give us additional tools to explore existing theories and also provide alternative description of these theories in addition to the widely used space-time formalism.

Here we will consider the use of twistors to describe irreducible representations of the Poincaré group in the $4D$ space-time, that is, in elementary particle physics.

Monographs/reviews on this subject

- R. Penrose, W. Rindler, Spinors and Space-Time: V. 2, Spinor and Twistor Methods in Space-Time Geometry, Cambridge University Press, 1986.
- R. Penrose, M.A.H. MacCallum, Twistor theory: an approach to the quantization of fields and spacetime, Phys. Rept. 6 (1972) 241.
- L.P. Hughston, Twistors and particles, Lecture Notes In Physics, Vol. 97, Springer-Verlag, Berlin, 1979.
- S.A. Hugget, K.P. Tod, Introduction to the Twistor Theory, Cambridge U. Press, 1994.

The lectures will also use the results obtained in our work, carried out in collaboration with *I.L. Buchbinder, J. de Azcarraga, E.A. Ivanov, A.P. Isaev, J. Lukierski, V.G. Zima.* (see arXiv and inSPIRES).

Plan

- Lecture 1:
  - Conformal symmetry and twistors. Twistor space.
  - Penrose twistor transform and twistor formulation of massless particles.
  - Field twistor transform.
  - Twistorial description of higher spin particle.
- Lecture 2:
  - Bitwistor formulation of massive particles and massless infinite spin particles.
  - Conformal supersymmetry and supertwistors.
  - Twistor description of massless superparticles.
  - Twistor transform for spinning particles.

Lecture 1

- Conformal symmetry and twistors. Twistor space.
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Conformal symmetry

The appearance of twistors and their wide application is associated with the study of systems with conformal symmetry.

Let us first consider massless particle of zero helicity as the simplest illustrative, but, at the same time, concrete and physically meaningful example.

Massless particle with zero helicity plays an important role in the subject of our study. In addition to its relative simplicity, massless particle of zero helicity has all the necessary formulations (purely space-time, purely twistor and mixed ones) and interrelations between them (Penrose twistor transformations), both at the classical (mechanical) and at the primary-quantized (field) level. This causes a constant reference to it when constructing more complex systems, such as spinning (super)particle, higher spin (super)particles or (super)strings, where the full twistor picture has not yet been established.

In space-time description the action of massless zero-helicity (spinless) particle looks like ($||\eta_{mk}|| = \mathrm{diag}(+1,-1,-1,-1)$, $m = 0,1,2,3$)

$$S_0^{s.-t.} = \int d\tau \left(p_m \dot{x}^m - \frac{1}{2}\, e p^2 \right) \qquad \Rightarrow \qquad \int d\tau \, \frac{1}{2e}\, \dot{x}_m \dot{x}^m$$

where $x^m(\tau)$ and $p_m(\tau)$ are the coordinate and momentum of the particle, $\{x^m, p_n\}_P = \delta^m_n$; τ is evolution parameter. The variable $e(\tau)$ is Lagrange multiplier for the mass constraint

$$p^2 \equiv p^m p_m \approx 0\,.$$

When quantized, this constraint produces the massless Klein-Gordon equation

$$\Box \Phi(x) \equiv \partial^m \partial_m \Phi(x) = 0\,.$$

The action is invariant under transformations (a^m, ℓ^{mn}, k_m, c are the constant parameters)

$$\delta x^m = a^m + \ell^{mn} x_n + c x^m + 2(k\cdot x) x^m - x^2 k^m \,,$$
$$\delta p_m = \ell_{mn} p^n - c p_m + 2(k\cdot p) x_m - 2(k\cdot x) p_m - 2(x\cdot p) k_m \,,$$
$$\delta e = 2ce + 4(k\cdot x) e \,,$$

whose generators (the Noether charges)

$$P_m = p_m \,, \qquad M_{mn} = x_m p_n - x_n p_m \,, \qquad D = x^m p_m \,, \qquad K_m = 2(x\cdot p) x_m - x^2 p_m$$

form the algebra with respect to Poisson brackets $\{x^m, p_n\}_P = \delta^m_n$:

$$\{M_{mn}, M_{lk}\}_P = \eta_{ml} M_{nk} - \eta_{mk} M_{nl} - (m \leftrightarrow n) \,, \qquad \{M_{mn}, P_l\}_P = \eta_{ml} P_n - (m \leftrightarrow n) \,,$$
$$\{P_m, K_n\}_P = 2M_{mn} - 2\eta_{mn} D \,, \qquad \{M_{mn}, K_l\}_P = \eta_{ml} K_n - (m \leftrightarrow n) \,,$$
$$\{P_m, D\}_P = -P_m \,, \qquad \{K_m, D\}_P = K_m \,.$$

This algebra is called the conformal algebra.

This conformal algebra has another representation, more convenient for our subject. After collection 15 generators into antisymmetric tensor $J_{\mathcal{M}\mathcal{N}} = -J_{\mathcal{N}\mathcal{M}}$, $\mathcal{M} = (1', 0'; m)$,

$$J_{mn} = M_{mn} \,, \quad J_{m0'} = \tfrac{1}{2}(P_m + K_m) \,, \quad J_{m1'} = \tfrac{1}{2}(P_m - K_m) \,, \quad J_{0'1'} = D \,,$$

the conformal algebra takes the form

$$\{J_{\mathcal{M}\mathcal{N}}, J_{\mathcal{K}\mathcal{L}}\}_P = \eta_{\mathcal{M}\mathcal{K}} J_{\mathcal{N}\mathcal{L}} - \eta_{\mathcal{M}\mathcal{L}} J_{\mathcal{N}\mathcal{K}} - (\mathcal{M} \leftrightarrow \mathcal{N}) \,,$$

where $\eta_{\mathcal{M}\mathcal{N}} = \eta_{\mathcal{N}\mathcal{M}}$ has the following components: η_{mn} and $\eta_{m0'} = \eta_{m1'} = \eta_{0'1'} = 0$, $\eta_{0'0'} = -\eta_{1'1'} = +1$, and, in fact, is the metric tensor of the 6-dimensional spaces with the signature $(++----)$.

This algebra is nothing but the $so(2,4)$ algebra. That is, conformal symmetry is described by the group $SO(2,4)$, which is the symmetry group of 6-dimensional space with two times.

Poincaré transformations, including Lorentz transformations (parameters ℓ^{mn}) and Poincaré translations (parameters a^m) are realized by linear transformations. Accounting this symmetry is well known: the use of Lorentz-covariant quantities and the presence of coordinates x^m outside the fields only through the derivative ∂_m in the field equations.

But conformal boosts are realized by nonlinear transformations. So, under conformal boosts

$$\delta\Box = -4(kx)\Box + 4k^m\partial_m \,,$$

Therefore, the conformal invariance of even the Klein-Gordon equation implies the following transformation of the massless scalar field

$$\delta\Phi = -2(kx)\,\Phi \,.$$

Already consideration of this simple system suggests to us the desire to have a formulation in which conformal $SO(2,4)$ transformations are realized by linear transformations. This becomes more relevant when considering more complex physical systems.

One way to the linear realization of conformal $SO(2,4)$ symmetry is consideration of the spaces (coordinate or field spaces) with $SO(2,4)$-tensors.

But to be able to describe all representations, including spinor representations, it is natural to consider the corresponding spinor group, homomorphic to the $SO(2,4)$ group, $SO(2,4) \cong Spin(2,4) \cong SU(2,2)$ (an analogue of $SL(2,C)$ for $SO(1,3)$):

$$G \in SU(2,2): \quad \det G = 1\,, \quad G^{+}gG = g\,, \quad g = \mathrm{diag}(\mathbb{1}_2, -\mathbb{1}_2)\,.$$

Then all conformal transformations (linear homogeneous, inhomogeneous and nonlinear) are realized as linear $SU(2,2)$-spinor transformations of the corresponding space.

The solution of this problem led R. Penrose to the twistor theory.

4D spinor notations used in these lectures

The space-time metric is $\eta_{mn} = \mathrm{diag}(+1, -1, -1, -1)$.

Totally antisymmetric tensor ε_{mnkl} has the component $\varepsilon_{0123} = -1$.

Four-component Dirac spinor Ψ is represented by two Weyl spinors $\Psi = \begin{pmatrix} \psi_\alpha \\ \bar{\chi}^{\dot\alpha} \end{pmatrix}$.

Two-component Weyl spinor indices are raised and lowered by $\epsilon_{\alpha\beta}$, $\epsilon^{\alpha\beta}$, $\epsilon_{\dot\alpha\dot\beta}$, $\epsilon^{\dot\alpha\dot\beta}$ with nonvanishing components $\epsilon_{12} = -\epsilon_{21} = \epsilon^{21} = -\epsilon^{12} = 1$: $\psi_\alpha = \epsilon_{\alpha\beta}\psi^\beta$, $\psi^\alpha = \epsilon^{\alpha\beta}\psi_\beta$, etc.

The Dirac matrices γ_m obey the Clifford algebra $\{\gamma_m, \gamma_n\} = 2\eta_{mn}$. In the Weyl representation they have the form $\gamma_m = \begin{pmatrix} 0 & (\sigma_m)_{\alpha\dot\beta} \\ (\tilde\sigma_m)^{\dot\alpha\beta} & 0 \end{pmatrix}$. Relativistic σ-matrices are $(\sigma_m)_{\alpha\dot\beta} = (\mathbf{1_2}; \sigma_1, \sigma_2, \sigma_3)_{\alpha\dot\beta}$, where $\sigma_1, \sigma_2, \sigma_3$ are the Pauli matrices. The matrices $(\tilde\sigma_m)^{\dot\alpha\beta} = \epsilon^{\dot\alpha\dot\delta}\epsilon^{\beta\gamma}(\sigma_m)_{\gamma\dot\delta} = (\mathbf{1_2}; -\sigma_1, -\sigma_2, -\sigma_3)^{\dot\alpha\beta}$ satisfy

$$\sigma^m_{\alpha\dot\gamma}\tilde\sigma^{n\,\dot\gamma\beta} + \sigma^m_{\alpha\dot\gamma}\tilde\sigma^{n\,\dot\gamma\beta} = 2\,\eta^{mn}\delta^\beta_\alpha\,, \qquad \sigma^m_{\alpha\dot\beta}\tilde\sigma^{\dot\beta\alpha}_n = 2\,\delta^m_n\,.$$

The link between Minkowski four-vectors and spinorial quantities is given by $A_{\alpha\dot\beta} = \frac{1}{\sqrt2}A_m(\sigma^m)_{\alpha\dot\beta}$, $A^{\dot\alpha\beta} = \frac{1}{\sqrt2}A_m(\tilde\sigma^m)^{\dot\alpha\beta}$, $A_m = \frac{1}{\sqrt2}A_{\alpha\dot\beta}(\tilde\sigma_m)^{\dot\beta\alpha}$, so that $A^mB_m = A_{\alpha\dot\beta}B^{\dot\beta\alpha}$.

The σ-matrices with two vector indices are defined by $(\sigma_{mn})_\alpha{}^\beta = -\frac14(\sigma_m\tilde\sigma_n - \sigma_n\tilde\sigma_m)_\alpha{}^\beta$, $(\tilde\sigma_{mn})^{\dot\alpha}{}_{\dot\beta} = -\frac14(\tilde\sigma_m\sigma_n - \tilde\sigma_n\sigma_m)^{\dot\alpha}{}_{\dot\beta}$ and satisfy the identities $\varepsilon^{mnkl}\sigma_{kl} = -2i\,\sigma^{mn}$, $\varepsilon^{mnkl}\tilde\sigma_{kl} = 2i\,\tilde\sigma^{mn}$. We represent the antisymmetric second rank vector tensor in the form

$$X_{[mn]} = (\sigma_{mn})^{\alpha\beta}X_{(\alpha\beta)} - (\tilde\sigma_{mn})^{\dot\alpha\dot\beta}\bar X_{(\dot\alpha\dot\beta)}\,.$$

Twistor space

In twistor theory, conformally invariant systems are formulated in the space parameterized by commuting $SU(2,2)$-spinor Z_A, $A=1,...,4$. As we will see below, this space actually replaces the usual phase space formed by 4-vectors x^m and p_m.

To obtain results in terms of ordinary $4D$ spin-tensor fields, it is convenient to consider the representation, in which the $SU(2,2)$-spinor

$$Z_A = (\pi_\alpha, \omega^{\dot\alpha})\,, \qquad Z_A \;\to\; G_A{}^B Z_A\,, \quad G\in SU(2,2)$$

is formed from two $4D$ Weyl spinors of opposite chirality π_α, $\omega^{\dot\alpha}$ ($\alpha=1,2$, $\dot\alpha=1,2$). Following R.Penrose, we use spinor $\omega^{\dot\alpha}$, which has dotted index, but without 'bar'. We point out that spinors are c-number.

Conjugate $4D$ spinors $\bar\pi_{\dot\alpha}=(\pi_\alpha)^*$, $\bar\omega^\alpha=(\omega^{\dot\alpha})^*$ form the $SU(2,2)$-spinor $\bar Z_{\dot A}=(\bar\pi_{\dot\alpha},\bar\omega^\alpha)\,,$ which transforms according to the complex conjugate representation.

Using $SU(2,2)$-invariant tensors $g^{A\dot B}$, $g_{A\dot B}$, which in the chosen representation have the form

$$g^{\dot AB} = \begin{pmatrix} 0 & -\delta^{\dot\alpha}{}_{\dot\beta} \\ \delta_\alpha{}^\beta & 0 \end{pmatrix}, \quad g_{A\dot B} = \begin{pmatrix} 0 & \delta_\alpha{}^\beta \\ -\delta^{\dot\alpha}{}_{\dot\beta} & 0 \end{pmatrix}, \qquad g_{A\dot C}g^{\dot CB}=\delta_A^B\,, \quad g^{\dot AC}g_{C\dot B}=\delta_{\dot A}^{\dot B}\,,$$

$\Big((G^+)^{\dot A}{}_{\dot B}g^{\dot BB}G_B{}^A = g^{\dot AA}\,, G\in SU(2,2)\Big)$ spinor $\bar Z_{\dot A}$ defines the $SU(2,2)$-spinor

$$\bar Z^A = \bar Z_{\dot B}g^{\dot BA} = (\bar\omega^\alpha, -\bar\pi_{\dot\alpha})\,,$$

which transforms using the inverse $SU(2,2)$-matrix: $\bar Z^A \;\to\; \bar Z^B\,(G^{-1})_B{}^A$,.

Contraction of the spinor Z_A and its conjugate $\bar{Z}^A$ defines the Hermitian form

$$\Lambda \equiv \frac{i}{2}\,\bar{Z}^A Z_A = \frac{i}{2}\,\bar{Z}_{\dot{B}} g^{\dot{B}A} Z_A = \frac{i}{2}\,(\bar{\omega}^{\alpha}\pi_{\alpha} - \bar{\pi}_{\dot{\alpha}}\omega^{\dot{\alpha}})$$

which is $SU(2,2)$-invariant and is the norm of the $SU(2,2)$-spinor Z_A.

By definition,
the twistor space $\mathbf{T}$ is the spinor space (space $\mathbf{C}^4$) of the conformal group $SU(2,2)$ with Hermitian form Λ.

The $SU(2,2)$-spinors Z_A, defined on this space, are called twistors.

Depending on the value of the Hermitian form, the following subsets of the twistor space are distinguished:

- the space of positive twistors $\mathbf{T}_+$, when $\Lambda > 0$;
- the space of negative twistors $\mathbf{T}_-$, when $\Lambda < 0$;
- the space of isotropic twistors $\mathbf{T}_0$, when $\Lambda = 0$.

Physical meaning of the twistor norm Λ will be given below.

Comment: In some papers, the imaginary unit i is used as additional factor in the definition of the $SU(2,2)$-metric $g^{\dot{A}B}$, and then this factor is absent in the definition of the twistor norm Λ. We use conventions where such a factor will be used less in further expressions.

Infinitesimal conformal transformations are realized in twistor space by linear transformations:

$$\delta Z_A = L_A{}^B Z_B\,, \qquad g^{\dot A B} L_B{}^A = (L^+)^{\dot A}{}_{\dot B}\, g^{\dot B A}\,,$$

or in spinor components

$$\delta\pi_\alpha = -\ell_\alpha{}^\beta \pi_\beta - \tfrac{1}{2}\,c\,\pi_\alpha - 2k_{\alpha\dot\beta}\,\omega^{\dot\beta}\,, \quad \delta\omega^{\dot\alpha} = \bar\ell^{\dot\alpha}{}_{\dot\beta}\,\omega^{\dot\beta} + \tfrac{1}{2}\,c\,\omega^{\dot\alpha} + a^{\dot\alpha\beta}\pi_\beta\,.$$

The generators of these transformations are found after giving symplectic structure in the twistor space. Defining Poisson brackets in twistor space

$$\{\bar Z^A, Z_B\}_P = \delta^A_B \qquad \Rightarrow \qquad \{\bar\omega^\alpha, \pi_\beta\}_P = \delta^\alpha_\beta\,, \qquad \{\omega^{\dot\alpha}, \bar\pi_{\dot\beta}\}_P = \delta^{\dot\alpha}_{\dot\beta}\,,$$

we get that conformal transformations are generated by the following bilinear combinations of twistor components

$$P_{\alpha\dot\alpha} = \pi_\alpha \bar\pi_{\dot\alpha}\,, \quad K^{\dot\alpha\alpha} = \omega^{\dot\alpha}\bar\omega^\alpha\,, \quad M_{\alpha\beta} = \pi_{(\alpha}\bar\omega_{\beta)}\,, \quad \bar M_{\dot\alpha\dot\beta} = \bar\pi_{(\dot\alpha}\omega_{\dot\beta)}\,, \quad D = \tfrac{1}{2}\,(\bar\omega^\alpha\pi_\alpha + \bar\pi_{\dot\alpha}\omega^{\dot\alpha})\,.$$

The generators form conformal algebra with respect to the Poisson twistor brackets and preserve the twistor norm. In terms of 4-component twistors, conformal generators are represented as traceless product of the twistor and its conjugate one:

$$\bar Z^A Z_B - \tfrac{1}{4}\,\delta^A_B\,\bar Z^C Z_C\,.$$

Having linear realization of conformal symmetry in terms of twistor variables, we can find twistor formulation of massless particle and its link with space-time description.

We use the notations $P_{\alpha\dot\alpha} = \frac{1}{\sqrt2}\,P_m\sigma^m_{\alpha\dot\alpha}$, $P_m = \frac{1}{\sqrt2}\,P_{\alpha\dot\alpha}\tilde\sigma_m^{\dot\alpha\alpha}$, that is $P_m \sim P_{\alpha\dot\alpha}$ and $M_{[mn]} = (\sigma_{mn})^{\alpha\beta}M_{(\alpha\beta)} - (\tilde\sigma_{mn})^{\dot\alpha\dot\beta}\bar M_{(\dot\alpha\dot\beta)}$.

Penrose twistor transform and twistor formulation of massless particles

Comparison of generators of conformal symmetry in space-time and twistor formulations leads to the following observations:
- four-momentum p_m is represented in the form of the product of the spinor π_α and its conjugate,
- second twistor spinor $\omega^{\dot\alpha}$ is proportional to the product of four-coordinate x^m and the spinor π_α.

In fact, exact expressions for the link space-time and twistor variables is determined by the relations of the Penrose transform:

$$p_{\alpha\dot\alpha} = \pi_\alpha \bar\pi_{\dot\alpha}\,, \qquad (a)$$

$$\omega^{\dot\alpha} = x^{\dot\alpha\beta}\pi_\beta\,, \qquad \bar\omega^{\alpha} = \bar\pi_{\dot\beta} x^{\dot\beta\alpha}\,. \qquad (b)$$

Characteristic features of Penrose transform:

- Important point is the Hermitianity of the matrix $x^{\dot\alpha\beta} = \frac{1}{\sqrt{2}} x^m \tilde\sigma_m^{\dot\alpha\beta}$ constructed by using the real vector x^m: $(x^{\dot\alpha\beta})^* = x^{\dot\beta\alpha}$.
- When performing the Penrose transformation, the twistor representation of the conformal generators goes over into the space-time representation.
- From the conformal transformations of twistors we obtain corresponding transformations of space-time variables.
- Relations are consistent with symplectic structures in twistor and space-time phase spaces. Namely, Poisson brackets for $p_{\alpha\dot\alpha}$ and $\omega^{\dot\alpha}$ are is the same if they are computed with twistor brackets $\{\bar\omega^\alpha, \pi_\beta\}_P = \delta_\beta^\alpha$ or Poisson brackets $\{x^{\dot\alpha\alpha}, p_{\beta\dot\beta}\}_P = \delta_\beta^\alpha \delta_{\dot\beta}^{\dot\alpha}$.

Twistor transform equations have transparent physical and geometric meaning:

- The equation (a) implies automatically that the particle four-momentum $p_{\alpha\dot\alpha} = \pi_\alpha \bar\pi_{\dot\alpha}$ is light-like

$$p^2 = p^{\alpha\dot\alpha} p_{\alpha\dot\alpha} = 0$$

due to the automatic fulfillment of the identity

$$\pi^\alpha \pi_\alpha = \epsilon^{\alpha\beta} \pi_\beta \pi_\alpha = \epsilon^{[\alpha\beta]} \pi_{(\beta} \pi_{\alpha)} \equiv 0\,,$$

which is valid for commuting $4D$ Weyl spinors.

- For fixed twistor $Z_A = (\pi^\alpha, \omega^{\dot\alpha})$, the solution of the equations (b) (incident conditions) with respect to space-time coordinate x^m

$$x^{\dot\alpha\alpha} = x_0^{\dot\alpha\alpha} + a\pi^\alpha \bar\pi^{\dot\alpha}\,, \qquad x_0^{\dot\alpha\alpha} = 2\omega^{\dot\alpha} \bar\omega^\alpha / (\pi_\beta \bar\omega^\beta + \bar\pi_{\dot\beta} \omega^{\dot\alpha})$$

contains an arbitrary real constant a, which parameterizes the light-like line (lightray) in the Minkowski space with the direction vector $\pi^\alpha \bar\pi^{\dot\alpha}$.

Incident conditions (b) have additional important consequence: the twistor appearing in them is isotropic:

$$\Lambda = \tfrac{i}{2}\,\bar{Z}^A Z_A = \tfrac{i}{2}\,(\bar{\omega}^{\alpha}\pi_{\alpha} - \bar{\pi}_{\dot{\alpha}}\omega^{\dot{\alpha}}) = 0\,.$$

This result is achieved due to the Hermiticity of the matrix $x^{\dot{\alpha}\alpha}$ in (b). This constraint generates local phase transformations

$$Z_A = (\pi_{\alpha}, \omega^{\dot{\alpha}}) \quad \rightarrow \quad e^{i\varphi} Z_A = (e^{i\varphi}\pi_{\alpha}, e^{i\varphi}\omega^{\dot{\alpha}})$$

and leaves the twistor transformations invariant.

What is the physical meaning of the twistor norm Λ?

The answer to this question is found after calculating the Pauli-Lubanski vector

$$W_m = \tfrac{1}{2}\,\varepsilon_{mnkl} P^n M^{kl}\,, \qquad W_{\alpha\dot{\alpha}} = i\left(P_{\alpha}{}^{\dot{\beta}}\bar{M}_{\dot{\beta}\dot{\alpha}} - P^{\beta}{}_{\dot{\alpha}} M_{\beta\alpha}\right).$$

In the twistor realization of Poincaré generators, we have

$$W_{\alpha\dot{\alpha}} \;=\; \Lambda\, P_{\alpha\dot{\alpha}}\,.$$

But as is known from the representation theory of the Poincaré group, "the proportionality operator" between the Pauli-Lubanski operator and the energy-momentum operator is exactly equal to the helicity in case of massless representations of fixed helicity.

Thus, the norm Λ of the twistor coincides with the helicity of the massless particle which is described by this twistor.

Thus, twistor transformations link the space-time and twistor formulations of massless particle of zero helicity.

- In space-time formulation there is mass constraint $p^2 \approx 0$, and the condition of equality to zero of helicity $\Lambda = 0$ is fulfilled automatically;
- In twistor formulation there is the spin (helicity) constraint $\Lambda \approx 0$ under resolved massless condition $p^2 = 0$.

In twistor formulation the action of massless particle of zero helicity has the form

$$S_0^{twistor} = \frac{1}{2} \int d\tau \left[\bar{Z}^A \dot{Z}_A - \dot{\bar{Z}}^A Z_A \; - \; \frac{i}{2} \, l \, \bar{Z}^A Z_A \right] ,$$

where $l(\tau)$ is Lagrange multiplier for twistor constraint $\Lambda \approx 0$. Up to the total derivative, this action in terms of $4D$ spinors takes the form

$$S_0^{twistor} = \int d\tau \left[\dot{\bar{\omega}}^{\alpha} \pi_{\alpha} + \bar{\pi}_{\dot{\alpha}} \dot{\omega}^{\dot{\alpha}} \; - \; \frac{i}{2} \, l (\bar{\omega}^{\alpha} \pi_{\alpha} - \bar{\pi}_{\dot{\alpha}} \omega^{\dot{\alpha}}) \right] .$$

We can check that the number of physical degrees of freedom is equal to six both in the space-time system and in twistor one
(the first class constraint eliminates two degrees of freedom in the phase space systems).

Twistor wave function

Let us find the twistor wave function and link it with the scalar field resulting from quantization of this model in space-time formulation.

In the transition to quantum theory, the Poisson brackets go over to the commutator

$$[\hat{\bar{Z}}^A, \hat{Z}_B] = i\delta^A_B\ : \qquad [\hat{\bar{\omega}}^\alpha, \hat{\pi}_\beta] = i\delta^\alpha_\beta\,, \qquad [\hat{\omega}^{\dot\alpha}, \hat{\bar{\pi}}_{\dot\beta}] = i\delta^{\dot\alpha}_{\dot\beta}\,.$$

It is convenient to quantize twistor particle in holomorphic representation (Penrose representation), when the operators $\hat{Z}_A$ are diagonal, and $\hat{\bar{Z}}^A$ are realized by differentiation operators

$$\hat{\bar{Z}}^A = i\frac{\partial}{\partial Z_A} \qquad \text{or in spinor components} \quad \hat{\bar{\pi}}_{\dot\alpha} = -i\frac{\partial}{\partial \omega^{\dot\alpha}}\,, \quad \hat{\bar{\omega}}^\alpha = i\frac{\partial}{\partial \pi_\alpha}\,.$$

In this holomorphic representation twistor wave function

$$\Psi(Z) = \Psi(\pi, \omega)$$

satisfies the equation

$$\hat{\Lambda}\Psi(Z) = 0\,,$$

which is quantum counterpart of the classical twistor constraint $\Lambda \approx 0$.

Carrying out the Weyl ordering in the helicity operator

$$\Lambda = \tfrac{i}{2}\,\bar{Z}^A Z_A \qquad \rightarrow \qquad \hat{\Lambda} = \tfrac{i}{4}\,(\hat{\bar{Z}}^A \hat{Z}_A + \hat{Z}_A \hat{\bar{Z}}^A) = \tfrac{i}{2}\,\hat{Z}_A \hat{\bar{Z}}^A - 1 = -\tfrac{1}{2}\,Z_A \frac{\partial}{\partial Z_A} - 1\,,$$

we obtain that the equation for the twistor wave function has the form

$$\tfrac{1}{2} Z_A \frac{\partial}{\partial Z_A} \Psi = -\Psi \qquad (*)$$

or, in writing through the spinor components of the twistor,

$$\tfrac{1}{2}(\pi_\alpha \frac{\partial}{\partial \pi_\alpha} + \omega^{\dot{\alpha}} \frac{\partial}{\partial \omega^{\dot{\alpha}}})\Psi = -\Psi\,.$$

Thus, the twistor wave function of the system under consideration is the holomorphic homogeneous function of the homogeneity degree (-2):

$$\Psi^{(-2)}(\alpha Z) = \alpha^{-2}\Psi^{(-2)}(Z)\,, \qquad (**)$$

where α is an arbitrary complex number.

Remark.
Equation $(*)$ is equivalent to the equation $(**)$:
Acting the operator $\alpha\frac{\partial}{\partial\alpha}$ on the left and right sides of the equation $(**)$ and after that putting the value $\alpha = 1$, we obtain the equation $(*)$.

Field twistor transform

In Penrose twistor approach, the usual space-time field is obtained from the twistor field by means of the Penrose twistor transform for the fields. It is constructed as follows.

As first step, the spinor ω in twistor field is resolved by using the incidence condition

$$\Psi^{(-2)}(Z)\Big|_{\omega^{\dot\alpha}=x^{\dot\alpha\alpha}\pi_\alpha} = \Psi^{(-2)}(\pi_\alpha, x^{\dot\alpha\alpha}\pi_\alpha)\,.$$

Due to the homogeneity of twistor field, this function is actually defined on the complex projective space $\mathbf{CP}^1$ and depends effectively on one complex variable if we take into account homogeneity. For example, from the ratio $z \equiv \pi_1/\pi_2$ at $\pi_2 \neq 0$.

Integrating the twistor field over this variable, we obtain the usual space-time field. In covariant notation, independent of the choice of coordinate on $\mathbf{CP}^1$, the field is integrated with the measure $\pi d\pi \equiv \pi^\alpha d\pi_\alpha$

$$\Phi(x) = \oint \pi d\pi\, \Psi^{(-2)}(\pi_\alpha, x^{\dot\alpha\alpha}\pi_\alpha)\,,$$

so the integrand is the invariant of the transformation $\pi \to \alpha\pi$ for the twistor field. In this integral transformation, the integration is carried out along the closed contour in the space of independent complex variable, enclosing the poles of the twistor field $\Psi^{(-2)}$.

This integral transformation is the Penrose twistor transform for scalar field. It is important that the field $\Phi(x)$ obtained in this way automatically satisfies the Klein-Gordon equation

$$\partial^{\alpha\dot\alpha}\partial_{\alpha\dot\alpha}\Phi(x) = 0\,.$$

This is the result of the dependence of the twistor field on $x^{\dot\alpha\alpha}$ only in combination $x^{\dot\alpha\alpha}\pi_\alpha$ with commuting spinor π_α, for which the identity $\pi^\alpha\pi_\alpha \equiv 0$ holds.

Twistor formulation of massless particle of arbitrary fixed helicity

In the twistor formulation, the particle helicity is determined by the twistor norm. Consequently, the phase space of massless particle of helicity s must contain the constraint

$$\Lambda - s = \tfrac{i}{2}\bar{Z}^A Z_A - s = \tfrac{i}{2}(\bar{\omega}^{\alpha}\pi_{\alpha} - \bar{\pi}_{\dot{\alpha}}\omega^{\dot{\alpha}}) - s \approx 0\,,$$

generalizing twistor constraint for zero-helicity particle.

The action

$$S_s^{twistor} = \int d\tau \left[\tfrac{1}{2}\,(\bar{Z}^A \dot{Z}_A - \dot{\bar{Z}}^A Z_A) - l\,(\tfrac{i}{2}\,\bar{Z}^A Z_A - s) \right]\,,$$

in which the constraint $\Lambda - s \approx 0$ is introduced through the term with the Lagrangian multiplier l, determines the twistor formulation of massless particle of helicity s.

After quantization, the twistor constraint $\Lambda - s \approx 0$ generates the equation for the twistor wave function

$$\tfrac{1}{2} Z_A \frac{\partial}{\partial Z_A}\Psi = -(1+s)\Psi\,.$$

Thus, the twistor field of massless helicity particle s is holomorphic homogeneous function of the homogeneity degree $(-2-2s)$:

$$\Psi^{(-2-2s)}(Z)\,, \qquad \Psi^{(-2-2s)}(\alpha Z) = \alpha^{-2-2s}\Psi^{(-2-2s)}(Z)\,.$$

Field twistor transform for arbitrary helicity

In the case of non-zero helicity, the space-time fields can be obtained from the twistor ones in the previously discussed way, by using the incidence conditions and the Penrose field transforms:

$$\Phi_{\alpha_1\ldots\alpha_{2s}}(x) = \oint (\pi d\pi)\, \pi_{\alpha_1}\ldots\pi_{\alpha_{2s}}\, \Psi^{(-2-2s)}(\pi_\alpha, x^{\dot\alpha\alpha}\pi_\alpha)\,.$$

In contrast to helicity-zero case, this integrand contains $2s$ components of the spinor π for compensation of the $U(1)$-charge of the twistor field $\Psi^{(-2-2s)}$.

The resulting space-time field is automatically symmetric with respect to the spinor indices due to the commutativity of the twistor components, $\Phi_{\alpha_1\ldots\alpha_{2s}} = \Phi_{(\alpha_1\ldots\alpha_{2s})}$, and satisfies automatically the Dirac-Weyl equation

$$\partial^{\dot\beta\alpha_1}\Phi_{\alpha_1\ldots\alpha_{2s}}(x) = 0\,.$$

That is, the complex field $\Phi_{\alpha_1\ldots\alpha_{2s}}(x)$ is the field strength of massless particle of helicity s.

helicity $1/2$: the Dirac equation

$$\partial_m\gamma^m\Psi(x) = 0\,, \quad \Psi = \begin{pmatrix} \psi_\alpha \\ \bar\chi^{\dot\beta} \end{pmatrix} \qquad \Rightarrow \qquad \partial^{\dot\beta\alpha}\psi_\alpha(x) = 0\,;$$

helicity 1: the Maxwell equation

$$\partial^n F_{mn}(x) = 0\,, \quad F_{mn} = -(\sigma_{mn})_\alpha{}^\beta F^\alpha{}_\beta + (\tilde\sigma_{mn})^{\dot\alpha}{}_{\dot\beta} F_{\dot\alpha}{}^{\dot\beta} \qquad \Rightarrow \qquad \partial^{\dot\gamma\alpha}F_{\alpha\beta}(x) = 0\,.$$

Coordinate twistor transform for arbitrary helicity (difficulties)

The incidence conditions constructed earlier assume zero helicity of the particle. That is, although in the twistor description everything is fine in description of nonzero helicity, but in the space-time picture it is not: something is missing to describe the helicity.

Description of the spinning particle requires consideration of extended spaces that include additional coordinates. From a physical point of view, these additional coordinates are intended to describe the spinning degrees of freedom.

One of the ways to introduce additional coordinates is to consider instead of the real Minkowski space with coordinates $(x^{\dot{\alpha}\beta})^* = x^{\dot{\beta}\alpha}$ ($x^m = (x^m)^*$) its complexification with coordinates $z^{\dot{\alpha}\beta} \neq (z^{\dot{\beta}\alpha})^*$ ($z^m \neq (z^m)^*$, i.e. $z^m = x^m + iy^m$). Modified incident conditions

$$\omega^{\dot{\alpha}} = z^{\dot{\alpha}\beta}\pi_\beta\,, \qquad \bar{\omega}^{\alpha} = \bar{\pi}_{\dot{\beta}}\bar{z}^{\dot{\beta}\alpha}\,, \qquad \text{where} \quad \bar{z}^{\dot{\beta}\alpha} \neq (z^{\dot{\alpha}\beta})^*$$

do not imply the zero norm of the twistor, which is defined by the imaginary part y^m of the complexified coordinate z^m. This way of describing the nonzero helicity is presented in the Penrose twistor theory, which is actually used at the level of the field approach.

But when using the complexified Minkowski space, some important element of the twistor program associated with the standard space-time description is completely lost.

There are other formulations of the massless spinning particle, in which the space-time formulation uses other additional variables of different type, for example, additional spinor coordinates. This formulation will be described later after the presentation of the twistor superparticle, since their descriptions are quite similar.

Shirafuji model

But there is possibility of obtaining nonzero helicity of particle after quantization if we use twistor variables in addition to the usual space-time coordinates in the space-time formulation. Then the twistors present here will carry the description of spinning degrees of freedom of the particle.

The well-known Shirafuji model is just such a model. This type of model will be useful in the presentation of twistorial formulation of the higher spin particles.

In the Shirafuji formulation, the Lagrangian of massless particle is, in fact, the term $p_m\dot{x}^m$ in which the momentum p_m is resolved through twistor spinors by using the Cartan-Penrose relation:

$$S_0^{mix} = \int d\tau \pi_\alpha \bar{\pi}_{\dot{\alpha}} \dot{x}^{\dot{\alpha}\alpha}\,.$$

In this system the Cartan-Penrose relation is reproduced as the constraint

$$p_{\alpha\dot{\alpha}} - \pi_\alpha \bar{\pi}_{\dot{\alpha}} \approx 0\,.$$

Additional constraints in the system are the following ones

$$\zeta^\alpha \approx 0\,, \qquad \bar{\zeta}^{\dot{\alpha}} \approx 0\,,$$

where ζ^α, $\bar{\zeta}^{\dot{\alpha}}$ are canonically conjugate variables to π_α, $\bar{\pi}_{\dot{\alpha}}$:

$$\{\zeta^\alpha, \pi_\beta\}_P = \delta^\alpha_\beta\,, \qquad \{\bar{\zeta}^{\dot{\alpha}}, \bar{\pi}_{\dot{\beta}}\}_P = \delta^{\dot{\alpha}}_{\dot{\beta}}\,.$$

From the eight constraints, six constraints are the second class, and two constraints are the first class. That is, the system has six physical degrees of freedom, like all massless systems considered earlier (the second class constraint eliminates one degree of freedom of the phase space, in contrast to the first class constraint, which eliminates two degrees).

The direct quantization of this system is rather complicated.
But we can move on to the equivalent system only with the first class constraints.

We introduce additional phase variables ν_α, κ^α, $\{\kappa^\alpha, \nu_\beta\}_P = \delta^\alpha_\beta$, and c.c.

Equivalent system only with first class constraints:

$$p_{\alpha\dot\alpha} - (\pi_\alpha - \nu_\alpha)(\bar\pi_{\dot\alpha} - \bar\nu_{\dot\alpha}) \approx 0\,, \qquad i\left(\pi_\alpha \zeta^\alpha + \nu_\alpha \kappa^\alpha - \bar\pi_{\dot\alpha}\bar\zeta^{\dot\alpha} - \bar\nu_{\dot\alpha}\bar\kappa^{\dot\alpha}\right) \approx 0\,,$$

$$\zeta^\alpha + \kappa^\alpha \approx 0\,, \qquad \bar\zeta^{\dot\alpha} + \bar\kappa^{\dot\alpha} \approx 0\,.$$

Initial system is reproduced in the gauge $\nu_\alpha \approx 0$, $\bar\nu_{\dot\alpha} \approx 0$, $\kappa^\alpha \approx 0$, $\bar\kappa^{\dot\alpha} \approx 0$.

But, in extended system we can make the exchange of the variables $\pi_\alpha \to \pi_\alpha - \nu_\alpha$, $\zeta^\alpha \to (\zeta^\alpha - \kappa^\alpha)/2$, $\nu_\alpha \to \pi_\alpha + \nu_\alpha$, $\kappa^\alpha \to (\zeta^\alpha + \kappa^\alpha)/2$. Than, the constraints take the form

$$p_{\alpha\dot\alpha} - \pi_\alpha\bar\pi_{\dot\alpha} \approx 0\,, \qquad i\left(\pi_\alpha\zeta^\alpha - \bar\pi_{\dot\alpha}\bar\zeta^{\dot\alpha}\right) \approx 0\,, \qquad \kappa^\alpha \approx 0\,, \qquad \bar\kappa^{\dot\alpha} \approx 0\,.$$

Variables ν_α, κ^α and c.c. are split off and fully gauged.

For the remaining variables, consider the representation in which ζ^α, $\bar\zeta^{\dot\alpha}$ are diagonal and $\pi_\alpha = i\partial/\partial\zeta^\alpha$, $\bar\pi_{\dot\alpha} = i\partial/\partial\bar\zeta^{\dot\alpha}$.

The wave function is $\Psi = \Psi(x^m, \zeta^\alpha, \bar\zeta^{\dot\alpha})$.

The wave function $\Psi(x^m, \zeta^\alpha, \bar\zeta^{\dot\alpha})$ is defined by the equations of the constraints

$$\textbf{a)} \quad \left(i\partial_{\alpha\dot\alpha} - \frac{\partial}{\partial\zeta^\alpha}\frac{\partial}{\partial\bar\zeta^{\dot\alpha}} \right)\Psi = 0\,, \qquad\qquad \textbf{b)} \quad \left(\zeta^\alpha \frac{\partial}{\partial\zeta^\alpha} - \bar\zeta^{\dot\alpha}\frac{\partial}{\partial\bar\zeta^{\dot\alpha}} \right)\Psi = 2s\,\Psi\,,$$

where $2s$ is some ordering constant. The uniqueness of the wave function Ψ requires that $2s$ be integer number: $2s \in \mathbb{Z}$.

Requiring the polynomial dependence of the field Ψ on the spinor variables ζ^α, $\bar\zeta^{\dot\alpha}$, we find that the equation **b)** has the following solution in the form of an infinite series:

$$\Psi(x,\zeta,\bar\zeta) \;=\; \zeta^{\alpha_1}\ldots\zeta^{\alpha_{2s}} \sum_{k=0}^{\infty} \zeta^{\beta_1}\bar\zeta^{\dot\beta_1}\ldots\zeta^{\beta_k}\bar\zeta^{\dot\beta_k}\,\psi_{\alpha_1\ldots\alpha_{2s}\beta_1\ldots\beta_k\dot\beta_1\ldots\dot\beta_k}(x)\,.$$

Equation **a)** leads to the following conclusions:

- Higher terms of expansion $\psi_{\alpha_1\ldots\alpha_{2s}\beta_1\ldots\beta_k\dot\beta_1\ldots\dot\beta_k}(x)$ at $k \neq 0$ are not independent:

$$\psi_{\alpha_1\ldots\alpha_{2s}\beta_1\ldots\beta_k\dot\beta_1\ldots\dot\beta_k} = i^k\,\partial_{\beta_1\dot\beta_1}\ldots\partial_{\beta_k\dot\beta_k}\psi_{\alpha_1\ldots\alpha_{2s}}\,,$$

- Independent field $\psi_{\alpha_1\ldots\alpha_{2s}}(x)$ satisfies Dirac-Weyl equation

$$\partial^{\dot\beta\alpha_1}\psi_{\alpha_1\ldots\alpha_{2s}}(x) = 0$$

  and describes massless particle of helicity s.

Twistorial description of higher spin particle

In a certain terminology, higher spin particle (HS particle) means the model which describes the states of all spins, from zero to infinity.
Most often, such system describes massless states with all possible helicities.

The simplest, but at the same time, very illustrative model is the HS generalization of the Shirafuji model. In this model, there is no constraint which fixes the helicity.
This is obtained by adding additional "kinetic terms" of spinor variables to the Shirafuji action, that is, by considering the action

$$S_{hs}^{mix} = \int d\tau \left(\pi_\alpha \bar{\pi}_{\dot{\alpha}} \dot{x}^{\dot{\alpha}\alpha} + \pi_\alpha \dot{\zeta}^\alpha + \bar{\pi}_{\dot{\alpha}} \dot{\bar{\zeta}}^{\dot{\alpha}} \right) .$$

The last terms in this action tell us that the commuting spinors $(\zeta^\alpha, \pi_\alpha)$, $(\bar{\zeta}^{\dot{\alpha}}, \bar{\pi}_{\dot{\alpha}})$ form pairs of canonically conjugate variables.

The model is described by only vector constraint of the first class:

$$p_{\alpha\dot{\alpha}} - \pi_\alpha \bar{\pi}_{\dot{\alpha}} \approx 0 .$$

There are no additional constraints in this system.

As before, let us consider the representation in which ζ^α, $\bar\zeta^{\dot\alpha}$ are diagonal and $\pi_\alpha = i\partial/\partial\zeta^\alpha$, $\bar\pi_{\dot\alpha} = i\partial/\partial\bar\zeta^{\dot\alpha}$ are realized by differentiation operators.

Requiring a polynomial dependence of the wave function, we have the following expression for it as the infinite series:

$$\Phi_{hs}(x^m, \zeta^\alpha, \bar\zeta^{\dot\alpha}) = \sum_{k=0}^{\infty}\sum_{n=0}^{\infty} \zeta^{\alpha_1}\ldots\zeta^{\alpha_k}\bar\zeta^{\dot\alpha_1}\ldots\bar\zeta^{\dot\alpha_k}\varphi_{\alpha_1\ldots\alpha_k\dot\alpha_1\ldots\dot\alpha_k}(x)\,.$$

Twistorial constraint yields the Vasiliev unfolded equation

$$\left(i\partial_{\alpha\dot\alpha} - \frac{\partial}{\partial\zeta^\alpha}\frac{\partial}{\partial\bar\zeta^{\dot\alpha}}\right)\Phi_{hs} = 0\,.$$

Independent space-time fields in the expansion of the field Φ are self-dual $\varphi_{\alpha_1\ldots\alpha_k}(x)$, $k = 0, 1, \ldots k$ and anti-self-dual $\varphi_{\dot\alpha_1\ldots\dot\alpha_k}(x)$, $n = 0, 1, \ldots k$ field strengths of all helicities. Basic unfolded equation leads to Klein-Gordon and Dirac equations for them.

All other component fields are expressed as x-derivatives of the basic fields.

Reality condition for the HS field $\Phi = (\Phi)^*$ leads to the reality conditions $\varphi_{\dot\alpha_1\ldots\dot\alpha_k} = (\varphi_{\alpha_1\ldots\alpha_k})^*$ for physical fields. Thus, the massless HS multiplet described by the real HS field $\Phi(x^m, \zeta^\alpha, \bar\zeta^{\dot\alpha})$ contains all helicities and each helicity appearing only once.

Twistor formulation of HS particle is obtained after passing to the variables

$$\omega^{\dot\alpha} = \bar\zeta^{\dot\alpha} + x^{\dot\alpha\beta}\pi_{\dot\beta}\,, \qquad \bar\omega^{\alpha} = \zeta^{\alpha} + \bar\pi_{\dot\beta}x^{\dot\beta\alpha}\,,$$

which are precisely the components of the twistors.

Up to total derivative in the Lagrangian, the action of HS particle takes the following form in twistor formulation

$$S_{hs} = \int d\tau \left(\dot{\bar\omega}^{\alpha}\pi_{\alpha} + \bar\pi_{\dot\alpha}\dot\omega^{\dot\alpha}\right) = \tfrac{1}{2}\int d\tau \left(\bar Z^{A}\dot Z_{A} - \dot{\bar Z}^{A} Z_{A}\right)\,.$$

Twistor wave function of this model is holomorphic twistor function

$$\Psi_{hs}(Z) = \Psi_{hs}(\pi_\alpha, \omega^{\dot\alpha})$$

without any additional equations of constraints.

This twistor field describes infinite tower of massless states of all helicities, which are described by homogeneous components in the expansion in spinor variables. Ordinary space-time fields with helicity s can be extracted by means of the integral transformation

$$\varphi_{\alpha_1\ldots\alpha_{2s}}(x) = \oint (\pi d\pi)\,\pi_{\alpha_1}\ldots\pi_{\alpha_{2s}}\,\Psi_{hs}(\pi_\alpha, x^{\dot\alpha\alpha}\pi_\alpha)\,.$$

In this integral, only the term with the correct degree of homogeneity contributes; other terms with different homogeneities do not contribute to this integral.

Therefore, this model describes the tower of massless states whose helicities start from zero and continue to infinity.

HS system contains infinite number of massless fields of arbitrary spins (helicities). Therefore, we can expect that such a system has infinite-dimensional symmetry, mixing all the spins with each other. The main and, in practice, the only requirement for such a symmetry is that it must be an extension of conformal symmetry. For this reason, twistors realizing conformal symmetry linearly play important role in describing the symmetry of higher spins. In other variables, for example, space-time variables, the full symmetry group of higher spins is hidden.

Symmetry in HS theory is usually characterized by its algebra, called HS algebra.

We have seen that twistor field is the function $\Psi_{hs}(Z_A)=\Psi_{hs}(\pi_\alpha,\omega^{\dot\alpha})$ in a twistor space.

This field space is preserved by 15 conformal algebra generators $\bar Z^A Z_B-\frac{1}{4}\,\delta^A_B\,\bar Z^C Z_C$, formed by all bilinear combinations of the twistor and its adjoint components:

$$P_{\alpha\dot\alpha}=\pi_\alpha\bar\pi_{\dot\alpha}\,,\quad K^{\dot\alpha\alpha}=\omega^{\dot\alpha}\bar\omega^{\alpha}\,,\qquad M_{\alpha\beta}=\pi_{(\alpha}\bar\omega_{\beta)}\,,\quad \bar M_{\dot\alpha\dot\beta}=\bar\pi_{(\dot\alpha}\omega_{\dot\beta)}\,,\quad D=\frac{1}{2}\,(\bar\omega^{\alpha}\pi_\alpha+\bar\pi_{\dot\alpha}\omega^{\dot\alpha})\,,$$

and the operator $\frac{i}{2}\bar Z^A Z_A=\frac{i}{2}\,(\bar\omega^{\alpha}\pi_\alpha-\bar\pi_{\dot\alpha}\omega^{\dot\alpha})\,.$
Other 20 second degree generators

$$R_{\alpha\beta}=\pi_\alpha\pi_\beta\,,\quad \bar R_{\dot\alpha\dot\beta}=\bar\pi_{\dot\alpha}\bar\pi_{\dot\beta}\,,\quad \tilde R^{\alpha\beta}=\bar\omega^{\alpha}\bar\omega^{\beta}\,,\quad \bar{\tilde R}^{\dot\alpha\dot\beta}=\omega^{\dot\alpha}\omega^{\dot\beta}\,,\quad F_\alpha{}^{\dot\beta}=\pi_\alpha\omega^{\dot\beta}\,,\quad \bar F_{\dot\alpha}{}^{\beta}=\bar\pi_{\dot\alpha}\bar\omega^{\beta}$$

are formed by the products of the twistor components between themselves and its conjugate in analogous way:

$$Z_A Z_B=(R_{\alpha\beta},\bar{\tilde R}^{\dot\alpha\dot\beta},F_\alpha{}^{\dot\beta})\,,\qquad \bar Z^A\bar Z^B=(\bar R_{\dot\alpha\dot\beta},\tilde R^{\alpha\beta},\bar F_{\dot\alpha}{}^{\beta})\,.$$

With respect to twistor Poisson brackets, all these 36 generators form the $Sp(8)$ algebra, which is one of the finite-dimensional extensions of the conformal algebra.

In fact, the twistors define oscillatory representation of the $SU(2,2)$ and $Sp(8)$ algebras.

A natural way to obtain infinite-dimensional HS symmetry is to relax the twistor bilinearity requirement for generators. Introducing the notation for twistor monomials of n-th degree

$$Z_{A(k)} \equiv Z_{A_1} \dots Z_{A_k} \,, \quad \bar{Z}^{B(l)} \equiv \bar{Z}^{B_1} \dots \bar{Z}^{B_l} \,,$$

and also for their spinor components $\pi_{\alpha(k)} \equiv \pi_{\alpha_1} \dots \pi_{\alpha_k}$, $\bar{\pi}_{\dot{\alpha}(l)} \equiv \bar{\pi}_{\dot{\alpha}_1} \dots \bar{\pi}_{\dot{\alpha}_l}$, etc.

Generators of infinite-dimensional symmetry that preserve HS field have the form

$$G^{B(r)}_{A(p)} = Z_{A(p)} \bar{Z}^{B(r)} \qquad \equiv \qquad G^{\beta(m),\dot{\alpha}(n)}_{\alpha(k),\dot{\beta}(l)} = \pi_{\alpha(k)} \bar{\pi}_{\dot{\beta}(l)} \bar{\omega}^{\beta(m)} \omega^{\dot{\alpha}(n)} \,, \; k+n=p, m+l=r \,.$$

These generators form infinite-dimensional Lie algebra:

$$\{G^{(N_1)}, G^{(N_2)}\}_P = G^{(N_1+N_2-2)}$$

in terms of the quantities $G^{(N)} \equiv G^{B(r)}_{A(p)}$, $N = p + r$.

Generators in $G^{(2)}$ form $Sp(8)$ subalgebra: $\{G^{(2)}, G^{(2)}\}_P = G^{(2)}$.

Even a minimal extension of the algebra by generators $G^{(3)}$ generates an infinite set of generators:

$$\{G^{(3)}, G^{(3)}\}_P = G^{(4)} \,, \qquad \{G^{(3)}, G^{(4)}\}_P = G^{(5)} \,, \cdots$$

The presented algebra is reducible and contains other (infinite-dimensional) subalgebras. For example, generators $G^{(N)}$ of even degree form a subalgebra. A further restriction arises when $SU(2,2)$-irreducible representations in generators are singled out.

Obtaining the $SU(2,2)$ irreducible representations occurs by selecting the trace parts, formed here by the twistor norm $(\frac{i}{2}\bar{Z}^A Z_A)$, and non-trace parts.

Irreducible parts of generators are generators

$$T^{(n)B(r)}_{\quad A(p)} = (\tfrac{i}{2}\bar{Z}^A Z_A)^n \, \langle Z_{A(p)}\bar{Z}^{B(r)} \rangle \,,$$

where the tensors in brackets $\langle\,\rangle$ are traceless by definition, $\langle M^{AB\cdots}_{AC\cdots} \rangle \equiv 0$. The generators $T^{(n)B(r)}_{\quad A(p)}$ at $r = p$ produce Fradkin-Linetsky-Vasiliev $hsc(2,2)$ algebra.

Note that the higher spin algebra, which is an extension of the conformal algebra, is not unique. Depending on the choice of the symmetry algebra, we obtain a different set of spin states, on which the transformations realizing this algebra are closed. For example, one way to extend conformal $su(2,2)$ algebra is to consider the $su(2,3)$ algebra rather than that $sp(8)$ algebra. In this way, it is obtained another HS algebra, based on the so-called bosonic supersymmetry (some discussion of it will be in the next lecture).

Lecture 2

- Bitwistor formulation of massive particles and massless infinite spin particles.
- Conformal supersymmetry and supertwistors.
- Twistor description of massless superparticles.
- Twistor transform for spinning particles.

The twistor approach is based on conformal symmetry, which leads to a natural description of conformally-invariant systems. But one of the tasks of the twistor approach is to present an alternative to the space-time description of the physical world, which includes conformally-non-invariant systems, for example, massive particles that possess, in general, non-zero spin.

Let us briefly recall the irreducible relativistic representations.

Irreducible unitary representations of the Poincaré group $ISO^{\uparrow}(1,3)$ are defined by values of the Casimir operators

$$C_2 := P^m P_m \,, \qquad C_4 := W^m W_m \,,$$

where $W_m = \frac{1}{2}\,\varepsilon_{mnkl} P^n M^{kl}$ is the Pauli-Lubanski vector.

Physically interesting unitary irreducible representations:

	$P^2 = P^n P_n$	$W^2 = W^n W_n$
Massless finite spin irreps.	0	0
Massless infinite spin irreps.	0	$-\mu^2$, where $\mu \in \mathbb{R}$, $\mu \neq 0$
Massive irreps.	$\mathrm{m}^2 \neq 0$	$-\mathrm{m}^2 j(j+1)$, where $j \in \mathbb{Z}_{\geq 0}/2$

So far, we have considered massless finite spin irreducible representations (helicity states). These are the standard massless representations that describe all currently known massless particles such as photon, gluons, graviton, massless helicity-1/2 fermion (until some time, it was believed that it is neutrino).

The necessity of bitwistor formalism

So far, we have considered the one-twistor case.

As it was said, the basic relation in the twistor description is the resolution of the 4-momentum through commutating Weyl spinor π_α: $P_{\alpha\dot{\alpha}} = \pi_\alpha \bar{\pi}_{\dot{\alpha}}$.

But from this we get important consequences:

- We get that the square of the 4-momentum is zero: $P^n P_n = 0$. Therefore, the description of massive particle is impossible in frame of the one-twistor formalism.
- Moreover, in the one-twistor case, we have seen that there is the expression

  $$W_n = \Lambda \cdot P_n$$

  for the Pauli-Lubanski vector, where Λ is helicity operator. Therefore, $W^n W_n = 0$ and we can only describe massless finite spin representations (helicity representations).

Thus, to describe massive states or massless infinite spin states, it is necessary to use more than one twistor.

For our purposes, it is enough to use two twistors: Z_A and Y_A, and we will consider this bitwistor description below.

Let us first consider the case of massless infinite spin particle.

Massless infinite spin particle: space-time formulation

Infinite (continuous) spin representations are infinite-dimensional ones.
In contrast to other irreps, infinite spin representation expands into infinite set of massless states with all possible helicities. Helicity in continuous spin representations takes standard discrete values: integer $0, \pm 1, \pm 2, \ldots, \pm\infty$ or half-integer $\pm 1/2, \pm 3/2, \ldots, \pm\infty$.

In Wigner-Bargmann space-time formulation infinite spin fields are described by the function $\Phi(x, y)$ defined on the space which is parametrized by

- commuting 4-vector x^m (the position coordinates on Minkowski space);
- additional commuting 4-vector y^m (describes the spin degrees of freedom).

Equations of motion of these fields (Wigner-Bargmann equations) have the form

$$\frac{\partial}{\partial x^m}\frac{\partial}{\partial x_m}\Phi = 0\,, \qquad \frac{\partial}{\partial x^m}\frac{\partial}{\partial y_m}\Phi = 0\,, \qquad \frac{\partial}{\partial y^m}\frac{\partial}{\partial y_m}\Phi = \mu^2\,\Phi\,, \qquad -i\,y^m\,\frac{\partial}{\partial x^m}\Phi = \Phi\,,$$

where $\mu \in \mathbb{R}$, $\mu \neq 0$ is a dimensionful parameter.

One can verify that for such fields the square of the Pauli-Lubanski vector is equal to

$$W^n W_n = -\mu^2\,.$$

The Wigner-Bargmann space-time field formulation of infinite spin particle is reproduced by means of one-dimensional dynamical model with the following Lagrangian

$$L_{\infty}^{sp.-time} = p_m\dot{x}^m + w_m\dot{y}^m + e\,p_m p^m + e_1\,p_m q^m + e_2\left(q_m q^m + \mu^2\right) + e_3\left(p_m y^m - 1\right).$$

Here, $p_m(\tau)$, $q_m(\tau)$ are the momenta for $x^m(\tau)$, $y^m(\tau)$:

$$\{x^m, p_n\} = \delta^m_n, \quad \{y^m, q_n\} = \delta^m_n\,.$$

In the Lagrangian the variables $e(\tau)$, $e_1(\tau)$, $e_2(\tau)$, $e_3(\tau)$ are the Lagrange multipliers for the first class constraints

$$p_m p^m \;\approx\; 0\,, \qquad p_m q^m \;\approx\; 0\,, \qquad q_m q^m + \mu^2 \;\approx\; 0\,, \qquad p_m y^m - 1 \;\approx\; 0\,.$$

After canonical quantization these constraints yield the Wigner-Bargmann equations.

Let us now construct physically equivalent system in the twistor formulation.

It is important to emphasize that the classical physical equivalence of systems does not imply their quantum equivalence. For example, the presence of spinor variables in the system will make it possible to obtain spinor representations after quantization.

Twistorial formulation of infinite spin particle

Following standard prescriptions of twistor approach in considered case we need to use

- twistor spinor π_α, $\bar\pi_{\dot\alpha} = (\pi_\alpha)^*$ for resolving the constraint $p_m p^m \approx 0$ by the Cartan-Penrose relation $p_{\alpha\dot\alpha} = \pi_\alpha \bar\pi_{\dot\alpha}$;
- spinor of 2-nd twistor ρ_α, $\bar\rho_{\dot\alpha} = (\rho_\alpha)^*$ for resolving the constraint $p_m q^m \approx 0$ in the form $q_{\alpha\dot\alpha} = \pi_\alpha \bar\rho_{\dot\alpha} + \rho_\alpha \bar\pi_{\dot\alpha}$.

Thus, in twistorial formulation infinite spin particle is described by 8 complex variables $(\bar\omega^\alpha, \pi_\alpha)$, $(\bar\eta^\alpha, \rho_\alpha)$ and c.c. ones, which obey the Poisson brackets $\{\bar\omega^\alpha, \pi_\beta\} = \{\bar\eta^\alpha, \rho_\beta\} = \delta^\alpha_\beta$ and are subjected to four first class (abelian) constraints

$$\mathcal{M} \ := \ \pi^\alpha \rho_\alpha \, \bar\rho_{\dot\alpha} \bar\pi^{\dot\alpha} - \mu^2/2 \ \approx \ 0 \, ,$$

$$\mathcal{F} \ := \ \bar\eta^\alpha \pi_\alpha - 1 \approx 0 \, , \qquad \bar{\mathcal{F}} \ := \ \bar\pi_{\dot\alpha} \eta^{\dot\alpha} - 1 \approx 0 \, ,$$

$$\mathcal{U} \ := \ \bar\omega^\alpha \pi_\alpha - \bar\pi_{\dot\alpha} \omega^{\dot\alpha} + \bar\eta^\alpha \rho_\alpha - \bar\rho_{\dot\alpha} \eta^{\dot\alpha} \approx 0 \, .$$

The Hamiltonian in the first order twistorial Lagrangian is linear combination of these constraints with Lagrange multipliers:

$$L^{tw}_\infty = \pi_\alpha \dot{\bar\omega}^\alpha \ + \ \bar\pi_{\dot\alpha} \dot\omega^{\dot\alpha} \ + \ \rho_\alpha \dot{\bar\eta}^\alpha \ + \ \bar\rho_{\dot\alpha} \dot\eta^{\dot\alpha} \ + \ l\,\mathcal{M} \ + \ k\,\mathcal{U} \ + \ \ell\,\mathcal{F} \ + \ \bar\ell\,\bar{\mathcal{F}} \, .$$

Link with the Wigner-Bargmann space-time formulation is carried out by using the generalized Cartan-Penrose relations

$$p_{\alpha\dot\alpha} = \pi_\alpha \bar\pi_{\dot\alpha}\,, \qquad q_{\alpha\dot\alpha} = \pi_\alpha \bar\rho_{\dot\alpha} + \rho_\alpha \bar\pi_{\dot\alpha}$$

and the incidence relations

$$\omega^{\dot\alpha} = x^{\dot\alpha\alpha}\pi_\alpha + y^{\dot\alpha\alpha}\rho_\alpha\,, \qquad \bar\omega^{\alpha} = \bar\pi_{\dot\alpha}x^{\dot\alpha\alpha} + \bar\rho_{\dot\alpha}y^{\dot\alpha\alpha}\,,$$

$$\eta^{\dot\alpha} = y^{\dot\alpha\alpha}\pi_\alpha\,, \qquad \bar\eta^{\alpha} = \bar\pi_{\dot\alpha}y^{\dot\alpha\alpha}\,.$$

We note that, in contrast to the fixed helicity particle, in the incidence conditions for the infinite helicity particle the y-dependent terms mix the spinors of different twistors.

Notice the following points:

- Twistor spinors in this formulation form two Penrose twistors

$$Z_A := \left(\pi_\alpha, \omega^{\dot\alpha}\right), \qquad Y_A := \left(\rho_\alpha, \eta^{\dot\alpha}\right); \qquad \bar Z^A := \begin{pmatrix} \bar\omega^\alpha \\ -\bar\pi_{\dot\alpha} \end{pmatrix}, \qquad \bar Y^A := \begin{pmatrix} \bar\eta^\alpha \\ -\bar\rho_{\dot\alpha} \end{pmatrix}.$$

So the description of infinite spin particles uses with necessity two twistors.

- The $U(1)$ twistor constraint has the form $\mathcal{U} = i\,(\bar Z^A Z_A + \bar Y^A Y_A) \approx 0$.

But the helicity operator is $\Lambda = \frac{i}{2}\,\bar Z^A Z_A$. So in the considered model of infinite (continuous) spin particle, helicity is not fixed since it is proportional to $-\bar Y^A Y_A$.

Performing operator quantization of the model we obtain the twistor wave function

$$\Psi^{(c)}(\pi,\bar\pi;\rho,\bar\rho) \;=\; \delta\Big((\pi\rho)(\bar\rho\bar\pi)-\mu^2/2\Big)\, e^{-iq_0/p_0}\,\hat\Psi^{(c)}(\pi,\bar\pi;\rho,\bar\rho)\,,$$

where $(\pi\rho):=\pi^\beta\rho_\beta$, $(\bar\rho\bar\pi):=\bar\rho_{\dot\beta}\bar\pi^{\dot\beta}$, $\quad q_0/p_0 \;=\; \sum\limits_{\alpha=\dot\alpha}(\pi_\alpha\bar\rho_{\dot\alpha}+\rho_\alpha\bar\pi_{\dot\alpha})\,/\,\sum\limits_{\beta=\dot\beta}\pi_\beta\bar\pi_{\dot\beta}$,

$$\hat\Psi^{(c)}(\pi,\bar\pi;\rho,\bar\rho) \;=\; \psi^{(c)}(\pi,\bar\pi)+\sum_{k=1}^{\infty}(\bar\rho\bar\pi)^k\,\psi^{(c+k)}(\pi,\bar\pi)+\sum_{k=1}^{\infty}(\pi\rho)^k\,\psi^{(c-k)}(\pi,\bar\pi)\,,$$

Constant c plays the role of the U(1) charge and takes (half-)integer values: $2c\;\in\;\mathbb{Z}$.

Fields $\psi^{(c+k)}(\pi,\bar\pi)$ are eigenvectors of the operator $\Lambda=-\frac{1}{2}\left(\pi_\beta\,\frac{\partial}{\partial\pi_\beta}-\bar\pi_{\dot\beta}\,\frac{\partial}{\partial\bar\pi_{\dot\beta}}\right)$:

$$\Lambda\,\psi^{(c+k)}(\pi,\bar\pi) \;=\; \lambda\,\psi^{(c+k)}(\pi,\bar\pi)\,,\qquad \lambda=-(c+k).$$

Helicity operator $\mathbb{\Lambda}=\vec{\mathbb{J}}\,\vec{\mathbb{P}}/\mathbb{P}_0$ where $\vec{\mathbb{J}}$ is total angular momentum, acts in the following way:

$$\mathbb{\Lambda}\,\Psi^{(c)}=\delta\Big((\pi\rho)(\bar\rho\bar\pi)-\mu^2/2\Big)\,e^{-iq_0/p_0}\;\left(\Lambda\psi^{(c)}+\sum_{k=1}^{\infty}(\bar\rho\bar\pi)^k\Lambda\psi^{(c+k)}+\sum_{k=1}^{\infty}(\pi\rho)^k\,\Lambda\psi^{(c-k)}\right)\,,$$

Thus, twistorial wave function of infinite spin particle $\Psi^{(c)}$ describes infinite number of massless states $\psi^{(c+k)}$ whose helicities $\lambda=-(c+k)$, $-\infty<k<\infty$ are equal to integer (for integer c) or half-integer (for half-integer c) values and run from $-\infty$ to $+\infty$.

Helicity content of the field $\Psi^{(c)}$ is the same for all integer or all half-integer values c. We can consider the twistorial field $\Psi^{(0)}(\pi,\bar{\pi};\rho,\bar{\rho})$ to describe infinite integer spin representation and $\Psi^{(-1/2)}(\pi,\bar{\pi};\rho,\bar{\rho})$ to describe infinite half-integer spin representation.

Twistor fields produce space-time fields by using the Penrose integral transform

The Wigner-Bargmann fields on the space with auxiliary 4-vector coordinate $y^{\alpha\dot{\alpha}}$ are obtained by the integral transformation

$$\Phi(x;y) \;=\; \int d^4\pi d^4\rho\, e^{\,i\pi_\alpha\bar{\pi}_{\dot{\alpha}}x^{\dot{\alpha}\alpha}}\, e^{\,i(\pi_\alpha\bar{\rho}_{\dot{\alpha}}+\xi_\alpha\bar{\rho}_{\dot{\alpha}})y^{\dot{\alpha}\alpha}}\,\Psi^{(0)}(\pi,\bar{\rho};\xi,\bar{\rho})\,,$$

where we perform integration over the twistor space with the integration measures $d^4\pi=\frac{1}{4}\,d\pi^\alpha\wedge d\pi_\alpha\wedge d\bar{\pi}^{\dot{\alpha}}\wedge d\bar{\pi}_{\dot{\alpha}}$, $d^4\rho=\frac{1}{4}\,d\rho^\beta\wedge d\rho_\beta\wedge d\bar{\rho}_{\dot{\beta}}\wedge d\bar{\rho}^{\dot{\beta}}$. Due to the twistor equations of motion for twistor field $\Psi^{(0)}(\pi,\bar{\pi};\rho,\bar{\rho})$, the fields $\Phi(x;y)$ satisfies automatically the Wigner-Bargmann equations.

Other space-time systems, but with additional spinor variables, can be derived in this way.

Twistor formulation of massive particle

To resolve the time-like 4-momentum, it is necessary to change the twistor relation $p_{\alpha\dot\alpha} = \pi_\alpha \, \bar\pi_{\dot\alpha}$, used earlier for the light-like momenta.

The only way to solve this problem is to replace the one-twistor formalism with two-twistor formalism. That is, to describe massive states, we use two spinors

$$\pi^i_\alpha \,, \qquad \bar\pi_{\dot\alpha\, i} = (\pi^i_\alpha)^* \,, \qquad i = 1, 2 \,,$$

that define halves of two twistors

$$Z^i_A := \left(\pi^i_\alpha, \omega^{\dot\alpha i}\right), \qquad \bar Z^A_i := \begin{pmatrix} \bar\omega^\alpha_i \\ -\bar\pi_{\dot\alpha i} \end{pmatrix} .$$

Note: it is convenient to combine two twistors into one $SU(2)$ spinor with $SU(2)$-spinor index $i = 1, 2$, since in the standard momentum frame (in the rest frame) of massive particle the small group is $SU(2) \cong SO(3)$.

Then the momentum of massive particle is represented in the twistor-like form

$$p_{\alpha\dot\alpha} = \pi^i_\alpha \, \bar\pi_{\dot\alpha\, i} \,.$$

But then we get the following corollary:
when describing massive particle with mass m determined by the mass condition $p^2 = m^2$, used two spinors π^i_α must be limited by the constraint

$$|\pi^{\alpha i}\pi_{\alpha i}|^2 = m^2 \qquad \text{or stronger constraints} \qquad \pi^{\alpha i}\pi_{\alpha i} = m \,, \qquad \bar\pi_{\dot\alpha i}\bar\pi^{\dot\alpha i} = m \,.$$

These conditions violate the conformal invariance.

Consider first the twistor formulation of the massive particle with spin.
The corresponding space-time formulation will be considered later.

In contrast to the massless case where the twistor description of the arbitrary helicity particle can be achieved by using only one twistor, in the massive case it is necessary to use some spinning variables in addition to the twistor ones.

Additional spin variables should give the description of the nonrelativistic integer or half-integer spin in in the rest frame. Therefore, we will use commuting spinors ξ^i as spin variables.

As a result, massive spinning particle in the twistor formulation is described by the variables

$$\pi^i_\alpha\,, \quad \bar\pi_{\dot\alpha\, i} = (\pi^i_\alpha)^*\,; \qquad \omega^{\dot\alpha i}\,, \quad \bar\omega^\alpha_i = (\omega^{\dot\alpha i})^*\,; \qquad \xi^i\,, \quad \bar\xi_i = (\xi^i)^*\,; \qquad i=1,2\,,$$

which satisfy the Poisson brackets $\{\bar\omega^\alpha_i, \pi^j_\beta\}_P = \delta^\alpha_\beta\delta^j_i$, $\{\omega^{\dot\alpha i}, \bar\pi_{\dot\beta j}\}_P = \delta^{\dot\alpha}_{\dot\beta}\delta^i_j$, $\{\xi^i, \bar\xi_j\}_P = -\frac{i}{2}\,\delta^i_j$,
and is subjected to the set of the first class constraints

$$\begin{aligned} h &:= \pi^{\alpha i}\pi_{\alpha i} - m \approx 0\,, \qquad \bar h := \bar\pi_{\dot\alpha i}\bar\pi^{\dot\alpha i} - m \approx 0\,, \\ D_a &:= (\sigma_a)_j{}^i \left[\tfrac{i}{2}\left(\bar\omega^\alpha_i\pi^j_\alpha - \bar\pi_{\dot\alpha i}\omega^{\dot\alpha j}\right) + \bar\xi_i\xi^j\right] \approx 0\,, \\ S &:= \bar\xi_i\xi^i - s \approx 0\,. \end{aligned}$$

The constraints $D_a \approx 0$ form SU(2) algebra with respect to the Poisson brackets.
The constraints $S \approx 0$ contained the constant s defines the particle spin.

The mass constraints are also presented in the form

$$h = Z^i_A I^{AB} Z_{Bi} - m \approx 0\,, \qquad \bar h = \bar Z^A_i I_{AB} Z^{Bi} - m \approx 0\,,$$

where SU$(2,2)$-noninvariant so-called infinity twistors (asymptotic twistors)

$$I^{AB} = \begin{pmatrix} \epsilon^{\alpha\beta} & 0 \\ 0 & 0 \end{pmatrix}, \qquad I_{AB} = \begin{pmatrix} 0 & 0 \\ 0 & \epsilon^{\dot\alpha\dot\beta} \end{pmatrix}.$$

are used.

Twistor fields of massive particles

Let us find twistor massive fields by canonical quantization of twistor massive particle.

We impose gauge-fixing conditions for the constraints $h \approx 0$, $\bar{h} \approx 0$. After the introduction of the Dirac brackets, these constraints are satisfied in the strong sense, that is, conditions

$$\pi^{\alpha i}\pi_{\alpha i} = m\,, \qquad \bar{\pi}_{\dot{\alpha} i}\bar{\pi}^{\dot{\alpha} i} = m$$

hold. These conditions state that determinant of the matrix $\Pi \equiv m^{-1/2}||\pi^i_\alpha||$ is equal to one, i.e. $\Pi \in SL(2,\mathbb{C})$.

In π-representations, twistor massive wave function $\Psi(\pi,\bar{\pi})$ is defined by the equations

$$(S-J)\,\Psi = \left(\tfrac{1}{2}\,a^{+i}a_i - J\right)\Psi = 0\,,$$

$$D_a\Psi = (\mathcal{D}_a + \Delta_a)\,\Psi = 0\,, \qquad a = 1,2,3\,,$$

where $$\mathcal{D}_a = \tfrac{1}{2}\left[\pi^i_\alpha(\sigma_a)_i{}^j\,\frac{\partial}{\partial\pi^j_\alpha} - \frac{\partial}{\partial\bar{\pi}_{\dot{\alpha} i}}(\sigma_a)_i{}^j\,\bar{\pi}_{\dot{\alpha} j}\right], \qquad \Delta_a = \tfrac{1}{2}\,a^{+i}(\sigma_a)_i{}^j a_j\,.$$

The operators $a_i \equiv \sqrt{2}\hat{\bar{\xi}}_i$ and $a^{+i} \equiv \sqrt{2}\hat{\xi}^i$ are usual annihilation and creation operators of two-dimensional oscillator; they are defined by the commutators $[a_i, a^{+j}] = \delta_i^j$.
The wave function Ψ is taken in filling numbers space of these operators.

Constant J is classical constant s renormalized by ordering constants.

By means direct calculations we obtain that square of Pauli-Lubanski vector takes the form

$$W^n W_n \Psi = -m^2 \mathcal{D}_a \mathcal{D}_a \Psi \, .$$

Since the condition $\mathcal{D}_a \Psi = (D_a - \Delta_a)\Psi$ holds and using $\Delta_a \Delta_a = \frac{1}{2}\, a^{+i} a_i (\frac{1}{2}\, a^{+j} a_j + 1)$, we obtain

$$W^n W_n \Psi = -m^2 J(J+1)\, ,$$

i.e. we have massive particle with fixed spin J in the spectrum of this model.

The operators Δ_a form su(2)-algebra which realized by two oscillators a_i, a^{+i}, $i = 1, 2$.
Let integer nonnegative numbers n_1 and n_2 are corresponding filling numbers
i.e. n_1 and n_2 are the eigenvalues of operators $a^{+1} a_1$ and $a^{+2} a_2$.

The constraints $(S - J)\, \Psi = 0$ gives us that $\frac{1}{2}(n_1 + n_2) = J$.

Then the number $\frac{1}{2}(n_1 - n_2) = M$ takes $(2J+1)$ values $M = -J, -J+1, ..., J-1, J$ and twistor field is $(2J+1)$-component field $\Psi_M(\pi, \bar{\pi})$.

By definition, this field satisfies

$$\Delta_3 \Psi_M = M \Psi_M \, , \qquad \Delta_\pm \Psi_M = (\Delta_1 \pm i \Delta_2) \Psi_M = \sqrt{(J \mp M)(J \pm M + 1)}\, \Psi_M \, .$$

From $(\mathcal{D}_a + \Delta_a)\, \Psi = 0$ we have

$$\mathcal{D}_3 \Psi_M = -M \Psi_M \, , \qquad \mathcal{D}_\pm \Psi_M = (\mathcal{D}_1 \pm i \mathcal{D}_2) \Psi_M = -\sqrt{(J \mp M)(J \pm M + 1)}\, \Psi_M \, .$$

The operators $\mathcal{D}_a$ are generators of SU(2)-transformations, acting on index i of π^i_α and last equations state that the wave function $\Psi_M(\pi,\bar\pi)$ is defined up to the transformations acting on index M:

$$\Psi'_M(\pi') = \mathbf{D}^J_{MN}(h)\Psi'_N(\pi)\,, \qquad \pi'^i_\alpha = h^i_j\pi^j_\alpha\,, \qquad h \in \mathrm{SL}(2,\mathbb{C})\,.$$

Here $\mathbf{D}^J_{MN}$ is the matrix of $SU(2)$-transformations of weight J.

Thus twistor wave function of massive spinning particle is defined on the homogeneous space $SL(2,\mathbb{C})/SU(2)$.

In the form of the $SU(2)$-indices $i, j, ... = 1, 2$, the index M is collective index $M = (i_1 \ldots i_{2J})$. Then the wave function (twistor field of massive spinning particle) is

$$\Psi_{i_1\ldots i_{2J}}(\pi,\bar\pi)\,,$$

which is completely symmetric with respect to SU(2)-indices:

$$\Psi_{i_1\ldots i_{2J}} = \Psi_{(i_1\ldots i_{2J})}\,.$$

Twistor transform for massive fields

The relation of the twistor fields with the usual space-time spin-tensor fields is established by means of the integral transformation in the following way. One constructs $SU(2)$-invariant expressions by contracting the twistor fields $\Psi_{i_1\ldots i_{2J}}(\pi,\bar\pi)$ with twistor spinors $\pi^{i_1}_{\alpha_1}\ldots\pi^{i_{2J}}_{\alpha_{2J}}$. After integration with invariant measure $d^3\pi$ on the coset space $\mathrm{SL}(2,\mathbb{C})/\mathrm{SU}(2)$ with the standard Fourier exponent $\exp(ix^m p_m)$ where $p_m = p_{\alpha\dot\alpha}\tilde\sigma_m^{\dot\alpha\alpha}$ we obtain space-time fields

$$\Phi_{\alpha_1\ldots\alpha_{2J}}(x) = \int d^3\pi\, e^{ix^m p_m}\,\pi^{i_1}_{\alpha_1}\ldots\pi^{i_{2J}}_{\alpha_{2J}}\Psi_{i_1\ldots i_{2J}}(\pi,\bar\pi)\,.$$

These fields are totally symmetric in spinor indices $\Phi_{\alpha_1\ldots\alpha_{2J}} = \Phi_{(\alpha_1\ldots\alpha_{2J})}$ and give us standard $(2J+1)$-component field description of massive spin J. Due to the presence of the exponent in the integrand, $\Phi_{\alpha_1\ldots\alpha_{2J}}$ satisfies automatically massive Klein-Gordon equation

$$\left(\partial^n\partial_n + m^2\right)\Phi_{\alpha_1\ldots\alpha_{2J}}(x) = 0\,.$$

Similarly, but with using the spinor $\bar\pi^{\dot\alpha i}$, twistor field produces the field with dotted indices:

$$\Phi^{\dot\alpha_1\ldots\dot\alpha_{2J}}(x) = \int d^3\pi\, e^{ix^m p_m}\,\bar\pi^{\dot\alpha_1 i_1}\ldots\bar\pi^{\dot\alpha_{2J} i_{2J}}\Psi_{i_1\ldots i_{2J}}(\pi,\bar\pi)\,.$$

It is easy to show that the fields $\Phi_{\alpha_1\ldots\alpha_{2J}}(x)$ and $\Phi^{\dot\alpha_1\ldots\dot\alpha_{2J}}(x)$ are related by the $(2J+1)$-order Weinberg equations

$$(i\partial_{n_1}\sigma^{n_1}_{\alpha_1\dot\beta_1})\ldots(i\partial_{n_{2J}}\sigma^{n_{2J}}_{\alpha_{2J}\dot\beta_{2J}})\Phi^{\dot\beta_1\ldots\dot\beta_{2J}}(x) = \Phi_{\alpha_1\ldots\alpha_{2J}}(x)\,.$$

Coordinate twistor transformation and corresponding space-time description of massive spin particle will be presented after consideration of twistor description of the superparticle.

Superparticle

Twistor constructions for a massless superparticle of zero superspin largely repeat the structural elements of an ordinary massless particle. New details that will arise are related to the twistor realization of supertranslations and superconformal boosts, which is reflected by the presence of additional Grassmann superspace coordinates.

Let us first describe superparticle model in which the target space is described by the superspace coordinates and whose quantization produces the superfield in the spectrum.

Note: We will consider only non-extended $\mathcal{N}=1$ supersymmetry.

Superspace formulation of massless superparticle

The action of the superparticle in the first-order formalism is similar to the action of bosonic massless particle:

$$S_0^{super} = \int d\tau \left(p_{\alpha\dot{\alpha}} \omega^{\dot{\alpha}\alpha} - e p_{\alpha\dot{\alpha}} p^{\dot{\alpha}\alpha} \right) ,$$

where instead of $\dot{x}^{\dot{\alpha}\alpha}$ is supertranslation-invariant vector

$$\omega^{\dot{\alpha}\alpha} \equiv \dot{x}^{\dot{\alpha}\alpha} - i\bar{\theta}^{\dot{\alpha}}\dot{\theta}^{\alpha} + i\dot{\bar{\theta}}^{\dot{\alpha}}\theta^{\alpha} .$$

The Weyl spinor θ^{α}, $\bar{\theta}^{\dot{\alpha}} = (\theta^{\alpha})^{*}$ is Grassmannian variable $((\theta^{\alpha})^2 \equiv 0 \ \forall\alpha)$, which, together with the usual coordinate $x^{\dot{\alpha}\alpha}$, parameterizes the trajectory of the superparticle in the superspace.

This supersymmetric system is invariant under the following global transformations:

- Poincaré transformations $\delta x^{\dot\alpha\alpha} = a^{\dot\alpha\alpha} + x^{\dot\alpha\beta} l_\beta{}^\alpha + \bar l^{\dot\alpha}{}_{\dot\beta} x^{\dot\beta\alpha}$,

  $\delta\theta^\alpha = \theta^\beta l_\beta{}^\alpha$, $\delta p_{\alpha\dot\alpha} = -l_\alpha{}^\beta p_{\beta\dot\alpha} - p_{\alpha\dot\beta}\bar l^{\dot\beta}{}_{\dot\alpha}$;
- dilatations $\delta x^{\dot\alpha\alpha} = c x^{\dot\alpha\alpha}$, $\delta\theta^\alpha = \frac12 c\theta^\alpha$, $\delta p_{\alpha\dot\alpha} = -c p_{\alpha\dot\alpha}$, $\delta e = 2ce$;
- conformal boosts $\delta x^{\dot\alpha\alpha} = x^{\dot\alpha\beta} k_{\beta\dot\beta} x^{\dot\beta\alpha} - 4\theta^\alpha\bar\theta^{\dot\alpha}\,\theta^\beta k_{\beta\dot\beta}\bar\theta^{\dot\beta}$, $\delta\theta^\alpha = \theta^\beta k_{\beta\dot\beta}(x^{\dot\beta\alpha} + i\bar\theta^{\dot\beta}\theta^\alpha)$;

  $\delta p_{\alpha\dot\alpha} = -(p_{\alpha\dot\beta}k_{\beta\dot\alpha} + k_{\alpha\dot\beta}p_{\beta\dot\alpha})x^{\dot\beta\beta} - 2i(p_{\alpha\dot\beta}k_{\beta\dot\alpha} - k_{\alpha\dot\beta}p_{\beta\dot\alpha})\bar\theta^{\dot\beta}\theta^\beta$, $\delta e = 2(x^{\dot\beta\beta}k_{\beta\dot\beta})e$,
- chiral transformations of spinors $\delta\theta^\alpha = -\frac12 i\phi\theta^\alpha$;
- supertranslations $\delta x^{\dot\alpha\alpha} = -(\bar\theta^{\dot\alpha}\epsilon^\alpha - \bar\epsilon^{\dot\alpha}\theta^\alpha)$, $\delta\theta^\alpha = \epsilon^\alpha$;
- superconformal boosts $\delta x^{\dot\alpha\alpha} = 2i(\bar\theta^{\dot\alpha}\bar\eta_{\dot\beta}x^{\dot\beta\alpha} - x^{\dot\alpha\beta}\eta_\beta\theta^\alpha) - 4\bar\theta^{\dot\alpha}\theta^\alpha(\theta^\beta\eta_\beta + \bar\eta_{\dot\beta}\bar\theta^{\dot\beta})$,

  $\delta\theta^\alpha = -4i\theta^\alpha\,\theta^\beta\eta_\beta + \bar\eta_{\dot\beta}(x^{\dot\beta\alpha} + i\bar\theta^{\dot\beta}\theta^\alpha)$, $\delta p_{\alpha\dot\alpha} = 4i(\eta_\alpha\theta^\beta p_{\beta\dot\alpha} - p_{\alpha\dot\beta}\bar\theta^{\dot\beta}\bar\eta_{\dot\alpha})$, $\delta e = -4ie\theta^\beta\eta_\beta + c.c.$

In supertranslation and superconformal boost transformations, the transformation parameters ϵ^α and η_α are Grassmann Weyl spinors.

The generators of these transformations (actually, Noether charges) form the $SU(2,2|1)$ superconformal algebra. In addition to the usual inhomogeneous translations and nonlinear conformal boosts, superconformal transformations include inhomogeneous supertranslations and non-linear superconformal boosts.

Below, by introducing supertwistors, the superconformal algebra $SU(2,2|1)$ will be realized by homogeneous linear transformations
(new (anti-)commutators of this superalgebra will be presented there).

Let us briefly describe the quantization of this superparticle model and the form of the obtained superfield.

Phase space x^m, p_m, θ^α, p_α, $\bar\theta^{\dot\alpha}$, $\bar p_{\dot\alpha}$ with Poisson brackets $\{x^{\dot\alpha\alpha}, p_{\beta\dot\beta}\}_{\mathrm{P}} = \delta^\alpha_\beta \delta^{\dot\alpha}_{\dot\beta}$, $\{\theta^\alpha, p_\beta\}_{\mathrm{P}} = \delta^\alpha_\beta$, $\{\bar\theta^{\dot\alpha}, \bar p_{\dot\beta}\}_{\mathrm{P}} = \delta^{\dot\alpha}_{\dot\beta}$ is limited by the constraints

$$p_{\alpha\dot\alpha} p^{\alpha\dot\alpha} \approx 0\,; \qquad D_\alpha \equiv p_\alpha + i p_{\alpha\dot\alpha}\bar\theta^{\dot\alpha} \approx 0\,, \qquad \bar D_{\dot\alpha} \equiv \bar p_{\dot\alpha} + i\theta^\alpha p_{\alpha\dot\alpha} \approx 0\,.$$

Since the nonzero Poisson brackets of the constraints are $\{D_\alpha, \bar D_{\dot\alpha}\}_{\mathrm{P}} = 2ip_{\alpha\dot\alpha}$ and matrix $p_{\alpha\dot\alpha}$ is singular for massless particle, the constraint $p^2 \approx 0$ is first class constraint, whereas odd constraints $D_\alpha \approx 0$, $\bar D_{\dot\alpha} \approx 0$ are the mixture of 2 first class constraints and 2 second class ones.

Not independent first class constraints are the constraints

$$F^{\dot\alpha} \equiv p^{\dot\alpha\alpha} D_\alpha \approx 0\,, \qquad \bar F^{\alpha} \equiv \bar D_{\dot\alpha} p^{\dot\alpha\alpha} \approx 0\,,$$

which generate κ-symmetry $\delta\theta^\alpha = \bar\kappa_{\dot\alpha} p^{\dot\alpha\alpha}$, $\delta\bar\theta^{\dot\alpha} = p^{\dot\alpha\alpha}\kappa_\alpha$ with local odd parameter $\kappa_\alpha(\tau)$.

With Gupta-Bleuler quantization, when all first class constraints and half, commuting in weak sense, second class ones are putted on the wave function, independent covariant equations for the wave function $\Phi(x,\theta,\bar\theta)$ have the form

$$\Box\,\Phi = 0\,, \qquad \bar D_{\dot\alpha}\,\Phi = 0\,, \qquad \partial^{\dot\alpha\alpha}D_\alpha\,\Phi = 0\,,$$

where $D_\alpha = i(\partial_\alpha - i\partial_{\alpha\dot\alpha}\bar\theta^{\dot\alpha})$, $\bar D_{\dot\alpha} = i(\bar\partial_{\dot\alpha} - i\theta^\alpha\partial_{\alpha\dot\alpha})$ are covariant derivatives.

When determining the complete system of equations for the wave function, it is necessary to require the preservation of symmetries in passing to quantum theory, in addition to standard procedure for the quantum realization of classical constraints. Under superconformal boosts, the transformation of the operator in one of the equations has the form

$$\delta\,(\partial^{\dot\alpha\alpha}D_\alpha) = 4i(\bar\eta\bar\theta)\,\partial^{\dot\alpha\alpha}D_\alpha + i\eta^\alpha D_\alpha\bar D^{\dot\alpha} - i\bar\eta^{\dot\alpha}D^\alpha D_\alpha - 2\eta_\alpha\partial^{\dot\alpha\alpha}\,.$$

That is, we obtain the necessary transformation of the wave function $\delta\,\Phi = -2i(\theta\eta)\,\Phi$, as well as the requirement to impose additional condition

$$D^\alpha D_\alpha\,\Phi = 0\,.$$

The solution of the equation $\bar D_{\dot\alpha}\,\Phi = 0$ is the chiral superfield

$$\Phi = \Phi(x_L,\theta) = A(x_L) + \theta^\alpha\psi_\alpha(x_L) + \theta^2 B(x_L)\,.$$

living on the chiral superspace with the supercoordinates

$$x_L^{\dot\alpha\alpha} = x^{\dot\alpha\alpha} + i\bar\theta^{\dot\alpha}\theta^\alpha\,, \qquad \theta^\alpha\,.$$

The remaining equations vanish the highest component in the $B=0$ expansion and lead to the Klein-Gordon and Dirac equations for complex scalar and spinor fields. That is, the spectrum of this superparticle model is described by massless supermultiplet with zero superhelicity, which includes massless particles of 0 and $1/2$ helicities.

Supertwistor formulation solves the main problem: superconformal transformations in it are realized by linear transformations. Moreover, reformulation of superparticle dynamics in terms of supertwistors leads to its description in physical variables, with resolution of local symmetries, including κ-invariance.

Supertwistor formulation of superparticle

By analogy with the purely bosonic case, supertwistors are defined as spinors of the superconformal group $SU(2,2|1)$. Among the five components of the supertwistor

$$\mathcal{Z}_{\mathcal{A}} = (Z_A;\, \chi) = (\pi_\alpha, \omega^{\dot\alpha};\, \chi)\,, \qquad \mathcal{A} = 1,\ldots,5$$

four c-numerical components are formed by ordinary twistor – $SU(2,2)$-spinor Z_A.
The fifth Grassmann component of the supertwistor is complex Lorentz-scalar

$$\chi\,, \quad \bar\chi = (\chi)^*\,, \qquad\qquad (\chi)^2 \equiv \chi\chi \equiv 0\,.$$

Conjugate supertwistor has the form $\bar{\mathcal{Z}}^{\mathcal{A}} = (\bar Z^A; 2i\bar\chi) = (\bar\pi^\alpha, -\bar\omega_{\dot\alpha}\,.\,2i\bar\chi)\,.$
It can be written using the complex conjugate twistor

$$\bar{\mathcal{Z}}^{A} = \bar{\mathcal{Z}}_{\dot{\mathcal{B}}} G^{\dot{\mathcal{B}}\mathcal{A}}\,, \qquad \bar{\mathcal{Z}}_{\dot{\mathcal{B}}} = (\bar\pi_{\dot\alpha}, \bar\omega^\alpha;\, \bar\chi)$$

after using $SU(2,2|1)$-invariant tensor $G^{\dot{\mathcal{A}}\mathcal{B}} = \begin{pmatrix} g^{\dot A B} & 0 \\ 0 & 2i \end{pmatrix}$, where $g^{\dot A B}$ is $SU(2,2)$-inv. tensor.

$SU(2,2|1)$-invariant twistor norm is defined by

$$\Lambda \equiv \tfrac{i}{2}\, \bar{\mathcal{Z}}^{\mathcal{A}} \mathcal{Z}_{\mathcal{A}} = \tfrac{i}{2}\, \bar{\mathcal{Z}}_{\dot{\mathcal{B}}} G^{\dot{\mathcal{B}}\mathcal{A}} \mathcal{Z}_{A} = \tfrac{i}{2}(\omega^\alpha \pi_\alpha - \bar\pi_{\dot\alpha}\bar\omega^{\dot\alpha}) - \bar\chi\chi\,.$$

$\mathsf{SU}(2,2)$ conformal transformations act only on the bosonic components of the supertwistor and were defined above.

Supertranslations and superconformal boosts are realized linearly in the supertwistor space and mix the bosonic and fermionic components of the supertwistor

$$\delta\pi_\alpha = 2i\eta_\alpha\chi\,, \qquad \delta\omega^{\dot\alpha} = 2i\bar\epsilon^{\dot\alpha}\chi\,, \qquad \delta\chi = \epsilon^\alpha\pi_\alpha - \bar\eta_{\dot\alpha}\omega^{\dot\alpha}\,.$$

Chiral transformations of the supertwistor components are

$$\delta\pi_\alpha = \tfrac{i}{2}\,\phi\,\pi_\alpha\,, \qquad \delta\omega^{\dot\alpha} = \tfrac{i}{2}\,\phi\,\omega^{\dot\alpha}\,, \qquad \delta\chi = i\phi\,\chi\,.$$

Introducing (graded) symplectic structure into supertwistor space by the previously used canonical Poisson brackets for bosonic components and $\{\chi,\bar\chi\}_{\mathrm{P}} = \frac{i}{2}$ for Grassmann components, we find the following expressions for generators of

- supertranslation $\quad Q_\alpha = 2i\,\bar\chi\pi_\alpha\,, \qquad \bar Q_{\dot\alpha} = -2i\,\chi\bar\pi_{\dot\alpha}\,,$
- superconformal boosts $\quad S^\alpha = 2i\,\chi\,\bar\omega^\alpha\,, \qquad \bar S^{\dot\alpha} = -2i\,\bar\chi\,\omega^{\dot\alpha}\,,$
- chiral transformations $\quad A = \frac{i}{2}(\bar\omega^\alpha\pi_\alpha - \bar\pi_{\dot\alpha}\omega^{\dot\alpha}) - 4\bar\chi\chi\,.$

Previously defined generators $P_{\alpha\dot\alpha}$,$K^{\dot\alpha\alpha}$,$L_\alpha{}^\beta$,$\bar L^{\dot\alpha}{}_{\dot\beta}$, together with generators Q_α,$\bar Q_{\dot\alpha}$,S^α,$\bar S^{\dot\alpha}$,A form superconformal algebra $\mathsf{SU}(2,2|1)$. In addition to $\mathsf{SU}(2,2)$ subalgebra considered above, it has nonzero Poisson brackets between the generators Q and S:

$$\{Q_\alpha,\bar Q_{\dot\alpha}\}_{\mathrm{P}} = 2iP_{\alpha\dot\alpha}\,, \quad \{S^\alpha,\bar S^{\dot\alpha}\}_{\mathrm{P}} = 2iK^{\dot\alpha\alpha}\,, \quad \{Q_\alpha,S^\beta\}_{\mathrm{P}} = -2iL_\alpha{}^\beta - i(D-iA)\delta_\alpha{}^\beta \text{ and c.c.}$$

That is, the closure of fermionic symmetries generates full superalgebra $\mathsf{SU}(2,2|\,1)$.

Other non-zero brackets of fermion generators are

$$\{Q_\alpha,K^{\dot\beta\beta}\}_{\mathrm{P}} = 2i\,\delta_\alpha^\beta\,\bar S^{\dot\beta}\,, \quad \{S^\alpha,P_{\beta\dot\beta}\}_{\mathrm{P}} = 2i\,\delta_\beta^\alpha\,\bar Q_{\dot\beta}\,, \quad \{Q_\alpha,A\}_{\mathrm{P}} = 2i\,Q_\alpha\,, \quad \{S^\alpha,A\}_{\mathrm{P}} = 2i\,S^\alpha$$

Supertwistor coordinate transform

The relationship between supertwistor variables and superspace variables is determined by supersymmetric generalization of the Penrose transform

$$p_{\alpha\dot{\alpha}} = \pi_\alpha \bar{\pi}_{\dot{\alpha}}\,;$$

$$\omega^{\dot{\alpha}} = x^{\dot{\alpha}\alpha}\pi_\alpha + i\,\bar{\theta}^{\dot{\alpha}}\chi\,, \qquad \bar{\omega}^{\alpha} = \bar{\pi}_{\dot{\alpha}} x^{\dot{\alpha}\alpha} - i\bar{\chi}\,\theta^\alpha\,;$$

$$\chi = \theta^\alpha \pi_\alpha\,, \qquad \bar{\chi} = \bar{\pi}_{\dot{\alpha}}\bar{\theta}^{\dot{\alpha}}\,.$$

- With such link of the supercoordinates of the two formulations, superconformal symmetries of supertwistor formulation go over into the corresponding symmetries of superspace approach. In addition, simple expressions for superconformal generators in supertwistor approach can easily be used to find expressions for them in space-time approach.
- As in the case of non-supersymmetric particle, the supertwistor transform includes the Cartan resolution of the lightlike momentum vector.
- Supersymmetric generalization of incidence conditions is "twistor shift" of spinor $\omega^{\dot{\alpha}}$ by the term depending on Grassmann variables χ. Note that in these incidence conditions, in fact, there is complex vector coordinate of the chiral superspace x_L:

$$\omega^{\dot{\alpha}} = x_L^{\dot{\alpha}\alpha}\pi_\alpha\,.$$

- Grassmann supertwistor variable χ, which is defined as λ-projection of θ-spinor, is invariant under κ-transformation: $\delta\chi = \delta\theta^\alpha\lambda_\alpha = \bar{\kappa}_{\dot{\alpha}} p^{\dot{\alpha}\alpha}\lambda_\alpha = 0$. That is, supertwistor description uses only one (complex) Grassmann degree of freedom, which is physical degree of freedom covariantly extracted from the space-time system. Therefore, there is no κ-invariance now in twistor twistor formulation. This, in a certain slang, can be defined as the resolution of κ-symmetries in the transition to supertwistors.

Supertwistor superparticle

Supertwistor transformations make it possible to reformulate the superparticle system in terms of supertwistor variables. As a result, we obtain the supertwistor action of the massless superparticle

$$S_{tw}^{super} = \tfrac{1}{2}\int d\tau \left[\bar{\mathcal{Z}}^{\mathcal{A}}\dot{\mathcal{Z}}_{\mathcal{A}} - \dot{\bar{\mathcal{Z}}}^{\mathcal{A}}\mathcal{Z}_{\mathcal{A}} - i\ell\,\bar{\mathcal{Z}}^{\mathcal{A}}\mathcal{Z}_{\mathcal{A}}\right],$$

That is, in the supertwistor formulation, the superparticle action is formally obtained from the non-supersymmetric particle action, in which the change $Z_A \to \mathcal{Z}_{\mathcal{A}}$ is made.

In the supertwistor components, the action has the form

$$S_{tw}^{super} = \int d\tau \left(\dot{\bar{\omega}}^{\alpha}\pi_{\alpha} + \bar{\pi}_{\dot{\alpha}}\dot{\omega}^{\dot{\alpha}} + i(\dot{\bar{\chi}}\chi - \bar{\chi}\dot{\chi}) - \ell \mathbb{A}\right),$$

where $\ell(\tau)$ is Lagrange multiplier for the constraint

$$\mathbb{A} \equiv \tfrac{i}{2}\bar{\mathcal{Z}}^{\mathcal{A}}\mathcal{Z}_{\mathcal{A}} = \tfrac{i}{2}(\omega^{\alpha}\pi_{\alpha} - \bar{\pi}_{\dot{\alpha}}\bar{\omega}^{\dot{\alpha}}) - \bar{\chi}\chi \approx 0.$$

This constraint is direct consequence of the expressions for twistor transform.

The supertwistor norm $\mathbb{A}$ coincides with the superhelicity of massless superparticle described by this supertwistor. Thus, the fundamental twistor transformations actually correspond to superparticle whose superhelicity is equal to zero.

Twistor superfield

Twistor superfield is found by quantizing twistor superparticle, by analogy with obtaining twistor field.

The Poisson brackets yield the (anti)commutators of basic operators

$$[\hat{\bar{\omega}}^{\alpha}, \hat{\pi}_{\beta}] = i\delta^{\alpha}_{\beta}\,, \qquad [\hat{\omega}^{\dot{\alpha}}, \hat{\bar{\pi}}_{\dot{\beta}}] = i\delta^{\dot{\alpha}}_{\dot{\beta}}\,, \qquad \{\hat{\chi}, \hat{\bar{\chi}}\} = -\tfrac{1}{2}\,.$$

We consider the representation, in which the operators $\hat{\mathcal{Z}}_{\mathcal{A}}$ are diagonal and $\hat{\bar{\mathcal{Z}}}^{\mathcal{A}}$ are realized by differentiation operators $\hat{\bar{\pi}}_{\dot{\alpha}} = -i\,\dfrac{\partial}{\partial\omega^{\dot{\alpha}}}$, $\hat{\bar{\omega}}^{\alpha} = i\,\dfrac{\partial}{\partial\pi_{\alpha}}$, $\hat{\bar{\chi}} = -\frac{1}{2}\,\dfrac{\partial}{\partial\chi}$.

Supertwistor wave function $\tilde{\Psi}(\mathcal{Z}) = \tilde{\Psi}(Z,\chi) = \tilde{\Psi}(\pi,\omega,\chi)$ is defined by the quantum counterpart of the classical supertwistor constraint:

$$\hat{\mathbb{\Lambda}}\tilde{\Psi}(Z) = 0\ : \qquad\qquad \tfrac{1}{2}\left(\pi_{\alpha}\frac{\partial}{\partial\pi_{\alpha}} + \omega^{\dot{\alpha}}\frac{\partial}{\partial\omega^{\dot{\alpha}}} + \chi\frac{\partial}{\partial\chi}\right)\tilde{\Psi} = -\tilde{\Psi}\,.$$

Thus, similarly to non-supersymmetric case, the twistor superfield of the system under consideration is the holomorphic homogeneous function of the homogeneity degree (-2):

$$\Psi^{(-2)}(\alpha\mathcal{Z}) = \alpha^{-2}\Psi^{(-2)}(\mathcal{Z})\,, \qquad \alpha\in\mathbb{C}\,.$$

In its expansion in respect to the Grassmann variable χ

$$\tilde{\Psi}^{(-2)}(Z,\chi) = \Psi^{(-2)}(Z) + \chi\Psi^{(-3)}(Z)$$

the fields $\Psi^{(-2)}(Z)$ and $\Psi^{(-3)}(Z)$ describe massless particles with helicities 0 and $1/2$ respectively.

Supertwistor superfield transform

Obtained twistor superfield produces the usual superspace-defined superfield through integral transformation, which is a supersymmetric generalization of Penrose field transform.

Similar to purely bosonic case, some of the variables in the twistor superfield $\tilde{\Psi}(\mathcal{Z}) = \tilde{\Psi}(Z, \chi) = \tilde{\Psi}(\pi, \omega, \chi)$ are resolved by using the incidence conditions:

$$\tilde{\Psi}^{(-2)}(\mathcal{Z})\Big|_{\left[\begin{array}{c}\omega^{\dot{\alpha}} = x_L^{\dot{\alpha}\alpha}\pi_\alpha \\ \chi = \theta^\alpha \pi_\alpha\end{array}\right]} = \tilde{\Psi}^{(-2)}(\pi_\alpha, x_L^{\dot{\alpha}\alpha}\pi_\alpha; \theta^\alpha \pi_\alpha)\,.$$

The subsequent contour integral over λ produces the superfield

$$\Phi(x_L, \theta) = \oint \lambda d\lambda\, \tilde{\Psi}^{(-2)}(\pi_\alpha, x_L^{\dot{\alpha}\alpha}\pi_\alpha; \theta^\alpha \pi_\alpha)\,.$$

The superfield obtained in this way is automatically chiral $\bar{D}_{\dot{\alpha}}\Phi = 0$ and automatically satisfies the mass shell equations $\Box\,\Phi = 0$, $\partial^{\dot{\alpha}\alpha}D_\alpha\Phi = 0$, $D^\alpha D_\alpha\Phi = 0$. That is, the supertwistor formalism give us the off-shell description of chiral supermultiplets.

After describing the twistor formulation of the superparticle, let us return to systems that describe massless/massive particles with nonzero helicity/spin.

We have so far left unanswered the question of the space-time formulation of such systems.

Modified twistor formulation of massless particle with nonzero helicity

In the twistor program of zero-helicity massless particle, its space-time description and its link through twistor Penrose transformations with twistor formulation are well defined. For particles with nonzero helicity, twistor formulation is also well defined. But two important questions arise regarding the other two elements of the twistor program: what kind of twistor transformations for non-zero helicity and what space-time system corresponds to the twistor one in this case?

To answer these two questions, we modify the twistor formulation and use the construction that has analogies with the case of the considered superparticle.

Let us construct dynamical system that is equivalent to the twistor formulation of spinning particle and is similar to the twistor model of superparticle.

As such a system, we consider the system described by the action

$$S_s^{tw} = \int d\tau \left[\tfrac{1}{2} \left(\bar{Z}^A \dot{Z}_A - \dot{\bar{Z}}^A Z_A \right) + i(\dot{\bar{\xi}}\xi - \bar{\xi}\dot{\xi}) - l \left(\tfrac{i}{2} \bar{Z}^A Z_A - \xi\bar{\xi} \right) - v \left(\xi\bar{\xi} - s \right) \right] .$$

In addition to the Penrose twistor Z_A, among the dynamical variables of this system there is complex c-number scalar $\xi\,, \quad \bar{\xi} = (\bar{\xi})\,,$
whose components are canonically self-conjugate: due to the kinetic term for ξ in action, their canonical brackets are $\{\xi, \bar{\xi}\}_{\mathrm{P}} = \frac{i}{2}\,.$

The action contains additional constraint

$$\xi\bar{\xi} - s \approx 0\,.$$

This constraint is the first class and gauges away exactly the two degrees of freedom present in ξ. After eliminating the variable ξ, we obtain the twistor system of the massless spin particle considered earlier.

Twistor transform in case of nonzero helicities

Starting from the modified formulation of the twistor particle of non-zero helicity, we can restore both the twistor transformations and the space-time formulation, if we use analogies with the superparticle formulation.

By analogy with supertwistor transformations for superparticles, twistor transformations for massless particle of non-zero helicity, connecting the twistor formulation with the corresponding space-time one, are defined as follows:

$$p_{\alpha\dot{\alpha}} = \pi_\alpha \bar{\pi}_{\dot{\alpha}}\,;$$

$$\omega^{\dot{\alpha}} = x^{\dot{\alpha}\alpha}\pi_\alpha + i\,\bar{\zeta}^{\dot{\alpha}}\xi\,, \qquad \bar{\omega}^{\alpha} = \bar{\pi}_{\dot{\alpha}}x^{\dot{\alpha}\alpha} - i\bar{\xi}\,\zeta^{\alpha}\,;$$

$$\xi = \zeta^\alpha \pi_\alpha\,, \qquad \bar{\xi} = \bar{\pi}_{\dot{\alpha}}\bar{\zeta}^{\dot{\alpha}}\,.$$

In these expressions, the Weyl spinor ζ^α, $\bar{\zeta}^{\dot{\alpha}} = (\zeta^\alpha)^*$ arises in a natural way, which belongs to the space-time description. But now, unlike to the supercase, spinor ζ^α is commuting c-numeric. It is intended to describe the spinning degrees of freedom of relativistic particle.

Modified twistor transformations solve the main problem, which is to describe the twistor of non-zero norm. Namely, these relations resolve the constraint

$$\tfrac{i}{2}\bar{Z}^A Z_A - \xi\bar{\xi} \approx 0\,,$$

present in the action, which is equivalent at $\xi\bar{\xi} \approx s$ to the definition of the (nonzero for $s \neq 0$) twistor norm.

Space-time formulation of massless spinning particle

Applying twistor transformations to the twistor system of massless spinning particle transforms it into equivalent system described by the space-time coordinate $x^{\dot\alpha\alpha}$ and commuting Weyl spinor ζ^α. The resulting action looks like

$$S^{s.-t.}_{0,s} = \int d\tau \left[p_{\alpha\dot\alpha}\tilde{w}^{\dot\alpha\alpha} - e p_{\alpha\dot\alpha}p^{\alpha\dot\alpha} - v(\zeta^\alpha p_{\alpha\dot\alpha}\bar\zeta^{\dot\alpha} - s) \right] ,$$

where the kinetic term is determined by the quantity

$$\tilde{w}^{\dot\alpha\alpha} = \dot{x}^{\dot\alpha\alpha} - i\bar\zeta^{\dot\alpha}\dot\zeta^\alpha + i\dot{\bar\zeta}^{\dot\alpha}\zeta^\alpha$$

and $e(\tau)$ and $v(\tau)$ are the Lagrange multipliers.

This system is like the superparticle. But instead of the Grassmann spinor θ^α, the commuting spinor ζ^α is used here. Also, there is additional term in action fixing the helicity.

In the Hamiltonian formalism, this system is described by 3 first class constraints and 2 second class constraints.

After quantization the system is described by the wave function

$$\Phi(x,\zeta) = \zeta^{\alpha_1}\ldots\zeta^{\alpha_{2s}}\Phi_{\alpha_1\ldots\alpha_{2s}}(x) ,$$

which links to twistor field by the field twistor transform presented above.

Twistor transform in massive case

The resulting space-time formulation of massless spinning particle has a natural generalization to massive case. When generalizing, it is necessary to make a natural replacement of the mass constraint $p^2 \to (p^2 - m^2)$. Therefore, the action of massive spin-s particle, in which spin degrees of freedom are described by the Weyl spinor ζ^α, has the form

$$S^{s.-t.}_{m,s} = \int d\tau \left[p_{\alpha\dot\alpha}\tilde{w}^{\dot\alpha\alpha} - e\left(p_{\alpha\dot\alpha}p^{\alpha\dot\alpha} - m^2\right) - v\left(\zeta^\alpha p_{\alpha\dot\alpha}\bar\zeta^{\dot\alpha} - s\right)\right],$$

where $\tilde{w}^{\dot\alpha\alpha} = \dot{x}^{\dot\alpha\alpha} - i\bar\zeta^{\dot\alpha}\dot\zeta^\alpha + i\dot{\bar\zeta}^{\dot\alpha}\zeta^\alpha$. In the Hamiltonian formalism, this system is described by 1 first class constraint and 4 second class ones. and after quantization the system is described by the wave function $\Phi(x,\zeta) = \zeta^{\alpha_1}\ldots\zeta^{\alpha_{2s}}\Phi_{\alpha_1\ldots\alpha_{2s}}(x)$, where $(2J+1)$-component field $\Phi_{\alpha_1\ldots\alpha_{2s}}(x) = \Phi_{(\alpha_1\ldots\alpha_{2s})}(x)$ satisfies $\Box\Phi_{\alpha_1\ldots\alpha_{2s}}(x) = 0$ and describes massive spin J.

This space-time formulation is interconnected with the previously considered twistor formulation of massive spinning particle through twistor transform:

$$p_{\alpha\dot\alpha} = \pi^i_\alpha\bar\pi_{\dot\alpha i}\,;$$

$$\omega^{\dot\alpha i} = x^{\dot\alpha\alpha}\pi^i_\alpha + i\,\bar\zeta^{\dot\alpha}\xi^i\,, \qquad \bar\omega^\alpha_i = \bar\pi_{\dot\alpha i}x^{\dot\alpha\alpha} - i\bar\xi_i\,\zeta^\alpha\,;$$

$$\xi^i = \zeta^\alpha\pi^i_\alpha\,, \qquad \bar\xi_i = \bar\pi_{\dot\alpha i}\bar\zeta^{\dot\alpha}\,.$$

Here there are used 2 twistors $Z^i_A = (\pi^i_\alpha, \omega^{\dot\alpha i})$, as was the case earlier in the massive case. The SU(2)-spinor ξ^i, used as the spin degrees of freedom of massive spinning particle, has now arisen in a very natural way.

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Thus, we have described all the twistor designs planned in these lectures.

Some issues not discussed (due to lack of time)

- Twistor formulation of strings, membranes,...
- Supertwistors in superstring theories.
- Twistors in diverse space-time dimensions.
- Twistors and helicity spinors.
- Twistors in calculating S-matrix and MHV formalizm.
- Momentum twistors.

Thank you very much for your attention !