

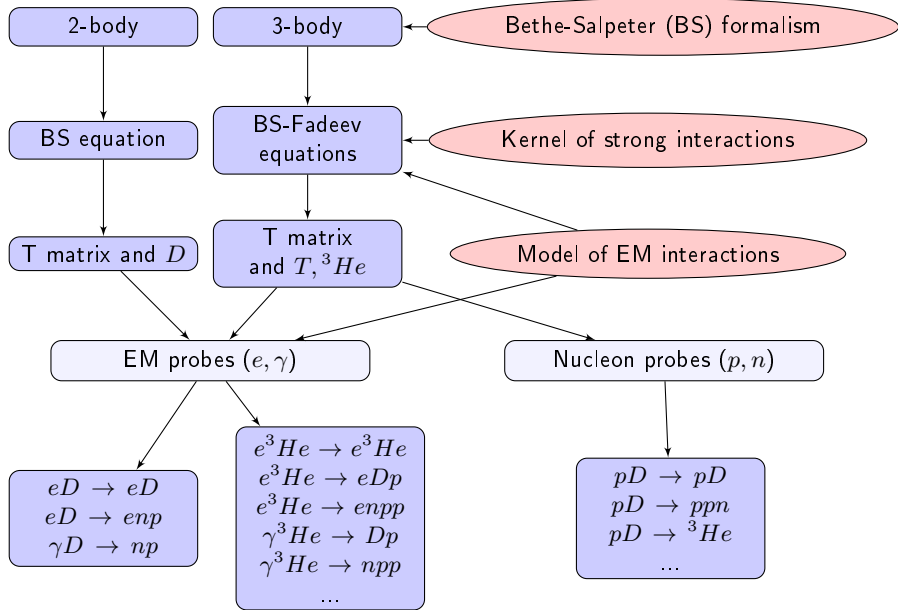
Three-nucleon EM form-factors in the Bethe-Salpeter-Faddeev approach

S. Bondarenko, V. Burov¹, S. Yurev

Joint Institute for Nuclear Research, Dubna, Russia

**International Workshop on Nuclear and Particles Physics,
Almaty, Kazakhstan, April 25-30, 2022**

¹deceased



Why a relativistic approach?

- Elastic electron-deuteron scattering experiments

“Large Momentum Transfer Measurements of the Deuteron Elastic Structure Function $A(Q^2)$ at Jefferson Laboratory”

JLab Hall A Collaboration, Phys.Rev.Lett.82:1374-1378,1999

$$Q^2 = 0.7-6.0 \text{ (GeV/c)}^2$$

Lorentz transformation factor: $\eta_{LOR} = -Q^2/4M_d^2 \sim 0.43$,

$$\sqrt{1 + \eta_{LOR}} \sim 1.19, \sqrt{\eta_{LOR}} \sim 0.65$$

- Exclusive disintegration of the deuteron experiments

JLab Hall C Deuteron Electro-Disintegration at Very High Missing Momenta (E12-10-003) proposal

https://www.jlab.org/exp_prog/proposals/10/PR12-10-003.pdf:

“We propose to measure the $D(e,e'p)n$ cross section at $Q^2 = 4.25 \text{ (GeV/c)}^2$ and $x_{bj} = 1.35$ for missing momenta ranging from $p_m = 0.5 \text{ GeV/c}$ to $p_m = 1.0 \text{ GeV/c}$ expanding the range of missing momenta explored in the Hall A experiment (E01-020)”

Lorentz transformation factor: $\eta_{LOR} = -Q^2/4s_{np} \sim 0.30$,

$$\sqrt{1 + \eta_{LOR}} \sim 1.14, \sqrt{\eta_{LOR}} \sim 0.55$$

Separable kernels of the NN interaction

The separable kernels of the nucleon-nucleon interaction are widely used in the calculations. The separable kernel as a *nonlocal* covariant interaction representing complex nature of the space-time continuum.

Separable rank-one *Ansatz* for the kernel

$$V_L(p'_0, |\mathbf{p}'|; p_0, |\mathbf{p}|; s) = \lambda^{[L]}(s) g^{[L]}(p'_0, |\mathbf{p}'|) g^{[L]}(p_0, |\mathbf{p}|)$$

Solution for the T matrix

$$T_L(p'_0, |\mathbf{p}'|; p_0, |\mathbf{p}|; s) = \tau(s) g^{[L]}(p'_0, |\mathbf{p}'|) g^{[L]}(p_0, |\mathbf{p}|)$$

with

$$[\tau(s)]^{-1} = [\lambda^{[L]}(s)]^{-1} + h(s),$$

$$h(s) = \sum_{\text{coupled } L} h_L(s) = -\frac{i}{4\pi^3} \int dk_0 \int |\mathbf{k}|^2 d\mathbf{k} \sum_L [g^{[L]}(k_0, |\mathbf{k}|)]^2 S(k_0, |\mathbf{k}|; s)$$

$g^{[L]}$ - the model function, $\lambda^{[L'L]}(s)$ - a model parameter.

What is a separable kernel?

The integral equations in the nuclear physics (Lippmann-Schwinger, Bethe-Salpeter) can be reduced to the **Fredholm (first or second) type** of equations. The separable kernel of the integral equation is **the degenerated** kernel. Fredholm integral equation of the second type:

$$\phi(x) = f(x) + \lambda \int dy K(x, y)\phi(y)$$

Degenerated kernel of the equation:

$$K(x, y) = \sum_i a_i(x)b_i(y)$$

Solution of the equation:

$$\phi(x) = f(x) + \lambda \sum c_i a_i(x)$$

Constants c_i can be found by solving the system of linear equations

$$c_i - \lambda \sum_j k_{ij} c_j = f_i$$

Matrix k_{ij} and f_i are:

$$k_{ij} = \int dy b_i(y)a_j(y), \quad f_i = \int dy f(y)b_i(y)$$

Lippmann-Schwinger equation \rightarrow Bethe-Salpeter equation

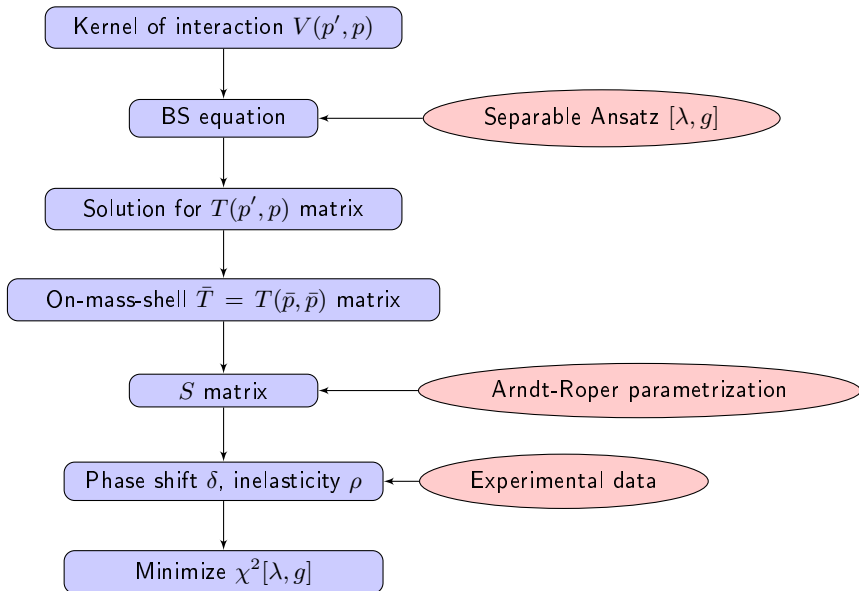
G. Rupp and J. A. Tjon "Relativistic contributions to the deuteron electromagnetic form factors" Phys. Rev. C41. 472 (1990)

$$\mathbf{p}^2 \rightarrow -p^2 = -p_0^2 + \mathbf{p}^2$$

$$g_p(p, P) = \frac{1}{-p^2 + \beta^2} \xrightarrow{\text{c.m.}} \frac{1}{-p_0^2 + \mathbf{p}^2 + \beta^2 + i\epsilon}$$

singularities: $p^0 = \pm \sqrt{\mathbf{p}^2 + \beta^2} \mp i\epsilon$

This procedure works well for reactions with 2-body bound state but failed for unbound np -state



Procedure ($J = 0 - 1$)

calculate the kernel parameters – $\lambda(s)$ -matrix and parameter of the g -functions – to minimize the function χ^2 :

$$\chi^2 =$$

$$\sum_{i=1}^n (\delta^{\text{exp}}(s_i) - \delta(s_i))^2 / (\Delta\delta^{\text{exp}}(s_i))^2 \quad \text{– for all partial-wave states}$$

$$\sum_{i=1}^n (\rho^{\text{exp}}(s_i) - \rho(s_i))^2 / (\Delta\rho^{\text{exp}}(s_i))^2 \quad \text{– for all partial-wave states}$$

$$+(a_0^{\text{exp}} - a_0)^2 / (\Delta a_0^{\text{exp}})^2 \quad \text{– for the } {}^1S_0^+ \text{ and } {}^3S_1^+ \text{ partial-wave states}$$

$$+(E_d^{\text{exp}} - E_d)^2 / (\Delta E_d^{\text{exp}})^2 \quad \text{– for the } {}^3S_1^+ \text{-} {}^3D_1^+ \text{ partial-wave states}$$

$$\{+\dots\}$$

δ - the phase shifts, a_0, r_0 - the low-energy parameters (the scattering length, the effective range), E_d - the deuteron binding energy

Covariant generalization of the *Yamaguchi*-functions

functions for $g^{[L]}(p_0, p)$:

$$g^{[S]}(p_0, |\mathbf{p}|) = \frac{1}{p_0^2 - \mathbf{p}^2 - \beta_0^2 + i0}$$

$$g^{[P]}(p_0, |\mathbf{p}|) = \frac{\sqrt{|-p_0^2 + \mathbf{p}^2|}}{(p_0^2 - \mathbf{p}^2 - \beta_1^2 + i0)^2}$$

$$g^{[D]}(p_0, |\mathbf{p}|) = \frac{C(p_0^2 - \mathbf{p}^2)}{(p_0^2 - \mathbf{p}^2 - \beta_2^2 + i0)^2}$$

The relativistic generalization of the NR Graz-II and Paris separable kernel:

- Graz-II: $^1S_0^+$ – rank 2, $^3S_1^+ - ^3D_1$ – rank 3
- Paris-1,2: $^1S_0^+$ – rank 3, $^3S_1^+ - ^3D_1$ – rank 4

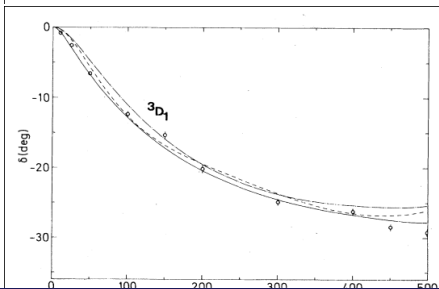
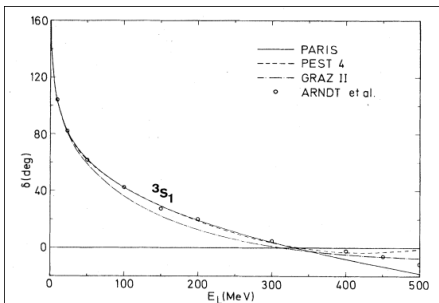
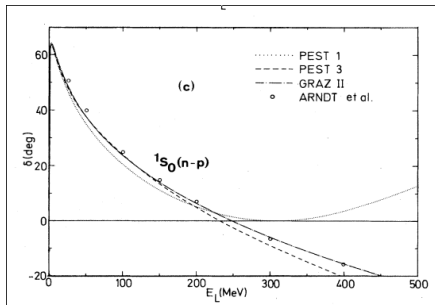
Results for $^1S_0^+$ channel

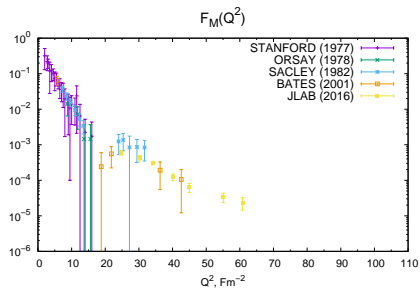
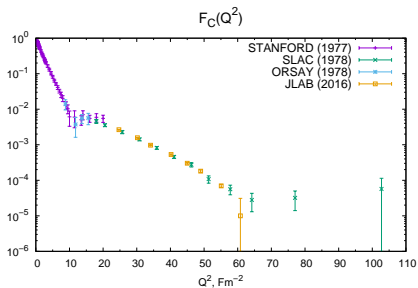
	Exp.	Graz-II	Paris-1	Paris-2
a (fm)	-23.748	-23.77	-23.72	-23.72
r_0 (fm)	2.75	2.683	2.810	2.817

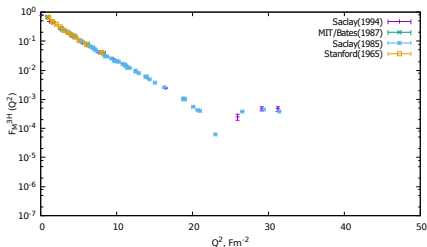
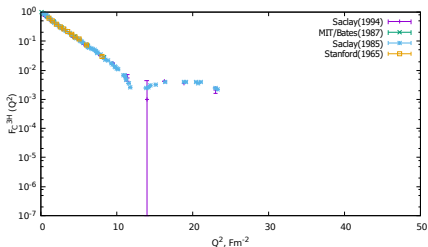
Results for $^3S_1^+ - ^3D_1$ channels

	Exp.	Graz-II	Graz-II	Graz-II	Paris-1	Paris-2
p_d (%)		4	5	6	5.77	5.77
a (fm)	5.424	5.419	5.420	5.421	5.426	5.413
r_0 (fm)	1.759	1.780	1.779	1.778	1.775	1.765
E_d (MeV)	2.2246	2.2254	2.2254	2.2254	2.2246	2.2250

Phase shifts



Experimental data for ${}^3\text{He}$ 

Experimental data for 3H 

The relativistic three-particle equation for T matrix

is considered in the **Faddeev form** with the following assumptions:

- no three-particles interaction $V_{123} = \sum_{i \neq j} V_{ij}$
- two-particles interaction is separable
- nucleon propagators are chosen in a scalar form
- the only strong interactions are considered (not EM), so ${}^3He \equiv T$

Bethe-Salpeter-Faddeev equation

$$\begin{bmatrix} T^{(1)} \\ T^{(2)} \\ T^{(3)} \end{bmatrix} = \begin{bmatrix} T_1 \\ T_2 \\ T_3 \end{bmatrix} - \begin{bmatrix} 0 & T_1 G_1 & T_1 G_1 \\ T_2 G_2 & 0 & T_2 G_2 \\ T_3 G_3 & T_3 G_3 & 0 \end{bmatrix} \begin{bmatrix} T^{(1)} \\ T^{(2)} \\ T^{(3)} \end{bmatrix},$$

where full three-particles T matrix $T = \sum_i T^{(i)}$, G_i is the free two-particles (j and n) Green function (ijn is cyclic permutation of (1,2,3)):

$$G_i(k_j, k_n) = 1/(k_j^2 - m_N^2 + i\epsilon)/(k_n^2 - m_N^2 + i\epsilon),$$

and T_i is the two-particles T matrix which can be written as following

$$T_i(k_1, k_2, k_3; k'_1, k'_2, k'_3) = (2\pi)^4 \delta^{(4)}(k_i - k'_i) T_i(k_j, k_n; k'_j, k'_n).$$

with $s_i = (k_j + k_n)^2 = (k'_j + k'_n)^2$.

Bethe-Salpeter-Faddeev equation

Orbital momentum of triton

$$\mathbf{L} = \mathbf{l} + \boldsymbol{\lambda}$$

\mathbf{l} – orbital momentum of NN -pair

$\boldsymbol{\lambda}$ – orbital momentum of 3d particle

Using [separable Ansatz](#) for two-particles T matrix one-rank

$$\Psi_{LM}(p, q; s) = \sum_{a\lambda} \Psi_{\lambda L}^{(a)}(p_0, |\mathbf{p}|, q_0, |\mathbf{q}|; s) \mathcal{Y}_{\lambda LM}^{(a)}(\hat{\mathbf{p}}, \hat{\mathbf{q}})$$

$$\mathcal{Y}_{\lambda LM}^{(a)}(\hat{\mathbf{p}}, \hat{\mathbf{q}}) = \sum_{m\mu} C_{lm\lambda\mu}^{LM} Y_{lm}(\hat{\mathbf{p}}) Y_{\lambda\mu}(\hat{\mathbf{q}}),$$

where $a \equiv {}^{2s+1}l_j$ is two-nucleon states of the NN -pair

Partial-wave three-nucleon functions

$$\Psi_{\lambda L}^{(a)}(p_0, |\mathbf{p}|, q_0, |\mathbf{q}|; s) = g^{(a)}(p_0, |\mathbf{p}|) \tau^{(a)} \left[\left(\frac{2}{3} \sqrt{s} + q_0 \right)^2 - \mathbf{q}^2 \right] \Phi_{\lambda L}^{(a)}(q_0, |\mathbf{q}|; s)$$

System of the integral equations

$$\Phi_{\lambda L}^{(a)}(q_0, |\mathbf{q}|; s) = \frac{i}{4\pi^3} \sum_{a'\lambda'} \int_{-\infty}^{\infty} dq'_0 \int_0^{\infty} \mathbf{q}'^2 d|\mathbf{q}'| Z_{\lambda\lambda'}^{(aa')} (q_0, q; q'_0, |\mathbf{q}'|; s) \frac{\tau^{(a')} \left[\left(\frac{2}{3} \sqrt{s} + q'_0 \right)^2 - \mathbf{q}'^2 \right]}{\left(\frac{1}{3} \sqrt{s} - q'_0 \right)^2 - \mathbf{q}'^2 - m^2 + i\epsilon} \Phi_{\lambda'L}^{(a')} (q'_0, |\mathbf{q}'|; s)$$

with effective kernels of equation

$$Z_{\lambda\lambda'}^{(aa')} (q_0, |\mathbf{q}|; q'_0, |\mathbf{q}'|; s) = C_{(aa')} \int d \cos \vartheta_{\mathbf{q}\mathbf{q}'} K_{\lambda\lambda'L}^{(aa')} (|\mathbf{q}|, |\mathbf{q}'|, \cos \vartheta_{\mathbf{q}\mathbf{q}'}) \frac{g^{(a)}(-q_0/2 - q'_0, |\mathbf{q}/2 + \mathbf{q}'|) g^{(a')}(q_0 + q'_0/2, |\mathbf{q} + \mathbf{q}'/2|)}{\left(\frac{1}{3} \sqrt{s} + q_0 + q'_0 \right)^2 - (\mathbf{q} + \mathbf{q}')^2 - m_N^2 + i\epsilon}$$

Singularities

Poles from one-particle propagator

$$q_{1,2}^{0'} = \frac{1}{3}\sqrt{s} \mp [E_{|\mathbf{q}'|} - i\epsilon]$$

Poles from propagator in Z-function

$$q_{3,4}^{0'} = -\frac{1}{3}\sqrt{s} - q^0 \pm [E_{|\mathbf{q}'+\mathbf{q}|} - i\epsilon]$$

Poles from Yamaguchi-functions

$$q_{5,6}^{0'} = -2q^0 \pm 2[E_{|\frac{1}{2}\mathbf{q}'+\mathbf{q}|,\beta} - i\epsilon]$$

and

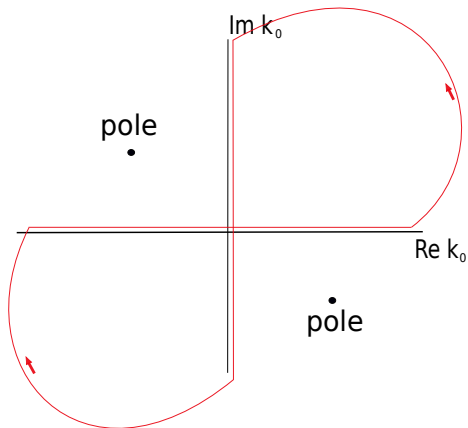
$$q_{7,8}^{0'} = -\frac{1}{2}q^0 \pm \frac{1}{2}[E_{|\mathbf{q}'+\frac{1}{2}\mathbf{q}|,\beta} - i\epsilon]$$

Cuts from two-particle propagator τ

$$q_{9,10}^{0'} = \pm\sqrt{q'^2 + 4m^2} - \frac{2}{3}\sqrt{s} \quad \text{and} \quad \pm\infty$$

Poles from two-particle propagator τ

$$q_{11,12}^{0'} = \pm\sqrt{q'^2 + 4M_d^2} - \frac{2}{3}\sqrt{s}$$



Method of solution

- Wick-rotation procedure: $q_0 \rightarrow iq_4$
- The Gaussian quadrature with $N_1 \times N_2 [q_4 \times |\mathbf{q}|]$ grid

$$q_4 = (1 + x)/(1 - x)$$

$$|\mathbf{q}| = (1 + y)/(1 - y)$$

- Iteration method to obtain the triton binding energy

$$\lim_{n \rightarrow \infty} \frac{\Phi_n(s)}{\Phi_{n-1}(s)} \Big|_{s=M_B^2} = 1$$

Triton binding energy (MeV)

Graz-II 4	8.628
Graz-II 5	8.223
Graz-II 6	7.832
Paris-1	7.545
Exp.	8.48

Electromagnetic form factors of three-nucleon systems:

$$2F_C(^3\text{He}) = (2F_C^p + F_C^n)F_1 - \frac{2}{3}(F_C^p - F_C^n)F_2,$$

$$F_C(^3\text{H}) = (2F_C^n + F_C^p)F_1 + \frac{2}{3}(F_C^p - F_C^n)F_2,$$

$$\mu(^3\text{He})F_M(^3\text{He}) = \mu_n F_M^n F_1 + \frac{2}{3}(\mu_n F_M^n + \mu_p F_M^p)F_2 + \frac{4}{3}(F_M^p - F_M^n)F_3,$$

$$\mu(^3\text{H})F_M(^3\text{H}) = \mu_p F_M^p F_1 + \frac{2}{3}(\mu_n F_M^n + \mu_p F_M^p)F_2 + \frac{4}{3}(F_M^n - F_M^p)F_3,$$

Electric and magnetic form factors of the proton and neutron $F_{C,M}^{p,n}$.

Impulse approximation:

$$F_i(\hat{Q}) = \int d^4\hat{p} \int d^4\hat{q} G'_1(\hat{k}'_1) G_1(\hat{k}_1) G_2(\hat{k}_2) G_3(\hat{k}_3) f_i(\hat{p}, \hat{q}, \hat{q}'; \hat{P}, \hat{P}')$$

Nucleon propagators:

$$G_i(\hat{k}_1) = \left[\hat{k}_i^2 - m_N^2 + i\epsilon \right]^{-1},$$

$$G'_1(q'_0, q') = \left[\left(\frac{1}{3}\sqrt{s} - q'_0 \right)^2 - \mathbf{q}'^2 - m_N^2 + i\epsilon \right]^{-1},$$

Three-nucleon vertex functions:

$$f_1 = \sum_{i=1}^3 \Psi_i^*(\hat{p}, \hat{q}; \hat{P}) \Psi_i(\hat{p}, \hat{q}'; \hat{P}')$$

$$f_2 = -3\Psi_1^*(\hat{p}, \hat{q}; \hat{P}) \Psi_2(\hat{p}, \hat{q}'; \hat{P}')$$

$$f_3 = \Psi_3^*(\hat{p}, \hat{q}; \hat{P}) \Psi_3(\hat{p}, \hat{q}'; \hat{P}')$$

Functions Ψ_i are the definite combinations of the partial state functions.

The Breit reference system

$$\hat{Q} = (0, \mathbf{Q}), \quad \hat{P} = (E_B, -\frac{\mathbf{Q}}{2}), \quad \hat{P}' = (E_B, \frac{\mathbf{Q}}{2}), \quad (1)$$

with $E_B = \sqrt{\mathbf{Q}^2/4 + s}$, $s = M_{3N}^2$.

$$\begin{aligned} \hat{P} &= L\hat{P}_{c.m.}, & \hat{p} &= L\hat{p}_{c.m.}, & \hat{q} &= L\hat{q}_{c.m.} \\ \hat{P}' &= L^{-1}\hat{P}'_{c.m.}, & \hat{p}' &= L^{-1}\hat{p}'_{c.m.}, & \hat{q}' &= L^{-1}\hat{q}'_{c.m.} \end{aligned}$$

The explicit form of the transformation L can be obtained by using (1). Let us assume the boost of the system to be along the Z axis:

$$L = \begin{pmatrix} \sqrt{1+\eta} & 0 & 0 & -\sqrt{\eta} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\sqrt{\eta} & 0 & 0 & \sqrt{1+\eta} \end{pmatrix}. \quad (2)$$

Relation of the arguments of initial and final 3N functions:

$$\begin{aligned}q'_0 &= (1 + 2\eta) q_0 - 2\sqrt{\eta}\sqrt{1 + \eta} q_z + \frac{2}{3}\sqrt{\eta} Q, \\q'_x &= q_x \quad q'_y = q_y \\q'_z &= (1 + 2\eta) q_z - 2\sqrt{\eta}\sqrt{1 + \eta} q_0 - \frac{2}{3}\sqrt{1 + \eta} Q,\end{aligned}\tag{3}$$

here $q_z = q \cos \theta_{qQ}$ is the projection of momentum \mathbf{q} onto the Z axis

Static approximation (SA):

$$q'_0 = q_0, \quad \mathbf{q}' = \mathbf{q} - \frac{2}{3}\mathbf{Q}$$

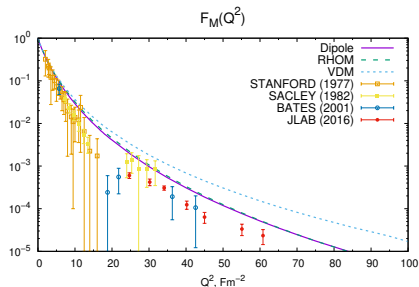
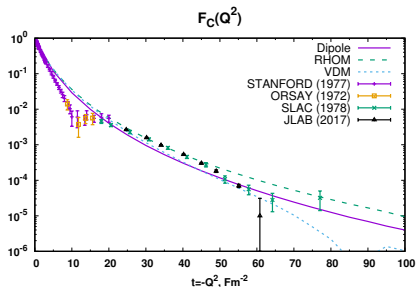
Propagator and final function:

$$G'_1(q'_0, q') \rightarrow \left[\left(\frac{1}{3}\sqrt{s} - q_0 \right)^2 - \mathbf{q}^2 - \frac{2}{3}\mathbf{q} \cdot \mathbf{Q} - \frac{4}{9}\mathbf{Q}^2 - m_N^2 + i\epsilon \right]^{-1}$$

$$\Psi_i(p_0, p, q'_0, q') \rightarrow \Psi_i(p_0, p, q_0, |\mathbf{q} - \frac{2}{3}\mathbf{Q}|)$$

with $\mathbf{q} \cdot \mathbf{Q} = qQ \cos \theta_{qQ}$.

The poles of G'_1 on q_0 do not cross the imaginary q_0 axis and always stay in the second and fourth quadrants. In this case, the Wick rotation procedure $q_0 \rightarrow iq_4$ can be applied.

One-rank relativistic kernel, static approximation, ${}^3\text{He}$ 

Beyond the SA:1. Exact propagator

$$G'_1 = \left[q_0^2 + \frac{2}{3}\sqrt{s}(1 + 6\eta)q_0 + 4\sqrt{1 + \eta}\sqrt{s}\sqrt{\eta}q_z - \frac{8}{3}\eta s + \frac{1}{9}s - \mathbf{q}^2 - m_N^2 + i\epsilon \right]$$

$$\Psi_i(p_0, p, q'_0, q') \rightarrow \Psi_i(p_0, p, q_0, |\mathbf{q} - \frac{2}{3}\mathbf{Q}|).$$

For any $t = -\hat{Q}^2 > -\hat{Q}_{min}^2 = 2/3\sqrt{s}(3m_N - \sqrt{s})$ the pole of G'_1 on q_0 crosses the imaginary q_0 axis and appears in the third quadrant.

Beyond the SA:

2. Additional term from residue inside the contour of integration

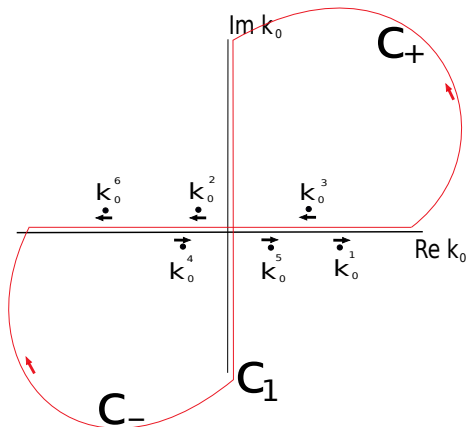
Using the Cauchy theorem, one can transform the integrals over p_0, q_0 as follows:

$$\begin{aligned} & \int_{-\infty}^{\infty} dp_0 \int_{-\infty}^{\infty} dq_0 \int_0^{\infty} dq \int_{-1}^1 dy \dots f(p_0, q_0, p, q, x, y) = \quad (4) \\ & - \int_{-\infty}^{\infty} dp_4 \int_{-\infty}^{\infty} dq_4 \int_0^{\infty} dq \int_{-1}^1 dy \dots f(ip_4, iq_4, p, q, x, y) \\ & + 2\pi \operatorname{Res}_{q_0=q_0^{(2)}} \int_{-\infty}^{\infty} dp_4 \int_{q_{min}}^{q_{max}} dq \int_{y_{min}}^1 dy \dots f(ip_4, q_0^{(2)}, p, q, x, y), \end{aligned}$$

where (...) means the two-fold integral $\int_0^{\infty} dp \int_{-1}^1 dx$ and

$$q_0^{(1,2)} = \frac{\sqrt{s}}{3}(1 + 6\eta) \pm \sqrt{4\eta(1 + \eta)s - 4\sqrt{s}\sqrt{\eta}\sqrt{1 + \eta}qy + \mathbf{q}^2 + m_N^2} \quad (5)$$

are the simple poles of the propagator G'_1 .



Beyond the SA:**3. Final function arguments transformation**

Remembering that the BSF solutions are known for real values of q_4 only, the following assumption was made:

$$\Psi(p_0, p, q'_0, q') \rightarrow g(p_0, p) \tau \left[\left(\frac{2}{3} \sqrt{s} + q_0^{(2)} \right)^2 - \bar{\mathbf{q}}'^2 \right] \Phi(0, \bar{\mathbf{q}}'),$$

where value $\bar{\mathbf{q}}'$ is obtained using (3) with $q_0 = q_0^{(2)}$.

The expansion of the function $\Phi(q'_4, q')$ up to the first order of the parameter η :

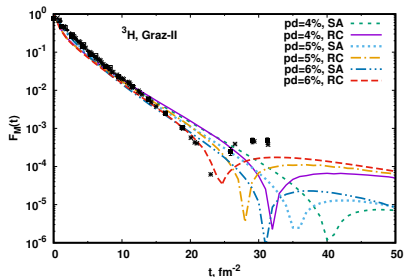
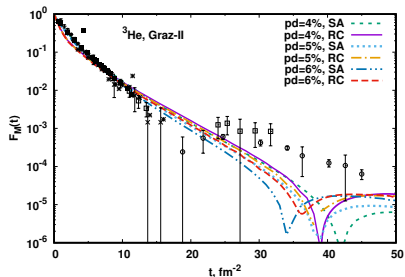
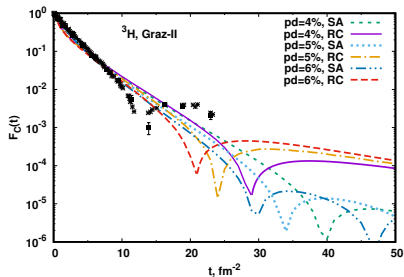
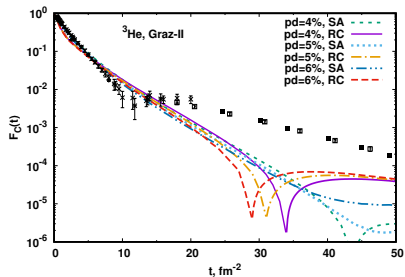
$$\begin{aligned} \Phi(iq'_4, q') &= \Phi(iq_4, |\mathbf{q} - \frac{2}{3}\mathbf{Q}|) + \left[C_{q_4} \frac{\partial}{\partial q_4} \Phi_j(iq_4, q) \right]_{q=|\mathbf{q}-\frac{2}{3}\mathbf{Q}|} \\ &\quad + \left[C_q \frac{\partial}{\partial q} \Phi_j(iq_4, q) \right]_{q=|\mathbf{q}-\frac{2}{3}\mathbf{Q}|}, \end{aligned}$$

where

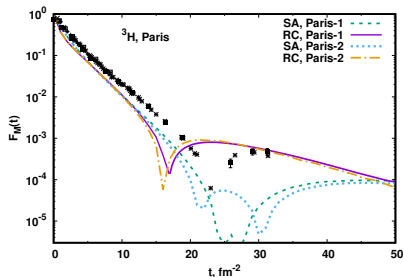
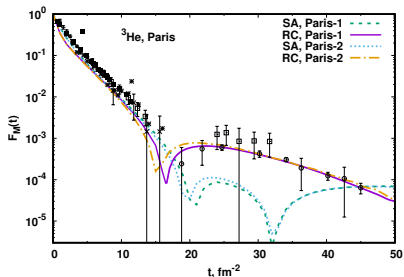
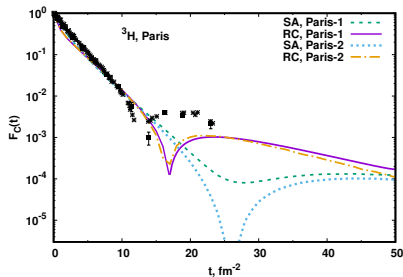
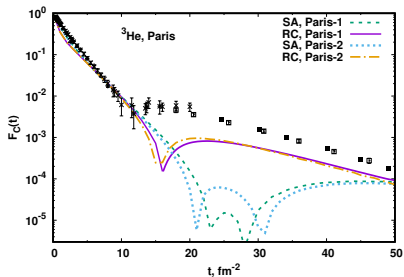
$$C_{q_4} = -i \left(2i\eta q_4 - 2\sqrt{\eta} \sqrt{1 + \eta} q \cos \theta_{qQ} + \frac{2}{3} \sqrt{\eta} Q \right),$$

$$C_q = \left(2\eta q \cos \theta_{qQ} - 2i\sqrt{\eta} \sqrt{1 + \eta} q_4 - \frac{2}{3} (\sqrt{1 + \eta} - 1) Q \right) \cos \theta_{qQ}.$$

Graz-II relativistic kernel



Paris relativistic kernel



Summary

- the relativistic three-nucleon vertex functions were found solving the BSF system of equations
- the charge and magnetic EM form factors of the 3N systems were calculated
- the static approximation and relativistic corrections were investigated
- the relativistic corrections were found to be significant in describing the experimental data