Effects of moving environment on self-organized critical behavior in anisotropic systems

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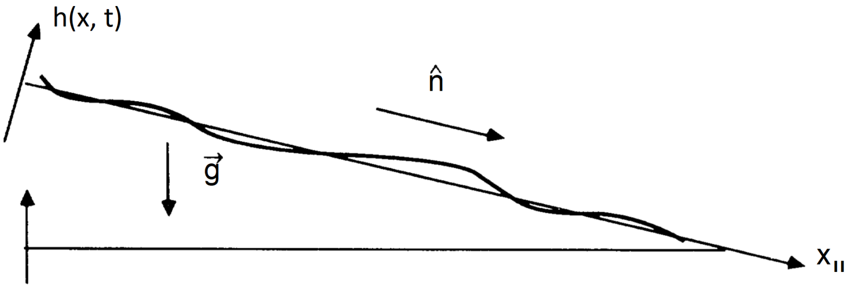

Vector $\mathbf{n}$ picks up a preferred direction of transport, so that coordinates can be split as

$$\mathbf{x} = \mathbf{x}_\perp + x_\parallel \mathbf{n}; \quad (\mathbf{x}_\perp \cdot \mathbf{n}) = 0$$

Symmetry of the system

$$x_\parallel \to -x_\parallel; \quad h \to -h$$

We are looking for infrared scaling behaviour

$$\langle h(\mathbf{x}, t) h(\mathbf{0}, 0) \rangle \simeq x_\perp^{-2\Delta_h} \mathcal{F}(t/r_\perp^{\Delta_\omega}, r_\parallel / r_\perp^{\Delta_\parallel})$$

The matter is conserved by the internal dynamics

$$\partial_t h + \partial \cdot \mathbf{j} = f(\mathbf{x}, t)$$

We aim to describe long-range physics, assuming fluctuations of the height to be relatively small

$$\mathbf{j} = -\nu_\perp \partial_\perp \, h - \nu_\parallel \, \mathbf{n} \, \partial_\parallel \, h + \frac{\lambda}{2} \, \mathbf{n} \, h^2$$

Equation of motion [1]

$$\partial_t h = \nu_\perp \, \partial_\perp^2 h + \nu_\parallel \, \partial_\parallel^2 h - \frac{\lambda}{2} \partial_\parallel h^2 + f$$

Balance between income and drain of particles implies

$$\langle f(\mathbf{x}, t) \rangle = 0; \quad \langle f(\mathbf{x}, t) f(\mathbf{x}, t) \rangle = C \, \delta(t - t') \, \delta^{(d)}(\mathbf{x} - \mathbf{x}')$$

[1] Hwa, T., Kardar, M. Phys. Rev. Lett., 62(16), 1813 (1989)

Any stochastic problem of the form

$$\partial_t \Phi(\mathbf{x},t) = U(\Phi,\mathbf{x}) + f(\mathbf{x},t); \quad \langle f(\mathbf{x},t) f(\mathbf{x}',t') \rangle = D(\mathbf{x}-\mathbf{x}', t-t')$$

can be recast into field theoretic problem with the action

$$S(\Phi,\Phi') = \frac{1}{2}\Phi' D \Phi' + \Phi'(-\partial_t \Phi + U(\Phi))$$

Hwa-Kardar Stochastic equation is equivalent to the fiels-theoretic model

$$S(h,h') = \frac{C}{2} h' h' + h' \Big(-\partial_t h + \nu_\parallel \partial_\parallel^2 h + \nu_\perp \partial_\perp^2 h - \frac{\lambda}{2} \partial_\parallel h^2 \Big)$$

Additional symmetry

$$\mathbf{x} \to \mathbf{x} + ut\mathbf{n}; \quad h'(\mathbf{x},t) \to h'(\mathbf{x}+ut\mathbf{n},t); \quad h(\mathbf{x},t) \to h(\mathbf{x}+ut\mathbf{n},t) - u$$

Parameter λ can be absorbed by rescaling of fields and parameters

$$\nu_\parallel = \nu_{\parallel R} Z_{\nu_\parallel}$$

Three independent canonical dimentions

$$[F] \sim [T]^{-d_F^{\omega}} \left[L_{\|}\right]^{-d_F^{\|}} [L_{\perp}]^{-d_F^{\perp}}$$

for each there is equation of canonical scale invariance

$$\left(\sum_i d_i \mathcal{D}_i - d_G\right) G = 0; \qquad \mathcal{D}_x = x\partial_x$$

The only dimensionless combination plays the role of expansion parameter

$$g = C\mu^{-\varepsilon}\nu_{\perp}^{-3/2}\nu_{\|R}^{-3/2}; \qquad \varepsilon = 4 - d$$

RG functions

$$\gamma_F = \widetilde{\mathcal{D}}_{\mu} \ln Z_F; \quad \beta_g = \widetilde{\mathcal{D}}_{\mu} g = -g\left(\varepsilon + \gamma_g\right)$$

All anomalous dimensions at the fixed point are known *exactly*

$$\gamma_{\nu_{\|}}^{*} = 2(4 - d)/3$$

Renormalization Group equation

$$\Big(\mathcal{D}_\mu + \beta(g)\partial_g - \gamma_{\nu_\|}\mathcal{D}_{\nu_\|} - \gamma_G\Big)G_R = 0$$

At fixed point

$$\Big(\mathcal{D}_\mu - \gamma^*_{\nu_\|}\mathcal{D}_{\nu_\|} - \gamma^*_G\Big)G_R = 0$$

Combining with canonical equations to exclude IR irrelevant parameters μ, $\nu_\perp$, $\nu_\|$

$$\Big(\mathcal{D}_{k_\perp} + \Delta_\|\mathcal{D}_{k_\|} + \Delta_\omega\mathcal{D}_\omega - \Delta_G\Big)G_r = 0$$

where

$$\Delta_\| = 1 + \gamma^*_{\nu_\|}/2; \quad \Delta_\omega = 2;$$

$$\Delta_G = d_g^\perp + d_G^\|\Delta_\| + d_g^\omega\Delta_\omega + \gamma^*_G$$

Exact values of scaling exponents

$$\Delta_h = (d-1)/3; \quad \Delta_{h'} = (d+5)/3; \quad \Delta_\omega = 2; \quad \Delta_\| = (7-d)/3$$

Advection introduced by the minimum coupling with the velocity field

$$\partial_t \to \nabla_t = \partial_t + (\mathbf{v} \cdot \partial)$$

Imcompressibility

$$\partial_i v_i = 0$$

Velocity statistics

$$\langle v_i(\mathbf{x}, t)\rangle = 0; \quad \langle v_i(\mathbf{x}, t) v_j(\mathbf{x}', t')\rangle = \delta(t - t')\, D_{ij}(\mathbf{x} - \mathbf{x}');$$

$$D_{ij}(\mathbf{x} - \mathbf{x}') = B \int_{k>m} \frac{d\mathbf{k}}{(2\pi)^d} \frac{1}{k^{d+\xi}} P^{\perp}_{ij}(\mathbf{k}) \exp(i\mathbf{k}(\mathbf{x} - \mathbf{x}'))$$

The action of corresponding field model [2]

$$S(h, h', \mathbf{v}, \mathbf{v}') = \frac{C}{2} h' h' + h' \{-\nabla_t h + \nu_{\|} \partial_{\|}^2 h + \nu_{\perp} \partial_{\perp}^2 h - \frac{1}{2} \partial_{\|} h^2\} + S_{\mathbf{v}};$$

$$S_{\mathbf{v}} = -\frac{1}{2} \int dt\ d\mathbf{x}\ d\mathbf{x}'\ v_i(\mathbf{x}, t) D^{-1}_{ij}(\mathbf{x} - \mathbf{x}') v_j(\mathbf{x}', t)$$

[2] N. V. Antonov, N. M. Gulitskiy, P. I. Kakin and G. E. Kochnev, Universe 6, 145 (2020)

Due to the transversity of the velocity field we can not introduce separate dimension along the preferred direction

$$[F] \sim [T]^{-d_F^\omega} [L_k]^{-d_F^k}$$

Augmented Galilean symmetry

$$\mathbf{x} \to \mathbf{x} + ut\mathbf{n}; \quad h'(\mathbf{x}, t) \to h'(\mathbf{x} + ut\mathbf{n}, t); \quad h(\mathbf{x}, t) \to h(\mathbf{x} + ut\mathbf{n}, t) - u;$$

$$\mathbf{v}(\mathbf{x}, t) \to \mathbf{v}(\mathbf{x} + ut\mathbf{n}, t)$$

Two renormalization constants

$$\nu_{\parallel} = \nu_{\parallel R} Z_{\nu_{\parallel}}; \quad \nu_{\perp} = \nu_{\parallel R} Z_{\nu_{\perp}}$$

Couplings

$$g = C\mu^{-\varepsilon} \nu_{\perp R}^{-3/2} \nu_{\parallel R}^{-3/2}; \quad u = B\mu^{-\xi} \nu_{\perp R}^{-1}; \quad x = \nu_{\parallel R} \nu_{\perp R}^{-1}$$

One-loop calculation gives

$$\beta_g = g\left(-\varepsilon + \frac{9}{32} g + \frac{9}{46}\frac{u}{x} + \frac{9}{16} u \right);$$

$$\beta_u = u\left(-\xi + \frac{3}{8} u \right);$$

$$\beta_x = x\left(-\frac{3}{16} g - \frac{3}{8}\frac{u}{x} + \frac{3}{8} u \right).$$

Four fixed points can be IR attractive

- $g^* = 0;$ $u^* = 0;$ $x^* = \forall;$ attractive for $d > 4;\ \xi < 0$
- $g^* = 0;$ $u^* = 8\,\xi/3;$ $x^* = 1;$ attractive for $\xi > 0;\ \xi > (4-d)/3$
- $g^* = 32\,\varepsilon/9;$ $u^* = 0;$ $x^* \to \infty;$ attractive for $d < 4;\ \xi < 0$
- $g^* = 32\,\varepsilon/9;$ $x^* = 8\,\xi/3;$ $u^* \to \infty;$ attractive for $\xi > 0;\ \xi < (4-d)/3$

At the pure Hwa-Kardar fixed point there are 2 IR-irrelevant parameters μ, $\nu_\perp$

$$\Big(\mathcal{D}_{k_\perp} + \mathcal{D}_{k_\|} + \Delta_\omega \mathcal{D}_\omega - \Delta_G\Big) G_r = 0$$

where $\Delta_\omega = 2 - \gamma^*_{\nu_\perp} = 2$ at Hwa-Kardar fixed point.

Lets introduce inverse coupling $\alpha = x^{-1} = \nu_{\perp R}\, \nu_{\| R}^{-1}$

$$\beta_\alpha(\alpha) = \beta_\alpha(\alpha^*) + \beta'_\alpha(\alpha^*)(\alpha - \alpha^*)$$

Is such approximation the equation of critical scaling takes the form

$$\Big(\mathcal{D}_{k_\perp} + \mathcal{D}_{k_\|} + \Delta_\omega \mathcal{D}_\omega + \beta'_\alpha(\alpha^*)\mathcal{D}_\alpha - \Delta_G\Big) G_r = 0$$

which allows us to incorporate third equation of canonical scale invariance and reproduce exact results of Hwa-Kardar

The next step toward the realism of the model is to describe the velocity field by the stochastic Navier-Stokes equation for incompressible fluid

$$\partial_t v_i + (\mathbf{v}\cdot\partial)v_i = \nu\partial^2 v_i - \partial_i P + \eta_i; \qquad \partial_i v_i = 0$$

Random force statistics

$$\langle \eta_i(\mathbf{x},t)\rangle = 0; \quad \langle \eta_i(t,\mathbf{x})\eta_j(t',\mathbf{x}')\rangle = B\,\delta(t-t')\int_{k>m}\frac{d\mathbf{k}}{(2\pi)^d}\,P^{\perp}_{ij}(\mathbf{k})\exp i(\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}'))\,.$$

The action of corresponding field model

$$S(h,h',\mathbf{v},\mathbf{v}') = \frac{C}{2}h'h' + h'\{-\nabla_t h + Z_1\,\nu_\parallel\partial_\parallel^2 h + Z_2\,\nu_\perp\partial_\perp^2 h - Z_4\,\frac{1}{2}\partial_\parallel h^2\} + S_{\mathbf{v}};$$

$$S_{\mathbf{v}} = \frac{B}{2}\,\mathbf{v}'\cdot\mathbf{v}' + \mathbf{v}'\cdot\{-\nabla_t\mathbf{v} + Z_3\,\nu\partial^2\mathbf{v}\}$$

Galilean symmetry

$$\mathbf{x}\to\mathbf{x}+ut\mathbf{n}; \quad h'(\mathbf{x},t)\to h'(\mathbf{x}+ut\mathbf{n},t); \quad h(\mathbf{x},t)\to h(\mathbf{x}+ut\mathbf{n},t);$$

$$\mathbf{v}'(\mathbf{x},t)\to\mathbf{v}'(\mathbf{x}+ut\mathbf{n},t); \qquad \mathbf{v}(\mathbf{x},t)\to\mathbf{v}(\mathbf{x}+ut\mathbf{n},t) - u\mathbf{n}$$

Couplings

$$g = C\mu^{-\varepsilon}\nu_{\perp R}^{-3/2}\nu_{\parallel R}^{-3/2}; \quad w = B\mu^{-\varepsilon}\nu_R^{-3}; \quad x_1 = \nu_{\parallel R}\nu_R^{-1}; \quad x_2 = \nu_{\perp R}\nu_R^{-1}$$

One-loop calculation

$$Z_1 = 1 - \frac{1}{8\pi^2\varepsilon}\left[g\frac{3}{16} + w\frac{\sqrt{x_2+1}(-3x_1+x_2-2)+2\left(x_1+1\right)^{3/2}}{2x_1\sqrt{x_2+1}(x_1-x_2)^2}\right];$$

$$Z_2 = 1 - \frac{1}{8\pi^2\varepsilon}\frac{w}{3}\frac{2\sqrt{x_1+1}(2x_1-3x_2-1)+\sqrt{x_2+1}\left(-3x_1+5x_2+2\right)}{2\,x_2\sqrt{x_2+1}(x_2-x_1)^2};$$

$$Z_3 = 1 - \frac{1}{8\pi^2\varepsilon}\frac{w}{8}; \qquad Z_4 = 0;$$

System of the beta functions

$$\begin{aligned}
\beta_g =& - g \left[\varepsilon - \frac{3}{2}\gamma_1 - \frac{3}{2}\gamma_2 + 2\gamma_4 \right] ; \\
\beta_w =& - w \left[\varepsilon - 3\gamma_3 \right] ; \\
\beta_{x_1} =& - x_1 \left[\gamma_1 - \gamma_3 \right] ; \\
\beta_{x_2} =& - x_2 \left[\gamma_2 - \gamma_3 \right] ;
\end{aligned}$$

At one-loop level there is a linear dependence between β functions

$$\beta_g = -g \left[\frac{3}{2} \frac{\beta_{x_1}}{x_1} + \frac{3}{2} \frac{\beta_{x_2}}{x_2} - \frac{\beta_w}{w} + 2\gamma_4 \right]$$

$\gamma_4 = 0 \;\; \Rightarrow$ there are line of the fixed points

$$w^* = 8\varepsilon/3, \quad g^*(x_2^*), \quad x_1^*(x_2^*), \quad x_2^* \in \left[0, (\sqrt{13} - 1)/2 \right]$$

The entire line is IR attractive and belong to the pure turbulence universality class.

Couplings

$$g = C\mu^{-\varepsilon}\nu_{\perp}^{-3/2}\nu_{\parallel}^{-3/2}; \quad w = B\mu^{-\varepsilon}\nu^{-3}; \quad x_1 = \nu_{\parallel}\nu^{-1}; \quad x_2 = \nu_{\perp}\nu^{-1}$$

Other possible fixed points

- $g^* = 0;$ $w = 0;$ $x_1 = \forall;$ $x_2 = \forall;$
- $g^* = 0;$ $w/(x_1 x_2) = 0;$ $x_1 = 0;$ $x_2 = 0;$
- $g^* = 32\,\varepsilon/9;$ $w = 0;$ $x_1 = \infty;$ $x_2 = \forall;$
- $g^* = 32\,\varepsilon/9;$ $w/x_1 = 0;$ $x_1 = 0;$ $x_2 = \forall;$
- $g^* = 32\,\varepsilon/9;$ $w/(x_1 x_2) = 0;$ $x_1 = 0;$ $x_2 = 0;$
- $g^* = 32\,\varepsilon/9;$ $w = 8\,\varepsilon/3;$ $x_1 \to \infty;$ $x_2 \to \infty;$
- $g^* = 0;$ $w = 8\,\varepsilon/3;$ $x_1 \to \infty;$ $x_2 \to \infty;$
- $g^* = 0;$ $w = \varepsilon\,\frac{2x_2^2\sqrt{1+x_2}}{2(1-\sqrt{1+x_2})+x_2\sqrt{1+x_2}};$ $x_1 = 0;$ $x_2 = \forall;$
- $g^* = 0;$ $w = \varepsilon\,\frac{6x_1^2}{6x_1+2(2x_1-1)\sqrt{1+x_1}};$ $x_1 = \forall;$ $x_2 = 0;$

Thank you for attention!