Physics Letters B 817 (2021) 136331

Contents lists available at ScienceDirect

Physics Letters B

www.elsevier.com/locate/physletb

Critical exponents from five-loop scalar theory renormalization near six-dimensions

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ARTICLE INFO

Article history:
Received 27 January 2021
Accepted 27 April 2021
Available online 29 April 2021
Editor: A. Ringwald

ABSTRACT

We present five-loop results for the renormalization of various models with a cubic interaction (in $d = 6 - 2\varepsilon$ dimensions). For the scalar model and its $O(n)$-symmetric extension we provide renormalization constants, anomalous dimensions and critical exponents. We discuss in detail the method of calculation, and provide all counterterms up to five loops. This allows one to consider generalizations of the φ^3 theory to other symmetries.

1. Introduction

In the quantum field theory there are two widely known models used as a playground for new techniques of multi-loop calculations, these are models with quadratic interaction ($\lambda\varphi^4$) and cubic interaction ($\lambda\varphi^3$). While these models are most simple among the non-trivial theories, they admit a number of generalizations which are interesting from the physical point of view. Most famous generalization is $O(n)$-symmetric models: $O(n)$-symmetric $\lambda\varphi^4$ theory describes critical behavior of different systems (among them are liquid-vapor transitions, superfluid transition of helium, uniaxial magnets, isotropic ferromagnets etc.) [1–3], while $O(n)$ φ^3 theory generalization is widely known from [4,5] as candidate for model dual to the Vasiliev higher-spin theory in AdS$_6$ [6,7]. Also many other generalizations are available for these theories, among them we can mention cubic [8] and chiral [9] φ^4 models which represent physically interesting universality classes laying outside of $O(n)$ model. For φ^3 theory most famous generalization is the q-state Potts model [10] and its special case ($q = 1$) percolation theory [11–13]. Recently significant progress was made in $\lambda\varphi^4$ model: six- [14,15,3] and even seven-loop [16] anomalous dimensions and critical exponents were calculated. For the second model and its generalizations only four-loop results are available [13] in analytical form and just recently five-loop calculations were presented [17]. In this paper we present a simple method to calculate renormalization group functions and critical exponents in any generalization of the $\lambda\varphi^3$ theory in $d = 6 - 2\varepsilon$ dimensions up to five loops.

The paper is organized as follows: in section 2 we describe the technique used for five-loop calculations of scalar $\lambda\varphi^3$ model. In section 3 using $O(n)$-symmetric model as an example we show how to perform calculations in generalizations of the scalar $\lambda\varphi^3$ model using results for counterterms for the individual diagrams available as a byproduct of calculations in the model 2. For this model we calculate critical dimensions of different operators allowing comparison with $1/n$ expansion results available in the literature.

2. Simple scalar model

We start with the 1-component scalar model in $d = 6 - 2\varepsilon$ dimensions, defined by the bare Lagrangian

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\varphi_B)^2 + \frac{1}{2}m_B^2\varphi_B^2 + g_B\frac{(4\pi)^3}{6}\varphi_B^3. \qquad (1)$$

Bare and renormalized parameters in the $\overline{\text{MS}}$ scheme are related by

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https://doi.org/10.1016/j.physletb.2021.136331

$$\varphi_B = Z_\varphi \varphi, \quad g_B = \mu^\varepsilon Z_g g, \quad m_B = Z_m m. \tag{2}$$

For our first step to the five-loop renormalization-group functions, we need to calculate the renormalization constants Z_i defined in eq. (2). Unfortunately, it is very difficult to extend the techniques used in Ref. [13] to the five-loop level.

Four-loop calculations [13] were performed with a relatively straightforward approach, by transforming all diagrams to 4-loop massless propagator type integrals. The such obtained integrals can then be reduced to a small set of master integrals using IBP (integration by parts) relations, as four-loop IBP reduction is well developed. Another important property of the four-loop propagator type integrals reduction is that it is possible to derive dimensional recurrence relations for master integrals in $d = 6 - 2\varepsilon$ from known $d = 4 - 2\varepsilon$ results [18,19]. Unfortunately, this approach cannot directly be applied to five-loop calculations, as there is no efficient way to reduce five-loop massless propagator integrals to the master integrals. It is especially difficult to derive dimensional recurrence relations to use the known $d = 4 - 2\varepsilon$ five-loop master integrals [20] in the $d = 6 - 2\varepsilon$ problem we are dealing with.

More sophisticated methods were applied in the breakthrough calculations in $\lambda\varphi^4$ theory in $d = 4 - 2\varepsilon$ [14,15,3,16]. In Ref. [14] anomalous dimension of the field was calculated by using the infrared $\mathcal{R}^*$-operation [21–25] and four-loop IBP reduction. Due to the specific structure of diagrams with only quartic vertices, the $\mathcal{R}^*$-operation is extremely effective, and permits to reduce each of the six-loop two-point diagrams to four-loop massless propagators, which can then be further reduced to known master integrals [18,19]. However, 12 non-primitive diagrams (diagrams containing subdivergences) of the four-point function cannot be calculated with the $\mathcal{R}^*$-operation and four-loop IPB reduction.

Therefore, a new method [15] using a distinct $\mathcal{R}'$ operation in combination with special one-scale subtractions and parametric integration of hyperlogarithms (`HyperInt` package) [26,27] was developed [15,3].

Seven-loop terms were calculated with an even more advanced *graphical functions* technique [16]. This most significant recent progress in $\lambda\varphi^4$ theory was possible due to the development of new powerful methods for multi-loop calculations. It should be noted, however, that it is possible due to the relative simplicity of diagrams constructed from quartic vertices only.

The methods mentioned above can be applied to $\lambda\varphi^3$ theory as well: graphical functions were applied recently [17] to the problem considered in the present paper. Parametric integration using hyperlogarithms also works, but the straightforward approach used in [15,3] meets performance problems due to the large number of integrations in parametric space, and the reach combinatorial structure of diagrams in the $\lambda\varphi^3$ theory.

Recently, five-loop QCD renormalization was performed [28–31]. Methods utilized there are much closer to the problem we are interested in here. One idea is to avoid calculation of five-loop integrals, by reducing the problem to the calculation of four-loop massless propagator integrals through application of the $\mathcal{R}^*$-operation to individual diagrams [29] or applying the $\mathcal{R}^*$-operation in a global form [28,31]. The rest utilizes fully massive five-loop tadpoles calculated numerically with high precision [30] at intermediate steps, and reconstructs the analytical answer in the end.

Our starting point was to use the $\mathcal{R}^*$ approach, as it guaranties reduction of the L-loop problem to a $(L-1)$-loop problem, in our case the reduction from five to four loops, where the latter can be solved using available IBP reduction tools. After a detailed analysis we realized that it is even possible to perform calculations without the $\mathcal{R}^*$-operation. Our strategy lies somewhere in between the two methods used in the QCD calculations mentioned above. Similar to [30] we reduce the problem to the calculation of the divergent parts of specially constructed fully massive five-loop tadpole integrals. But instead of calculating integrals directly as in [30], we calculate their divergences with the help of the *infra-red rearrangement* (IRR) trick, which requires knowledge of four-loop integrals only. This IRR technique is valid due to the independence of the renormalization constants Z_i of masses and external momenta [32].

2.1. Details of calculation

Due to the moderate number of diagrams in model (1), we apply the $\mathcal{R}'$-operation separately to each diagram. This approach allows us to study generalizations of $\lambda\varphi^3$ theory to different symmetries or field contents, by simply changing the symmetry factor of each diagram.

The $\mathcal{R}'$-operation, also known as the incomplete $\mathcal{R}$-operation, subtracts all subdivergences from a diagram, except the superficial divergence of the diagram itself. Working in the $\overline{\text{MS}}$ scheme, it is useful to introduce the $\mathcal{K}$ operation, which extracts the divergent part of the Laurent series expansion in ε. For each diagram G, the result $\mathcal{K}\mathcal{R}'G$ gives the contribution to the corresponding renormalization constant in the $\overline{\text{MS}}$ scheme. Its recursive definition has the form

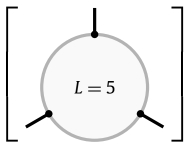
$$\mathcal{K}\mathcal{R}'G = \mathcal{K}G - \mathcal{K}(1-\mathcal{R}')G = \mathcal{K}G + \sum_{\{\gamma\}} \mathcal{K}\Big[\prod_{\gamma_i \in \{\gamma\}} (-\mathcal{K}\mathcal{R}'\gamma_i) * G/\{\gamma\}\Big]. \tag{3}$$

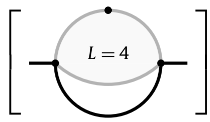
The sum is taken over all non-empty subsets $\{\gamma\}$ of non-intersecting divergent subgraphs γ_i. The product is taken over all subgraphs in each subset. The recursion stops when no $\{\gamma\}$ contains a diagram with subdivergences. In this case, $\mathcal{K}\mathcal{R}'\gamma = \mathcal{K}\gamma$ (pole part of the diagram γ). The operation $*$ inserts the appropriate counterterm into the co-graph $G/\{\gamma\}$. For a logarithmically divergent subgraph, it reduces to a scalar multiplication, while for a quadratically divergent subgraph γ_i it inserts the full (momentum) structure $Ap^2 + Bm^2$. In equation (3), we introduce the formal operation $(1-\mathcal{R}')G$, which for an L-loop diagram G contains only terms with less than L loops. Assuming our ability to calculate arbitrary loop integrals at level $(L-1)$, such a splitting allows us to separate the complicated part containing L-loop integrals ($\mathcal{K}G$ in (3)) from the rest.

The renormalization constants Z_φ, Z_g and Z_m are finally calculated from the following Green functions with all subdivergences subtracted,

$$Z_\varphi^2 = 1 + \mathcal{K}\mathcal{R}'\partial_{q^2}\Gamma_{\varphi\varphi}, \quad Z_m^2 Z_\varphi^2 = 1 + \mathcal{K}\mathcal{R}'\partial_{m^2}\Gamma_{\varphi\varphi}, \quad Z_g Z_\varphi^3 = 1 - \mathcal{K}\mathcal{R}'\overline{\Gamma}_{\varphi\varphi\varphi}, \quad \overline{\Gamma}_{\varphi\varphi\varphi} = \Gamma_{\varphi\varphi\varphi}/(-g\mu^\varepsilon). \tag{4}$$

In order to realize this program, we need to compute the divergent parts of diagrams contributing to the three-point function, and derivatives (with respect to external momenta and mass) of the two-point function. Since we are working in a scalar theory, integrals with numerators (all propagators give denominators) only appear in the calculation of Z_φ after differentiation with respect to q^2. Other renormalization constants are calculated using scalar integrals only. The most complicated part in the calculations of the renormalization constants (4) is to compute the divergent part of the five-loop diagrams. The independence of the $\overline{\text{MS}}$ renormalization constants of masses and external momenta allows us to set to zero all masses and external momenta of the original diagram, and to introduce artificial external momenta flowing through a single edge,

$\mathcal{KR}'$


$= \mathcal{KR}'$


. (5)

The result of the $\mathcal{KR}'$ operation does not change, if such a transformation is applied to logarithmically divergent diagrams, and no IR divergences appear after rearrangement. We have checked explicitly that for all three-point integrals at least one such edge can be identified. This edge assigns external momenta routing. As we can see in equation (5), the rearranged diagram has the form of a four-loop propagator inserted into a one-loop propagator, and can easily be calculated since four-loop propagators are known. The same technique is appropriate for the calculation of Z_m, since differentiation with respect to m^2 is equal to the attachment of an additional leg to one of the edges of the original two-point diagram.

The calculation of the field renormalization constant Z_φ is more complicated than that of Z_m, since differentiation with respect to the external momenta produces a large number of integrals with numerators, and it becomes difficult to apply the IRR trick in the form given in equation (5). To overcome this difficulty, we developed another approach, reducing the problem to the calculation of fully massive tadpoles instead of massless propagators. This approach is applicable to the calculation of all renormalization constants of interest. Our starting point is to split the $\mathcal{KR}'G$ operation acting on a L-loop diagram G into the two parts given in equation (3). The first term $\mathcal{K}G$ contains divergences of L-loop diagrams, while the second term $\mathcal{K}(1-\mathcal{R}')G$ contains subtraction terms with maximal loop order $(L-1)$. Assuming that we know how to calculate an arbitrary integral at loop order $(L-1)$ present in $\mathcal{K}(1-\mathcal{R}')G$, the main difficulty is to calculate the L-loop part $\mathcal{K}G$.

To overcome the most complicated part, namely evaluating the poles of the L-loop diagrams $\mathcal{K}G$, we consider divergences of the related integrals instead. In comparison to diagrams, integrals satisfy more relations allowing to reduce their complexity. One class of relations is *Integration By Parts* (IBP) [33,34], widely used for the reduction of large numbers of integrals to a smaller set of *master integrals* (MI).

IBP relations connect m original integrals $J_a(a=1,\dots,m)$ with another set of n $(n \le m)$ integrals $I_b(b=1,\dots,n)$ through a matrix M_{ab} with rational polynomial coefficients in masses, scalar products of external momenta and space-time dimension $d = 6-2\varepsilon$. For single-scale integrals it simplifies to

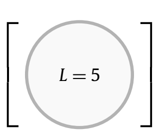
$$J_a(\varepsilon) = M_{ab}(\varepsilon) I_b(\varepsilon) \tag{6}$$

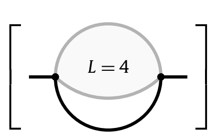
For arbitrary choice of integrals I_b, the matrix $M_{ab}(\varepsilon)$ may contain poles in ε, and the calculation of $\mathcal{K}J_a$ requires not only pole parts for some integrals from I_b, but also their finite parts, and even higher-order terms in the ε-expansion. To avoid such complications, we focus on candidate integrals I_b, for which the matrix M_{ab} is regular in the limit of $\varepsilon \to 0$. A basis of master integrals with such properties was considered in Ref. [35] and called *ε-finite*.

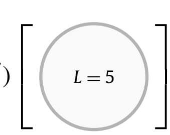
Since we decided to reduce the problem to the calculation of $\mathcal{KR}'$ for fully massive tadpole diagrams, we collect into the set J_a all integrals originated from vertex and propagator diagrams. In three-point integrals, we set to zero all external momenta while treating all internal lines as massive. The same procedure is used for two-point functions, after differentiation in squared external momenta. Such an IR rearrangement is allowed, since all integrals are logarithmically divergent, IR divergences are regularized by masses on all internal lines, and the action of the $\mathcal{KR}'$ operation is independent of external momenta. As a result, all integrals J_a are fully massive tadpoles, possibly with numerators, and including all 5-loop integrals needed for the $\mathcal{K}G$ operation in equation (4). Integrals in the set I_b satisfy the conditions:

1. the matrix M_{ab} does not have poles in ε,
2. $\mathcal{K}I_b$ can be calculated using four-loop integrals only using IRR trick,
3. the integrals I_b do not have numerators and allow an interpretation as diagrams in scalar field theory.

Since we only need the divergent parts of the integrals I_b (due to condition 1), we choose candidates for I_b computable from four-loop integrals (condition 2). Calculation of the divergent part using IRR is based on the independence of counterterms of the logarithmic diagrams on the infrared regularization (5). It can be rewritten as

$\mathcal{K}$


$_{m\neq 0} = \mathcal{KR}'$


$_{m=0} - \mathcal{K}(1-\mathcal{R}')$


$_{m\neq 0}$, (7)

where both terms on the right-hand-side can be calculated using integrals with four loops or less only.

The reduction of five-loop tadpole integrals is the most time-consuming part of the calculation, and our choice of candidate integrals for the set I_b is limited. Therefore we try to include in the candidate list as few integrals as possible, and mostly reuse results of IBP reduction for integrals J_a (notations in equation (6)). A natural choice fulfilling all three conditions is the set $J^{(3)}$ of the integrals needed for the three-point function. As discussed before (condition 2) their divergences can be calculated using equation (5), they are free from numerators, and are in one-to-one correspondence with scalar theory diagrams (condition 3). Unfortunately, it is not enough to include integrals from the set $J^{(3)}$ satisfying condition 1 and we need to extend the list of candidates. Our choice is to include integrals $J^{(4)}$ resulting from four-point functions in φ^3-theory (1). Formally, the IRR trick (7) cannot be applied to such diagrams (as they are not logarithmically divergent), but due to the absence of superficial divergences ($\mathcal{KR}'$ of such diagrams is always zero) it is still possible to calculate their divergent parts from knowledge of lower-loop integrals only,

$$\mathcal{K} J_k^{(4)} = \underbrace{\mathcal{KR}' J_k^{(4)}}_{=0} + \mathcal{K}(1-\mathcal{R}') J_k^{(4)}. \tag{8}$$

Combining integrals from $J^{(3)}$ with integrals from $J^{(4)}$ we are able to construct the set of integrals I_b meeting all three conditions. Trying to maximize the number of integrals from the set $J^{(4)}$, only 14 integrals from the set $J^{(3)}$ need to be calculated. From these 14 integrals 8 are free from subdivergences, thereby the $\mathcal{R}'$ operation acts trivially on them, and their divergences can be calculated from the massless diagrams via rule (5). For the remaining 6 diagrams, where knowledge of the $\mathcal{KR}'$ operation is required, details of the $\mathcal{KR}'$ calculation using the IRR trick are presented in Appendix C.

As an illustration, let us apply the described procedure to the most complicated class of diagrams,

$$\mathcal{KR}'\partial_{q^2} J_i = \mathcal{K}\partial_{q^2} J_i - \mathcal{K}(1-\mathcal{R}')\partial_{q^2} J_i = \mathcal{K}\left[A_{ik}(\varepsilon)\mathcal{K}J_k^{(3)} + B_{ik}(\varepsilon)\mathcal{K}J_k^{(4)}\right] - \partial_{q^2}\mathcal{K}(1-\mathcal{R}')J_i. \tag{9}$$

By using $\mathcal{KR}'\partial_{q^2} G = \partial_{q^2}\mathcal{KR}' G$, we interchanged the order of differentiation in external momenta, and divergences subtraction, in the last term. This allows us to get rid of the $\mathcal{KR}'$ operation applied to diagrams with numerators in all steps of the calculation. The matrices $A_{ik}(\varepsilon)$ and $B_{ik}(\varepsilon)$ produced as a result of the IBP reduction in equation (9) are free from poles in ε.

As a result, we have the following workflow to calculate all renormalization constants (4) up to five-loop order. First we generate all three-point and two-point diagrams in scalar theory using `DIANA` [36]. At this step we assign equal masses to all internal lines. For logarithmically divergent three-point functions we set all external momenta to zero, for two-point functions we keep the routing of external momenta. To each scalar diagram we apply the $\mathcal{KR}'$ operation, implemented as a private `C++` code generating the appropriate `FORM` [37] input for the calculation of divergent subgraphs and co-subgraphs according to equation (3). At this step, co-subgraphs needed for the calculation of $\partial_{q^2}\mathcal{KR}'$ are differentiated in external momenta, and external momenta are set to zero, reducing the problem to the calculation of fully massive tadpoles only. All required integrals up to three-loop order are calculated with the help of the `MATAD-ng`[1] version of the `MATAD` package [38], operating in arbitrary space-time dimension d, and at four-loop order using the package `FMFT` [39]. With the same technique, using relation (8), we obtain the divergences of all four-point integrals $\mathcal{K}J^{(4)}$ from equation (9). Another ingredient included in equation (9) are divergences of the three-point functions $\mathcal{K}J^{(3)}$, which, with the help of IRR, are reduced to the calculation of the $\mathcal{KR}'$ operation on properly constructed massless five-loop propagator diagram, in accordance with equation (5). Details of the calculation for the six diagrams with subdivergences from the set $\mathcal{K}J^{(3)}$ are presented in Appendix C. Four-loop massless propagator integrals are calculated with the package `FORCER` [40], and at lower loop orders with the `MINCER`[2] package [41,42].

As a last step to apply (9), we need to reduce five-loop tadpole integrals to the minimal set and construct matrices $A_{ik}(\varepsilon)$ and $B_{ik}(\varepsilon)$. To perform this reduction, we use the Laporta algorithm [43] implemented in the recent `C++` version of the `FIRE6` package [44]. To overcome the appearance of lengthy d-dependent coefficients in front of integrals during the reduction process, we activated the `FIRE6` reduction functionality, using modular arithmetic for the polynomial algebra at intermediate steps. Symmetry relations among different integrals used in the IBP reduction process were provided by the `LiteRed` [45] package. To check divergent parts of five-loop tadpoles numerically, we use the code `FIESTA` [46].

All our results for the divergent parts of 3-point and 4-point integrals, of $\mathcal{KR}'$ applied to each of the three-point diagrams, and of $\mathcal{KR}'\partial_{q^2}$ and $\mathcal{KR}'\partial_{m^2}$ applied to each of the two-point diagrams are available as ancillary files with the arXiv version of the paper.

2.2. Results for the scalar model

From the renormalization constants in equation (2) we extract the renormalization group functions

$$\beta_g = -\frac{\varepsilon g}{1 + g\partial_g \log Z_g}, \quad \gamma_\varphi = \beta_g\, \partial_g \ln Z_\varphi, \quad \gamma_m = \beta_g\, \partial_g \ln Z_m. \tag{10}$$

While in the above expressions the renormalization constants Z_i have poles, the final results are free from poles, which is a strong check for the correctness of our calculation. Expressions for anomalous dimensions (A.1) and (A.2) are provided in Appendix A. The β function is related to the above results via $\beta_g = -\varepsilon g + g\left(\gamma_\varphi - 2\gamma_m\right)$. From $\beta_g(g^*) = 0$ we can find the position of the fixed point g^*. Substituting the value for g^* into the anomalous dimension of the field, we obtain the critical exponent for the dimension of the field,

$$\eta = 2\gamma_\varphi^* = -0.222222\,\varepsilon - 0.23594\,\varepsilon^2 + 0.34952\,\varepsilon^3 - 1.26327\,\varepsilon^4 + 6.66413\,\varepsilon^5 + O\left(\varepsilon^6\right). \tag{11}$$

Due to the relation $2\gamma_m = \gamma_\varphi - \left(\frac{\beta}{g} + \varepsilon\right)$ between anomalous dimensions and the β function in the model (1), another critical exponent, corresponding to the critical dimension of the dimension-two operator, is trivially connected with equation (11) through the relation

$$\frac{1}{\nu} = 2 + 2\gamma_m^* = 2 - \varepsilon + \gamma_\varphi^* = 2 - \varepsilon + \frac{\eta}{2}. \tag{12}$$

For the correction-to-scaling exponent ω we have

$$\omega = \left(\frac{\partial \beta}{\partial g}\right)_{g=g_*} = 2\varepsilon - 3.08642\varepsilon^2 + 12.7253\varepsilon^3 - 72.522\varepsilon^4 + 493.942\varepsilon^5 + O\left(\varepsilon^6\right). \tag{13}$$

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[1] Available from https://github.com/apik/matad-ng.

[2] "Mincer exact" package available from https://www.nikhef.nl/~form/maindir/packages/mincer/mincerex.tgz.

The value found in equation (11) is in agreement with the four-loop results [47,48,13] and with the recent five-loop results obtained using the graphical-functions technique [17]. With the first five terms of expansion (11) at hand, we can compare them to predictions for the asymptotic behavior of higher-order terms derived in Ref. [49]:

$$\eta_{\text{as}} = \sum_{n=1}^{\infty} f_n \varepsilon^n, \quad f_n = -0.000586 \cdot \sqrt{2\pi} \left(-\frac{5}{9} \cdot \frac{n}{e}\right)^n n^5 \left(1 + O\left(\frac{1}{n}\right)\right). \tag{14}$$

Comparing the asymptotic expansion (14) with equation (11), we see that the low-order difference is huge ($f_3 = 0.0822729$ and $f_4 = -0.67183$) but becomes smaller for the five-loop term ($f_5 = 5.11508$). It will be interesting to compare with six-loop results once they become available.

3. Model with $O(n)$ symmetry and $1/n$ expansion

As previously mentioned, one can use the results of section 2.1 for calculations in more advanced theories. In this section we focus on the $O(n)$-symmetric generalization of the model (1). Such a model was considered in Ref. [4], and is interesting for the investigation of the universality class with $O(n)$-symmetry above four dimensions. Another important property of $O(n)$ symmetric models is that critical exponents of the corresponding universality class can be calculated in the framework of a $1/n$ expansion for arbitrary space-time dimension, thus providing an additional check for the results obtained.

The bare Lagrangian of the model has the form

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \varphi_{i,B})^2 + \frac{1}{2}(\partial_\mu \sigma_B)^2 + g_{1,B}\frac{(4\pi)^3}{2}\sigma_B(\varphi_{i,B})^2 + g_{2,B}\frac{(4\pi)^3}{6}\sigma_B^3, \tag{15}$$

where field φ_i is an n-component vector, and σ is a scalar. We start from the massless model and extract anomalous dimensions for dimension-two operators. This is in contrast to the model (1) where we calculated the anomalous dimension of the mass in a theory where the mass term is in the Lagrangian. Anomalous dimensions are now calculated from diagrams with φ^2 and σ^2 insertions. Similar to equation (2) we define renormalization constants connecting bare and renormalized parameters,

$$\varphi_{i,B} = Z_\varphi \varphi_i, \quad \sigma_B = Z_\sigma \sigma, \quad g_{1,B} = \mu^\varepsilon Z_{g_1} g_1, \quad g_{2,B} = \mu^\varepsilon Z_{g_2} g_2. \tag{16}$$

Taking into account symmetry coefficients and group-theory factors for the model (15) and utilizing $\mathcal{KR}'$-operation results from the section 2.1, we obtain renormalization constants for the following Green functions

$$Z_{\varphi\varphi} = 1 + \partial_{q^2}\mathcal{KR}'\Gamma_{\varphi\varphi}, \quad Z_{\sigma\sigma} = 1 + \partial_{q^2}\mathcal{KR}'\Gamma_{\sigma\sigma}, \quad Z_{\sigma\varphi\varphi} = 1 - \mathcal{KR}'\overline{\Gamma}_{\sigma\varphi\varphi}, \quad Z_{\sigma\sigma\sigma} = 1 - \mathcal{KR}'\overline{\Gamma}_{\sigma\sigma\sigma}. \tag{17}$$

Here $\overline{\Gamma}_{\sigma\varphi\varphi} = \Gamma_{\sigma\varphi\varphi}/(-g_1\mu^\varepsilon)$ and $\overline{\Gamma}_{\sigma\sigma\sigma} = \Gamma_{\sigma\sigma\sigma}/(-g_2\mu^\varepsilon)$. From these functions, the renormalization constants (16) are extracted via the relations

$$Z_\varphi = \sqrt{Z_{\varphi\varphi}}, \quad Z_\sigma = \sqrt{Z_{\sigma\sigma}}, \quad Z_{g_1} = \frac{Z_{\sigma\varphi\varphi}}{Z_\sigma Z_\varphi^2}, \quad Z_{g_2} = \frac{Z_{\sigma\sigma\sigma}}{Z_\sigma^3}. \tag{18}$$

Charge renormalization constants Z_{g_1} and Z_{g_2} provide us with beta-functions for the two coupling constants of the theory,

$$\begin{aligned}
\beta_{g_1} = \frac{\partial g_1}{\partial \ln \mu} &= \frac{\varepsilon g_1\left(1 + g_2\partial_{g_2}\ln Z_{g_2} - g_2\partial_{g_2}\ln Z_{g_1}\right)}{g_1 g_2\left(\partial_{g_2}\ln Z_{g_1}\right)\left(\partial_{g_1}\ln Z_{g_2}\right) - (1 + g_1\partial_{g_1}\ln Z_{g_1})(1 + g_2\partial_{g_2}\ln Z_{g_2})} \\
\beta_{g_2} = \frac{\partial g_2}{\partial \ln \mu} &= \frac{\varepsilon g_2\left(1 + g_1\partial_{g_1}\ln Z_{g_1} - g_1\partial_{g_1}\ln Z_{g_2}\right)}{g_1 g_2\left(\partial_{g_2}\ln Z_{g_1}\right)\left(\partial_{g_1}\ln Z_{g_2}\right) - (1 + g_1\partial_{g_1}\ln Z_{g_1})(1 + g_2\partial_{g_2}\ln Z_{g_2})}
\end{aligned} \tag{19}$$

Renormalization constants for the field yield the anomalous field dimensions

$$\gamma_\varphi = \beta_{g_1}\,\partial_{g_1}\ln Z_\varphi + \beta_{g_2}\,\partial_{g_2}\ln Z_\varphi, \quad \gamma_\sigma = \beta_{g_1}\,\partial_{g_1}\ln Z_\sigma + \beta_{g_2}\,\partial_{g_2}\ln Z_\sigma. \tag{20}$$

As in the previous section, cancellation of poles in equations (19) and (20) is a strong test of our calculations. Results for the beta-functions and anomalous dimensions are quite lengthy to present here, but are available in the supplementary material to the paper. They are in agreement with existing three-loop [5] and four-loop [13] calculations.

With results for the RG-functions at hand, we proceed with the analysis of the critical behavior of the model (15): Zeroes of the beta-functions (19) $\beta_1 = \beta_2 = 0$ define multiple fixed points (g_1^*, g_2^*). We choose the unique IR-stable one corresponding to the large-n solution considered in Ref. [4].

Since our goal is to compare results of our calculation with available large-n expansions for the critical exponents, we make an ansatz for the solution of the fixed point (g_1^*, g_2^*) as a double expansion in ε and $1/n$. Expansions up to $O(1/n^4)$ and up to five loops, i.e. $\mathcal{O}(\varepsilon^5)$, are presented in equation (B.1) and (B.2). Substituting the found large-n solution for the critical point as a $1/n$ expansion into the anomalous field dimension (20), we obtain the critical dimensions of operators Δ_φ and Δ_σ as

$$\Delta_\varphi = \frac{d}{2} - 1 + \gamma_\varphi^* = \frac{d}{2} - 1 + \frac{\eta}{2}, \quad \Delta_\sigma = \frac{d}{2} - 1 + \gamma_\sigma^* = d - \frac{1}{\nu}. \tag{21}$$

Results for η up to $O(1/n^3)$ can be found in Ref. [50] and for $1/\nu$ up to order $O(1/n^2)$ in [51]. Using the expressions for γ_φ^* and γ_σ^*, and the results for the critical exponents, the critical dimensions as given in equation (21) are equal to all known orders in $1/n$ expansion.

In addition to the critical dimensions of fields (21), we consider critical dimensions of operators with canonical dimension two. The main complication in this case is operator mixing under the renormalization procedure. We start with the definition of two operators with equal canonical dimension,

$$\mathcal{O}_1 = \varphi_i \varphi_i \quad \mathcal{O}_2 = \sigma^2. \tag{22}$$

The renormalization of the operators is defined as

$$Z^{\mathcal{O}}_{ij} \mathcal{O}^R_j = \mathcal{O}_i. \tag{23}$$

The matrix of the renormalization constants $Z = (Z^{\mathcal{O}})^{-1}$ can be determined from the set of one-particle irreducible (1PI) Green functions with operator insertions $\{\langle \mathcal{O}_1 \varphi_i \varphi_i \rangle, \langle \mathcal{O}_1 \sigma\sigma \rangle, \langle \mathcal{O}_2 \varphi_i \varphi_i \rangle, \langle \mathcal{O}_2 \sigma\sigma \rangle\}$. As in the previous sections, we subtract all subdivergences with the help of the $\mathcal{KR}'$-operation,

$$Z = \begin{pmatrix} (1 - \mathcal{KR}' \langle \mathcal{O}_1 \varphi_i \varphi_i \rangle) Z_\varphi^{-2} & -\mathcal{KR}' \langle \mathcal{O}_1 \sigma\sigma \rangle Z_\sigma^{-2} \\ -\mathcal{KR}' \langle \mathcal{O}_2 \varphi_i \varphi_i \rangle Z_\varphi^{-2} & (1 - \mathcal{KR}' \langle \mathcal{O}_2 \sigma\sigma \rangle) Z_\sigma^{-2} \end{pmatrix}. \tag{24}$$

The anomalous dimensions matrix is then defined as

$$\gamma^{\mathcal{O}}_{ij} = (Z^{\mathcal{O}})^{-1}_{ik} \left[\beta_1 \partial_{g_1} + \beta_2 \partial_{g_2}\right] Z^{\mathcal{O}}_{kj} = -\left(\left[\beta_1 \partial_{g_1} + \beta_2 \partial_{g_2}\right] Z_{ik}\right) Z^{\mathcal{O}}_{kj}. \tag{25}$$

Five-loop results for the matrix elements γ_{ij} are attached to the paper in a computer readable form, and the four-loop part is in agreement with previous calculations [13].

Critical dimensions of the mixing operators are determined as eigenvalues $\Delta_\pm$ of the matrix $\Delta^{\mathcal{O}} = d^{\mathcal{O}} + \gamma^{\mathcal{O}}(g_1^*, g_2^*)$, where $d^{\mathcal{O}}$ is a matrix of the canonical dimensions of the operators, and $\gamma^{\mathcal{O}}(g_1^*, g_2^*)$ are anomalous dimensions from equation (25) at the fixed point. In our case, the canonical dimensions of the operators are equal and proportional to the unit matrix $d^{\mathcal{O}} = (d-2) \cdot \mathbb{1}$. Dimensions of these operators can be identified with critical exponents in the $O(n)$-universality class. While the definition of the anomalous dimensions depends on the particular model (non-linear sigma model, $O(n)$-symmetric φ^4 model, or $O(n)$-symmetric φ^3 model), the critical dimensions of operators can easily be mapped from one model to another. One can show that $\Delta_\pm$ must coincide with $\Delta^{\mathrm{NLSM}}_{\psi^2}$ and $\Delta^{\mathrm{NLSM}}_{\phi^2}$ from the non-linear sigma model [2]. This allows us to relate these dimensions to critical exponents of the $O(n)$-universality class, and to compare with results of the $1/n$ expansion,

$$\Delta_+ = \Delta^{\mathrm{NLSM}}_{\psi^2} = d + \omega, \quad \Delta_- = \Delta^{\mathrm{NLSM}}_{\phi^2} = d - \Delta_\sigma = \frac{1}{\nu}. \tag{26}$$

Here critical dimension Δ_+ agrees with the correction-to-scaling exponent ω calculated to order $1/n^2$ in Refs. [52,53], while the critical dimension Δ_- is the already calculated exponent $1/\nu$.

Another question we can address with our results for the beta functions (19) is about n_{cr}: the critical value of n below which the fixed point becomes complex. A very large value $n_{\mathrm{cr}} = 1038$ was determined at leading order for the first time in Ref. [4] and refined using higher-loop results for the beta-functions in [5,13]. It was further considered in [54] with functional renormalization-group methods. Let us follow Ref. [4], where the following conditions determine n_{cr},

$$\beta_1 = 0, \quad \beta_2 = 0, \quad \frac{\partial \beta_1}{\partial g_1}\frac{\partial \beta_2}{\partial g_2} - \frac{\partial \beta_1}{\partial g_2}\frac{\partial \beta_2}{\partial g_1} = 0. \tag{27}$$

Using an ansatz for n_{cr} as a series in ε, and rescaled coupling constants (x, y), defined as $g_1 = 2\sqrt{\frac{3\varepsilon}{n}}x$, $g_2 = 2\sqrt{\frac{3\varepsilon}{n}}y$, we determine a set of critical values,

$$\begin{aligned} n_{\mathrm{cr}} &= 1038.2661 - 1219.6796\varepsilon - 1456.6933\varepsilon^2 + 3621.6847\varepsilon^3 + 986.2232\varepsilon^4 + O\left(\varepsilon^5\right) \\ x_{\mathrm{cr}} &= 1.01804 - 0.0187953\varepsilon + 0.0276061\varepsilon^2 - 0.0258653\varepsilon^3 + 0.188983\varepsilon^4 + O\left(\varepsilon^5\right) \\ y_{\mathrm{cr}} &= 8.90305 - 0.420480\varepsilon + 4.06718\varepsilon^2 - 2.00946\varepsilon^3 - 62.4036\varepsilon^4 + O\left(\varepsilon^5\right) \end{aligned} \tag{28}$$

From the multiple solutions of the system (27), we have chosen the one corresponding to the large-n limit of the theory.

To get an estimate for n_{cr} we use Pade approximants. It is well known that for finite values of the expansion parameter estimates obtained from different approximants may differ significantly. This happens when approximants have spurious poles or a powerlaw asymptotics becomes dominant. It is hard to estimate the influence of these factors, and the choice of a particular approximant becomes intricate, and biased by already known estimates from other methods. To avoid this, we use the method suggested in Ref. [8], excluding approximants for which the influence of poles or asymptotics is obvious, and consider estimates from the remaining approximants as "independent measurements". This allows us to get unbiased estimate as well as error bars. As a result, $n_{\mathrm{cr}} = 340 \pm 200$ at four loops and $n_{\mathrm{cr}} = 420 \pm 120$ at five loops, both in dimension $d = 5$. These values are in agreement with earlier estimates [4,5,13].

4. Conclusion

We calculated the renormalization-group functions and the corresponding critical exponents for the simplest scalar cubic theory and its $O(n)$-extension in $d = 6 - 2\varepsilon$ space-time dimensions at five-loop order. To achieve this, we constructed an ε-finite basis of fully massive five-loop tadpole integrals, sufficient for computation of all needed renormalization constants. For each diagram in the scalar theory, we provide counterterms, which will help extend our results to more advanced field theory models and obtain the critical dimensions of other observables of interest. The critical exponents in the $O(n)$-symmetric model were checked to agree with results of the $1/n$-expansion available in the literature.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Acknowledgements

We thank A. Bednyakov, G. Kalagov, N. Lebedev and K. Wiese for fruitfull discussions and careful reading of the manuscript, J. Gracey for his comments on the large-n limit of the $O(n)$ model, as well as M. Borinsky and O. Schnetz for sharing their results prior to publication. We are grateful to the Joint Institute for Nuclear Research for letting us use their supercomputer "Govorun". This work is supported by the Foundation for the Advancement of Theoretical Physics and Mathematics "BASIS".

Appendix A. Anomalous dimensions for scalar theory

$$\gamma_\varphi = \frac{1}{12}g^2 + \frac{13}{432}g^4 + \left(\frac{5195}{62208} - \frac{\zeta_3}{24}\right)g^6 + \left(\frac{53449}{248832} + \frac{35\zeta_3}{864} + \frac{7\zeta_4}{96} - \frac{5\zeta_5}{18}\right)g^8 + \left(\frac{16492987}{20155392} + \frac{56693\zeta_3}{62208} + \frac{5651\zeta_4}{27648} - \frac{4471\zeta_5}{10368} + \frac{25\zeta_3^2}{144} + \frac{125\zeta_6}{288} - \frac{147\zeta_7}{64}\right)g^{10} + O\left(g^{11}\right) \tag{A.1}$$

$$\gamma_{m^2} = \frac{5}{6}g^2 + \frac{97}{108}g^4 + \left(\frac{52225}{31104} + \frac{7\zeta_3}{12}\right)g^6 + \left(\frac{445589}{93312} + \frac{821\zeta_3}{144} - \frac{19\zeta_4}{48} - \frac{35\zeta_5}{18}\right)g^8 + \left(\frac{40331135}{2519424} + \frac{839129\zeta_3}{15552} - \frac{66953\zeta_4}{13824} + \frac{225457\zeta_5}{5184} + \frac{229\zeta_3^2}{36} + \frac{25\zeta_6}{9} - \frac{2821\zeta_7}{32}\right)g^{10} + O\left(g^{11}\right) \tag{A.2}$$

Appendix B. Results for the $O(n)$ symmetric model

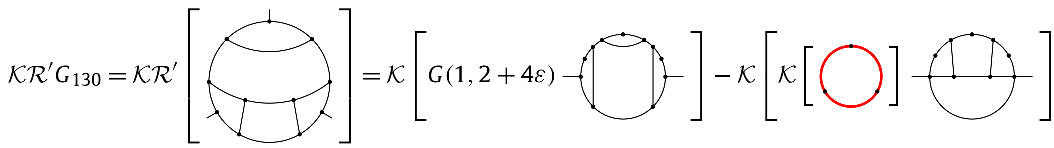
$$\begin{aligned} g_1^* = 2\sqrt{\frac{3\varepsilon}{n}}\Bigg[&1 + \left(\frac{22}{n} + \frac{726}{n^2} - \frac{326180}{n^3} + \mathcal{O}\left(\frac{1}{n^4}\right)\right) \\ &+ \left(-\frac{155}{3n} - \frac{3410}{n^2} + \frac{1825090}{n^3} + \mathcal{O}\left(\frac{1}{n^4}\right)\right)\varepsilon \\ &+ \left(\frac{1777}{36n} + \left(\frac{29093}{9} - 4680\zeta_3\right)\frac{1}{n^2} + \left(-\frac{106755739}{18} - 912240\zeta_3\right)\frac{1}{n^3} + \mathcal{O}\left(\frac{1}{n^4}\right)\right)\varepsilon^2 \\ &+ \Bigg(-\frac{217}{324n} + \left(\frac{709151}{108} + 25008\zeta_3 - 4680\zeta_4 + 2880\zeta_5\right)\frac{1}{n^2} \\ &+ \left(\frac{779869165}{54} + 17839320\zeta_3 - 912240\zeta_4 - 19509120\zeta_5\right)\frac{1}{n^3} + \mathcal{O}\left(\frac{1}{n^4}\right)\Bigg)\varepsilon^3 \\ &+ \Bigg(\left(\frac{6973}{2592} - \frac{155\zeta_3}{6}\right)\frac{1}{n} + \left(-\frac{49050023}{7776} - \frac{81742\zeta_3}{3} + 1296\zeta_3^2 + 28134\zeta_4 - 69120\zeta_5 + 5400\zeta_6\right)\frac{1}{n^2} \\ &+ \Bigg(-\frac{17059272503}{972} - \frac{77297317\zeta_3}{3} - 5870880\zeta_3^2 + 20039010\zeta_4 + 92059740\zeta_5 \\ &-36579600\zeta_6 - 23528232\zeta_7\Bigg)\frac{1}{n^3} + \mathcal{O}\left(\frac{1}{n^4}\right)\Bigg)\varepsilon^4 + O(\varepsilon^5)\Bigg] \end{aligned} \tag{B.1}$$

$$\begin{aligned} g_2^* = 2\sqrt{\frac{3\varepsilon}{n}}\Bigg[&6 + \left(\frac{972}{n} + \frac{412596}{n^2} + \frac{247346520}{n^3} + \mathcal{O}\left(\frac{1}{n^4}\right)\right) \\ &+ \left(-\frac{1290}{n} - \frac{1036020}{n^2} - \frac{908667180}{n^3} + \mathcal{O}\left(\frac{1}{n^4}\right)\right)\varepsilon \\ &+ \left(\frac{2781}{2n} + (1083644 - 628560\zeta_3)\frac{1}{n^2} + (1156981601 - 632020320\zeta_3)\frac{1}{n^3} + \mathcal{O}\left(\frac{1}{n^4}\right)\right)\varepsilon^2 \\ &+ \Bigg(-\frac{3461}{18n} + \left(\frac{25209239}{18} + 5561280\zeta_3 - 628560\zeta_4 - 3369600\zeta_5\right)\frac{1}{n^2} \\ &+ \left(\frac{18166643735}{9} + 9509192880\zeta_3 - 758121120\zeta_4 - 8137843200\zeta_5\right)\frac{1}{n^3} + \mathcal{O}\left(\frac{1}{n^4}\right)\Bigg)\varepsilon^3 \\ &+ \Bigg(\left(\frac{5945}{144} - 645\zeta_3\right)\frac{1}{n} + \Bigg(-\frac{1204206197}{432} - 11677092\zeta_3 - 1516320\zeta_3^2 + 6256440\zeta_4 + 10069920\zeta_5 \end{aligned}$$

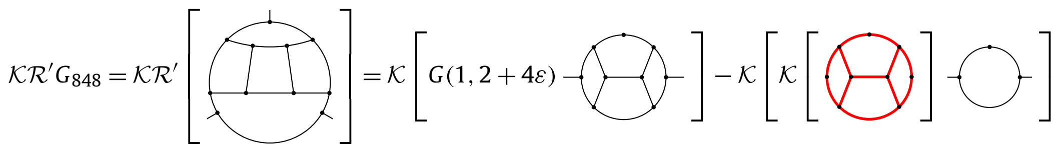
$$-6318000\zeta_6\Big)\frac{1}{n^2}-\left(\frac{143586895124}{27}+11476374450\zeta_3+2300337792\zeta_3^2-11687633640\zeta_4\right.$$
$$\left.-48410098200\zeta_5+16147512000\zeta_6+30589940304\zeta_7\right)\frac{1}{n^3}+\mathcal{O}\left(\frac{1}{n^4}\right)\Bigg)\varepsilon^4+\mathcal{O}(\varepsilon^5)\Bigg] \tag{B.2}$$

Appendix C. Diagrams calculated manually

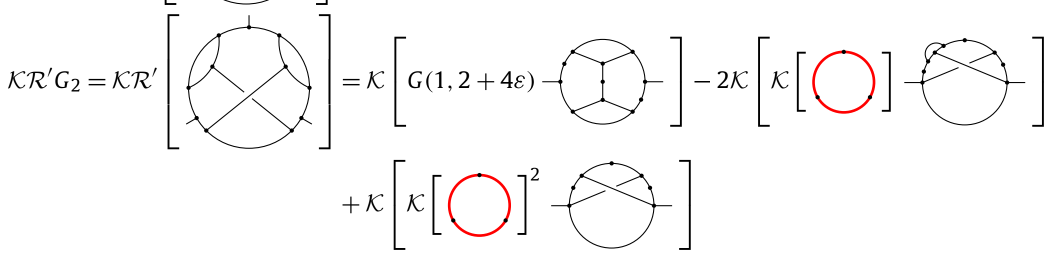
Here we present details of the $\mathcal{KR}'$ operation applied to six integrals with subdivergencies from the set $J^{(3)}$. Since $\mathcal{KR}'$ is independent of external momenta and masses, we made use of results for $\mathcal{KR}'$ for logarithmically divergent massive tadpole integrals from previous steps. Thin black lines are massless propagators, while thick red ones are massive.



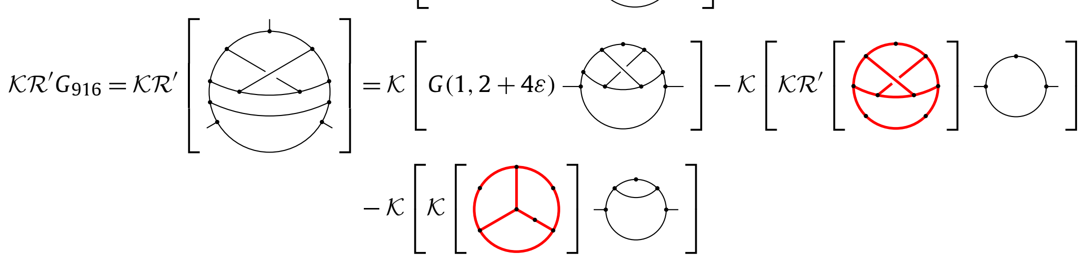
(C.1)



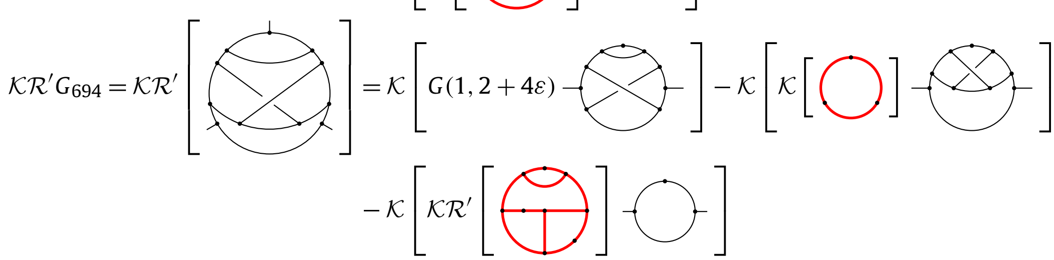
(C.2)



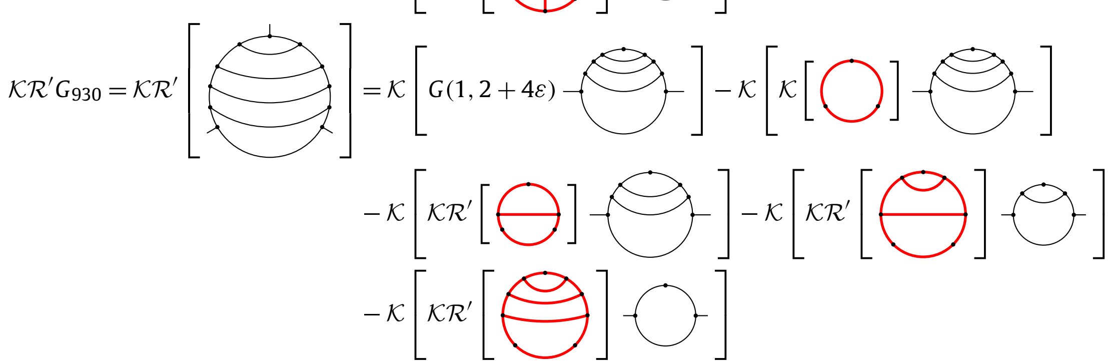
(C.3)



(C.4)

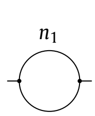


(C.5)



(C.6)

Here $G(n_1, n_2)$ means the one-loop two-point function defined as

$G(n_1,n_2)=$ 

$$=\frac{\Gamma(\frac{d}{2}-n_1)\Gamma(\frac{d}{2}-n_2)\Gamma(n_1+n_2-\frac{d}{2})}{\Gamma(n_1)\Gamma(n_2)\Gamma(d-n_1-n_2)}. \tag{C.7}$$

Appendix D. Supplementary material

Supplementary material related to this article can be found online at https://doi.org/10.1016/j.physletb.2021.136331.

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