Superfluidity in multicomponent fermions via the functional renormalization group

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Outline

1. Investigated systems;
2. Machinery of the functional renormalization group;
3. The outcomes achieved.

Multicomponent cold fermions

$$\underbrace{\ell_T \sim T^{-1/2}}_{\text{de Broglie wavelength}} \gtrsim \underbrace{\ell \sim (\text{density})^{-1/3}}_{\text{interparticle distance}}$$

Atom Species

- ^{171}Yb $I = 1/2$ $SU(2)$
- ^{173}Yb $I = 5/2$ $SU(6)$ (Rey, Rep. Prog. Phys. 2014)
- ^{87}Sr $I = 9/2$ $SU(10)$ (Ye, Nature Phys. 2020)
- $^{171}Yb +^{173} Yb$ $SU(2) \times SU(6)$ (Taie, PRL 2010)

$SU(n = 2I + 1)$ **interaction** $\Longleftrightarrow$ **spin-independent scattering length** a_s

Effective description

The system scales $\quad a_{Borh} \sim 0.5\text{Å}$

– the Van der Waals lenght	ℓ_{VdW}	$(10 \div 100) \times a_{Borh}$
– s-wave scattering length	a_s	$(10 \div 200) \times a_{Borh}$
– interparticle distance	ℓ	$(800 \div 3000) \times a_{Borh}$
– thermal de Broglie wavelength	ℓ_T	$(10\,000 \div 40\,000) \times a_{Borh}$
– size of the system	L	$100\,000 \times a_{Borh}$

$$\ell_{VdW} \lesssim a_s \lesssim \ell \ll \ell_T \ll L$$

Fermionic model

Ultra-Violet scales:

- ✓ $\Lambda_{UF} \sim 1/\ell_{VdW}$ - reciprocal of the Van der Waals lenght (atomic systems)
- ✓ $\Lambda_{UF} \sim \omega_D$ - the Debye frequency (electorons in solids)

The starting point – effective fermionic action

The pairwise interaction for scales $\ell > \ell_{VdW}$ reads $U(x) = \lambda\,\delta(x)$, and the action cast in the form

$$\mathcal{S}[\psi] = \int_0^{1/T} \mathrm{d}t \int \mathrm{d}x \left(\psi_i^* \left\{\partial_t - \nabla^2 - \mu\right\} \psi_i + \frac{\lambda}{2}(\psi_i^* \psi_i)^2\right), \quad i = 1, \ldots, n,$$

with the strength $\lambda = 4\pi a_s/m$ and the chemical potential $\mu \sim T_{Fermi}$.

General aims

- ✓ Investigation of the model in the Cooper channel, where

$$\phi_{ij} \sim \psi_i \, \psi_j \neq 0$$

- ✓ Obtaining possible phase transitions
- ✓ Estimation of thermodynamic quantities

$$\underbrace{\mathcal{S}[\psi] \,\text{on scale} \sim 1/a_s}_{\text{“micro” } \psi} \quad \Longrightarrow \quad \underbrace{\underbrace{\mathcal{S}[\phi] \,\text{on scale} \sim 1/\ell_T}_{\text{“meso” } \phi \sim \psi\,\psi} \quad \Longrightarrow \quad \underbrace{\Gamma[\varphi] \,\text{on scale} \sim 1/L}_{\text{“macro” } \varphi \sim \langle \phi \rangle}}_{\text{Bosonic field models}}$$

Knowledge of $\Gamma[\varphi]$ – “solution” of the model

Bosonic model

Integrating fermions out gives rise to the pure bosonic theory (put forward by Gor'kov for BCS, 1959)

$$\mathcal{S}[\phi] = \int \mathrm{d}x \Big[\mathrm{tr}(\partial\phi^\dagger\,\partial\phi) + m_0^2\,\mathrm{tr}(\phi^\dagger\phi) + g_{01}(\mathrm{tr}(\phi^\dagger\phi))^2 + g_{02}\,\mathrm{tr}(\phi^\dagger\phi\phi^\dagger\phi) \Big]$$

- ✓ $\mathcal{S}[\phi]$ – the G.-L.-like "Hamiltonian" for skew-symmetric field ϕ size of $n \times n$
- ✓ The bare parameters m_0^2, g_{01}, g_{02} – functions of μ, a_s, T, and Λ_{UF}.

Business as usual

- New nontrivial phase for $m_0^2 < 0$
- Continuous phase transition at the temperature T_0: $m_0^2|_{T=T_0} = 0$

$$T_0 \approx 0.61\,T_{\mathrm{Fermi}}\,\exp\left(\frac{\pi}{2a_s p_{\mathrm{Fermi}}}\right)$$

Let's treat fluctuations!

1PI-functional

- ✓ The generating functional of connected Green's functions

$$W[J] = \ln \int \mathcal{D}\phi \exp\left\{-S[\phi] + J\phi\right\}.$$

- ✓ The Legendre transformation – 1PI Green's functions

$$\Gamma[\varphi] = J_\varphi \varphi - W[J_\varphi],$$

where J_φ meets the equation

$$\left.\frac{\delta W[J]}{\delta J}\right|_{J=J_\varphi} = \varphi.$$

Mode decoupling

✓ The generating functional of connected Green's functions

$$W_k[J] = \ln \int \mathcal{D}\phi \exp\left\{-S[\phi] - \Delta S_k[\phi] + J\phi\right\},$$

with the quadratic additive

$$\Delta S_k[\phi] = \frac{1}{2}\phi\, R_k\, \phi.$$

✓ The Legendre transformation – 1PI Gren's functions

$$\Gamma_k[\varphi] = J_{k,\varphi}\varphi - W[J_{k,\varphi}] - \Delta S_k[\varphi],$$

where $J_{k,\varphi}$ meets the equation

$$\left.\frac{\delta W_k[J]}{\delta J}\right|_{J=J_{k,\varphi}} = \varphi.$$

The cut-off kernel R_k

Properties of R_k:

✓ $R_k(\mathbf{p}) \to \infty$ as $k \to \Lambda$ (or ∞): fluctuations are frozen, thus $\Gamma_{k\to\Lambda}[\varphi] \to S[\varphi]$ – the mean-field free energy.

✓ $R_k(\mathbf{p}) \to 0$ as $k \to 0$: all fluctuations are integrated out, thus $\Gamma_{k\to 0}[\varphi] \to \Gamma[\varphi]$ – the full free energy.

Widely used kernels:

- the exponential shape

$$R_k(\mathbf{p}) = \frac{p^2}{e^{p^2/k^2} - 1}$$

- the theta-regulator (Litim, 2021)

$$R_k(\mathbf{p}) = (k^2 - p^2)\Theta(k^2 - p^2)$$

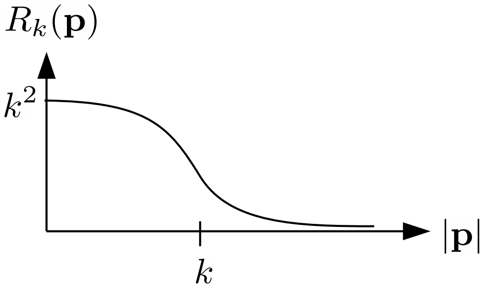


Figure 1: A typical shape of the cut-off function[1].

[1] N. Dupuis et al. "The nonperturbative functional renormalization group and its applications". In: *Physics Reports* 910 (2021), pp. 1–114.

The Wetterich equation

✓ Flow in functional space (Wetterich, 1990's)

$$\partial_k \Gamma_k[\varphi] = \frac{1}{2} \mathrm{Tr} \left\{ (\Gamma_k^{(2)}[\varphi] + R_k)^{-1} \partial_k R_k \right\},$$

where $\Gamma_k^{(2)}[\varphi]$ is given by the second functional derivative of $\Gamma_k[\varphi]$.

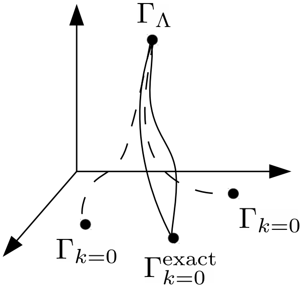


Figure 2: Schematic flows for two different cut-off shapes R_k.

Widely used truncations:

- derivative expansion

$$\Gamma_k[\varphi] = Z_k(\varphi)(\partial\varphi)^2 + U_k(\varphi) + \text{higher order deriv.}$$

- vertex expansion

$$\Gamma_k[\varphi] = \sum_n \frac{1}{n!} \int_x \Gamma_k^{(n)}(x_1, \ldots, x_n)\varphi(x_1)\ldots\varphi(x_n)$$

Invariant expansion

- ✓ Geometry of the vacuum expectation value $\langle\Phi\rangle = \varphi\, t_1$

$$t_1 = \begin{pmatrix} 0 & \mathrm{I}_{n/2} \\ -\mathrm{I}_{n/2} & 0 \end{pmatrix}$$

- ✓ The group invariants we use here are defined as

$$\rho_1 = \mathrm{tr}(\Phi^\dagger\Phi), \quad \rho_a = \mathrm{tr}\left(\Phi^\dagger\Phi - \frac{\rho_1}{n}\right)^a, \quad a \in \mathbb{N}$$

- ✓ Invariant expansion

$$U_k(\Phi, \Phi^\dagger) \approx \mathrm{V}_k(\rho_1) + \mathrm{W}_k(\rho_1)\,\rho_2 + \text{etc.}$$

Flow equations

The flow system

$$\partial_k \, \mathrm{V}_k(\rho_1) = C_d \, k^{d+1} \left\{ \frac{1/n^2}{k^2 + \mathrm{V}_k' + 2\rho_1 \, \mathrm{V}_k''} + \frac{(n+1)(n-2)/(2\,n^2)}{k^2 + \mathrm{V}_k' + 4\rho_1 \, \mathrm{W}_k} + \frac{(n-1)/(2\,n)}{k^2 + \mathrm{V}_k'} \right\},$$

$$\partial_k \, \mathrm{W}_k(\rho_1) = C_d \, k^{d+1} \left\{ \frac{(1-2/n)\,\mathrm{W}_k^2}{(k^2 + \mathrm{V}_k')^3} + \frac{9(n+2)(n-4)\,\mathrm{W}_k^2}{n^2(k^2 + \mathrm{V}_k' + 4\rho_1 \, \mathrm{W}_k)^3} - \frac{(\mathrm{W}_k + 2\,(1-1/n)\rho_1 \, \mathrm{W}_k')}{4\rho_1(k^2 + \mathrm{V}_k')^2} \right.$$

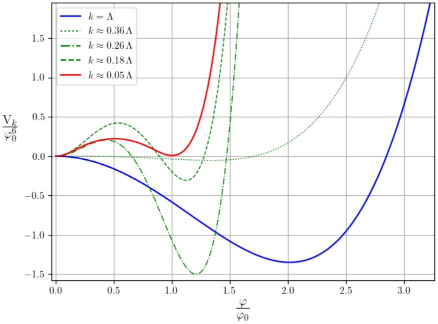
$$- \frac{1/n^2}{(k^2 + \mathrm{V}_k' + 2\rho_1 \, \mathrm{V}_k'')^2} \left(2\mathrm{W}_k'' \, \rho_1 + 5\,\mathrm{W}_k' + \frac{\mathrm{W}_k}{\rho_1} - \frac{\mathrm{V}_k''}{2\rho_1} + \frac{\left(\mathrm{V}_k'' + 4\,\mathrm{W}_k' \, \rho_1 + 4\,\mathrm{W}_k\right)^2}{2\rho_1 \left(\mathrm{V}_k'' - 2\,\mathrm{W}_k\right)} \right)$$

$$\left. + \frac{1/n^2}{(k^2 + \mathrm{V}_k' + 4\rho_1 \, \mathrm{W}_k)^2} \left(\frac{n^2+4}{4\rho_1} \mathrm{W}_k - \frac{n^2+6}{2} \mathrm{W}_k' - \frac{\mathrm{V}_k''}{2\rho_1} + \frac{\left(\mathrm{V}_k'' + 4\,\mathrm{W}_k' \, \rho_1 + 4\,\mathrm{W}_k\right)^2}{2\rho_1 \left(\mathrm{V}_k'' - 2\,\mathrm{W}_k\right)} \right) \right\}$$

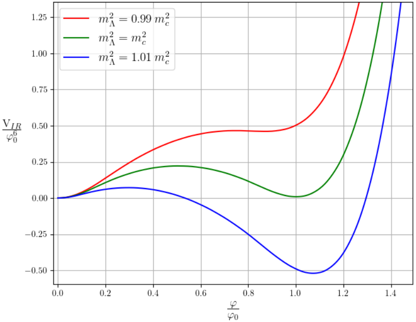
was solved with the UV initial conditions

$$\mathrm{V}_\Lambda(\rho_1) = -m_\Lambda^2 \, \rho_1 + g_{1\Lambda} \, \rho_1^2 \qquad \& \qquad \mathrm{W}_\Lambda(\rho_1) = g_{2\Lambda}.$$

Numerical solution



Flowing down potential at the phase transition point $m_\Lambda^2 = m_c^2$.



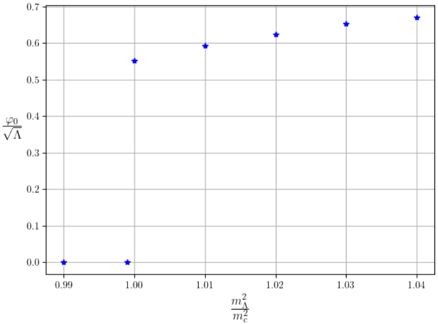
The full IR potential at $k = 0.05\Lambda$ for different m_Λ^2 values.

The flow was stored on a grid using pseudo-spectral methods and then run within Runge-Kutta schemes.

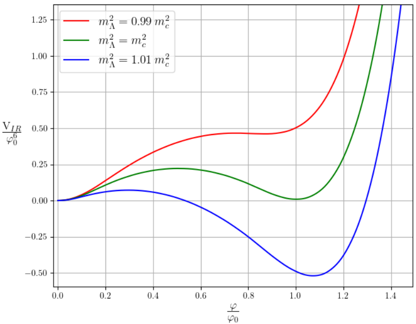
The first order phase transition

The potential $V_{IR}(\varphi, m_\Lambda^2)$ acquires a non-trivial stable minimum φ_0 when

$$\partial_\varphi V_{IR}(\varphi, m_c^2)|_{\varphi=\varphi_0} = 0 \quad \& \quad V_{IR}(0, m_c^2) = V_{IR}(\varphi_0, m_c^2).$$



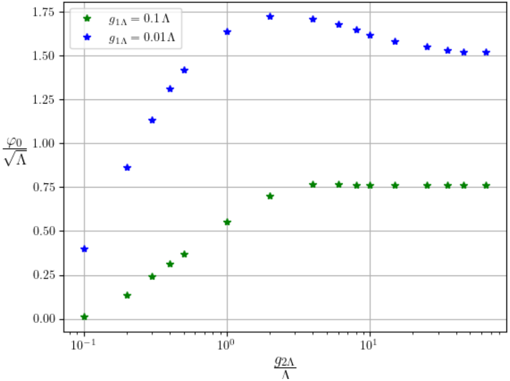
Order parameter as a function of "temperature".



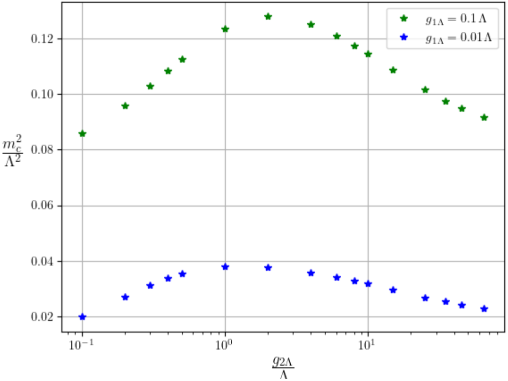
The full IR potential at $k = 0.05\Lambda$ for different m_Λ^2 values.

$$m_\Lambda^2(T) \approx \underbrace{m_\Lambda^2(T_c)}_{m_c^2} + \alpha \times (T_c - T)$$

Weak and strong phase transitions



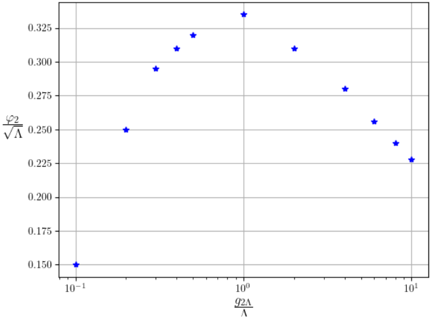
Order parameter as a function of the coupling constant $g_{2\Lambda}$



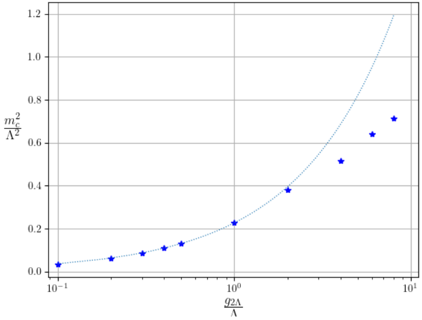
Phase transition “temperature” as a function of the coupling constant $g_{2\Lambda}$

Weak and strong phase transitions: a special case

In the original potential (after fermions elimination) we get $g_{1\Lambda} = g_{2\Lambda}/N$.



Order parameter as a function of the coupling constant $g_{2\Lambda}$



Phase transition “temperature” as a function of the coupling constant $g_{2\Lambda}$

$$\frac{m_c^2}{\Lambda^2} \sim \left(\frac{g_{2\Lambda}}{\Lambda}\right)^{0.8}$$

Nonperturbative findings

1. One should expect *the first order* phase transition to superfluity in $SU(N)$ fermions

2. The obtained picture preserves at the large N limit $N \to \infty$

3. No universal scaling behaviour with the standard critical exponents ν, η, but the *pseudo-scaling* with respect to a coupling constant

4. Quasi-universal behaviour in both weak and strong phase transitions

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2. The obtained picture preserves at the large N limit $N \to \infty$
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The end!