Four-loop critical properties of polymerized membranes

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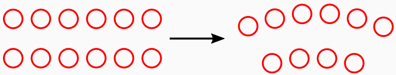
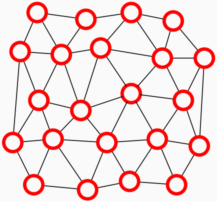
Polymerized or crystaline membranes

- Fluid membranes



  - Very weak interaction between molecules
  - Free diffusion inside the membrane plane
- Polymerized/crystalline/tethered membranes



  - A common working model of elastic membranes is the crystalline or polymerized membrane
  - Two-dimensional membranes made of linked molecules
  - Mechanical properties of thermalized elastic sheets

From crumpled phase to flat phase

Flat phase

Crumpling transition

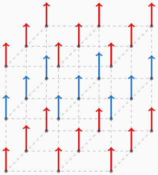
Crumpled phase

- Transition at low temperatures to a flat phase
- On average flat, strong fluctuations about spontaneously selected plane

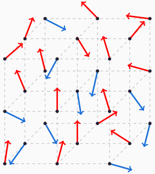
From spins to membranes

- Similar to the three-dimensional $O(3)$ symmetric magnet case:

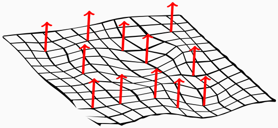
Ordered phase, $\langle \varphi(x) \rangle \neq 0$

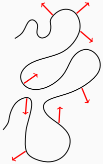


Normal phase, $\langle \varphi(x) \rangle = 0$

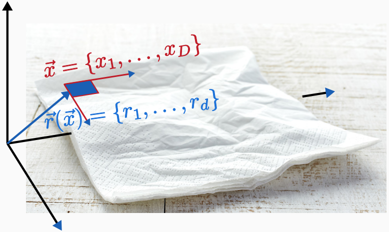


- Polymerized membranes, $\langle h(x) \rangle \neq 0$ in a flat phase





Coordinates and mean-field analysis



- D-dimensional manifold

$$x_i,\ i = 1 \ldots D$$

- Embedding d-dimesional space

$$r_a(\vec{x}),\ a = 1 \ldots d$$

- Tangent vector-field

$$t_i(\vec{x}) = \partial_i \vec{r}(\vec{x})$$

Standard φ^4 mean-field part for the field t_i, $\vec{r}(\vec{x}) = t_{\text{MF}} \cdot \left(x_1 \ldots x_D, \vec{0}\right)$:

$$V(t) = \frac{1}{2}\tau\,(t_i t_i) + u\left(t_i t_j\right)\left(t_i t_j\right) + v\,(t_i t_i)\,(t_i t_i)$$

$$t_{\text{MF}}^2 = \begin{cases} 0 & ,\tau \geq 0 \\ \frac{-\tau}{4(u+Dv)} & ,\tau < 0 \end{cases}, \quad \tau = T - T_c$$

One exponent to rule them all

- Beyond mean field, expansion around perfect flat configuration:

$$\vec{r}(\vec{x}) = t\left(x_1 + u_1(\vec{x}) \dots x_D + u_D(\vec{x}), h_1 \dots h_{d_c}\right), \quad d_c = d - D$$

- In-plane displacements $u_i(\vec{x})$ and out of plane modes $h_\alpha(\vec{x})$
- d_c massless Goldstone fields $h_\alpha(\vec{x})$ due to symmetry breaking
- Low-p limit of corellation functions

$$\Gamma_{uu}(\vec{p}) \sim |\vec{p}|^{2+\eta_u} \qquad \Gamma_{hh}(\vec{p}) \sim |\vec{p}|^{4-\eta}$$

- Ward identities provide scaling relation

$$\eta_u = 4 - D - 2\eta$$

- Roughness exponent $\langle \left(\vec{h}(\vec{x}) - \vec{h}(0)\right)^2 \rangle \sim x^{2\zeta}$

$$\zeta = 2 - \frac{D}{2} - \frac{\eta}{2}$$

Known methods of calculation

- Monte-Carlo simulation (MC) [Bowick et al.'96]
  - Most accurate estimate with errors
  - Network with strong nearest-neighbour bonds

- Self-Consist. Screening Approx. (SCSA) [LeDoussal,Radzihovsky,92]
  - Result is exact to leading order in $1/d_c$ + all order resummation in $1/d_c$
  - Exact results for limiting cases $D \to 4$ and $D = d$

- Nonperturbative RG (NPRG) [Kownacki,Mouhanna'09]
  - Effective average action $\Gamma_k[\vec{r}]$
  - Covers both flat phase and crumpled phase

- Perturbative RG within ε-expansion framework [Nelson,Peliti'87]
  - Two-loop [Coquand,Mouhanna,Teber'20][Mauri,Katsnelson'21]
  - Three-loop [Metayer,Mouhanna,Teber'21]
  - Four-loop - This work

Two-field model

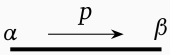
$$S = \frac{1}{2}\int d^D x \left[(\partial^2\vec{h})^2 + 2\mu u_{ij}^2 + \lambda u_{ii}^2\right], \quad u_{ij} = \frac{1}{2}\left(\partial_i u_j + \partial_j u_i + \partial_i\vec{h}\cdot\partial_j\vec{h}\right)$$

- u_i - D-dim. in plane modes
- h_α - d_c-dim. transverse modes, Goldstones analog
- Upper critical dimension $D_{uc} = 4$
- Renormalized parameters: $h_B = \sqrt{Z}h$, $u_B = Zu$, $\lambda_B = Z_\lambda \lambda$, $\mu_B = Z_\mu \mu$

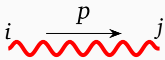
Calculation plan:

- Calculate Z_i in the model, derive γ_i
- Determine fixed points(FP) from $\beta_\lambda = 0$ and $\beta_\mu = 0$
- Critical exponent η from $\eta = \gamma(\mu^*, \lambda^*)$

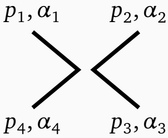
Feynman rules



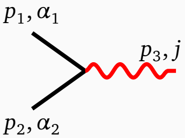
$$= \frac{\delta_{\alpha\beta}}{p^4}$$



$$= \frac{1}{p^2}\left[\frac{1}{\mu}\left(\delta_{ij} - \frac{p_i p_j}{p^2}\right) + \frac{1}{\lambda + 2\mu}\frac{p_i p_j}{p^2}\right]$$



$$= \begin{array}{l} -\frac{1}{4}\delta_{\alpha_1\alpha_4}\delta_{\alpha_2\alpha_3}\Big[\lambda(\vec{p}_1 \cdot \vec{p}_4)(\vec{p}_2 \cdot \vec{p}_3) + \mu(\vec{p}_1 \cdot \vec{p}_3)(\vec{p}_2 \cdot \vec{p}_4) \\ +\mu(\vec{p}_1 \cdot \vec{p}_2)(\vec{p}_3 \cdot \vec{p}_4)\Big] + \{1324\} + \{1234\} \end{array}$$



$$= \delta_{\alpha_1\alpha_2}\Big[\lambda(\vec{p}_1 \cdot \vec{p}_2)\vec{p}_{3,j} + \mu\Big((\vec{p}_3 \cdot \vec{p}_1)\vec{p}_{2,j} + (\vec{p}_3 \cdot \vec{p}_2)\vec{p}_{1,j}\Big)\Big]$$

- Higher powers of denominators, $1/p^4$
- Scalar products in numerator from projectors and vertices

Symmetries and Ward identities

After symmetry breaking we are left with the following set of symmetries:

- Translations in d-dimensional space
- Rotations in $d_c = (d - D)$-dimensional othogonal space
- Distance-preserving transformations in D-dimensional space of plane

Invariance of the action under linearized version of d-dimensional rotations:

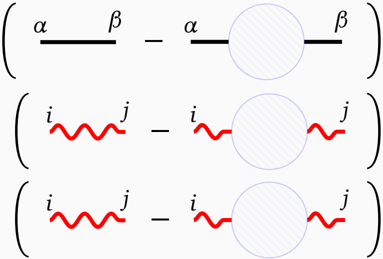
$$\vec{h}(\vec{x}) \to \vec{h}(\vec{x}) + \vec{A}_i \cdot x_i$$

$$u_i(\vec{x}) \to u_i(\vec{x}) - \left(\vec{A}_i \cdot \vec{h}(\vec{x})\right) - \frac{1}{2}\left(\vec{A}_i \cdot \vec{A}_j\right) x_j$$

Ward identities, need two-point functions only!

Two-point functions calculation

All higher-loop contributions inside scalar functions Π_{hh}, Π^T_{uu} and Π^L_{uu}:



$$\cdot \frac{\delta_{\alpha\beta}}{d_c} = 1 - \Pi_{hh}$$

$$\cdot \frac{P^T_{ij}(q)}{D-1} = \mu - \Pi^T_{uu}$$

$$\cdot P^L_{ij}(q) = \lambda + 2\mu - \Pi^L_{uu}$$

	1-loop	2-loop	3-loop	4-loop
Π_{hh}	1	6	45	516
Π_{uu}	1	3	23	237

Table: Number of calculated diagrams

Diagrams generated with `DIANA` [Tentyukov,Fleischer'99] and integrals up to 4-loops calculated with `FORCER` [Ruijl,Ueda,Vermaseren'17]

Renormalization procedure

- Calculate 4-loop diagrams contributing to Π_{hh} , Π_{uu}^{T} and Π_{uu}^{L} upto $\mathcal{O}(\varepsilon^{-1})$, 3-loop to $\mathcal{O}(\varepsilon^{0})$, 2-loop to $\mathcal{O}(\varepsilon^{1})$ and 1-loop to $\mathcal{O}(\varepsilon^{2})$
- Replace bare parameters with renormalized using $\mu_B \to \mu Z_\mu, \lambda_B \to \lambda Z_\lambda$
- Require finitness of renormalized expressions after multiplication with corresponding field renormalization constants:

$$Z^2\left[\mu Z_\mu - \Pi_{uu}^{T}(\mu_B \to \mu Z_\mu, \lambda_B \to \lambda Z_\lambda)\right] = \mathcal{O}(\varepsilon^0)$$

$$Z^2\left[2\mu Z_\mu + \lambda Z_\lambda - \Pi_{uu}^{L}(\mu_B \to \mu Z_\mu, \lambda_B \to \lambda Z_\lambda)\right] = \mathcal{O}(\varepsilon^0)$$

$$Z\left[1 - \Pi_{hh}(\mu_B \to \mu Z_\mu, \lambda_B \to \lambda Z_\lambda)\right] = \mathcal{O}(\varepsilon^0)$$

- System of three equations determines Z, Z_μ, Z_λ as series in $\varepsilon = 2 - D/2$

RG functions and critical exponents

- Beta functions $\beta_X = \frac{\partial X}{\partial \log \mathfrak{m}}$, with $\overline{\text{MS}}$ scale parameter $\mathfrak{m}$:

$$\beta_\mu = \frac{2\varepsilon \partial_\lambda \log \frac{\mu Z_\mu}{\lambda Z_\lambda}}{\det \begin{pmatrix} \partial_\lambda \log \lambda Z_\lambda & \partial_\mu \log \lambda Z_\lambda \\ \partial_\lambda \log \mu Z_\mu & \partial_\mu \log \mu Z_\mu \end{pmatrix}} \quad \beta_\lambda = \frac{2\varepsilon \partial_\mu \log \frac{\lambda Z_\lambda}{\mu Z_\mu}}{\det \begin{pmatrix} \partial_\lambda \log \lambda Z_\lambda & \partial_\mu \log \lambda Z_\lambda \\ \partial_\lambda \log \mu Z_\mu & \partial_\mu \log \mu Z_\mu \end{pmatrix}}$$

- Field anomalous dimension, $\gamma = \frac{\partial \log Z}{\partial \log \mathfrak{m}}$:

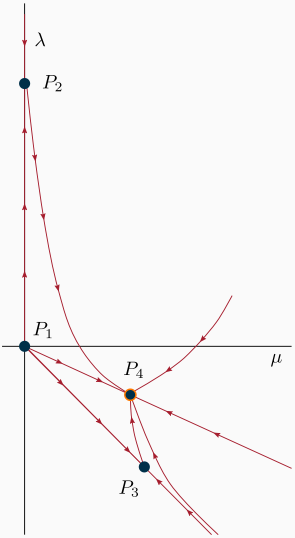
$$\gamma = \beta_\lambda \partial_\lambda \log Z + \beta_\mu \partial_\mu \log Z$$

- Fixed points (μ^*, λ^*) from $\beta_\mu(\mu^*, \lambda^*) = 0, \beta_\lambda(\mu^*, \lambda^*) = 0$

- Field anomalous dimension with substituted FP coordinates provides us with the critical exponent:

$$\eta = \gamma(\mu^*, \lambda^*)$$

Phase diagram for the flat phase



$$\beta_\mu = -2\varepsilon\mu + \frac{10\mu^2(\lambda+\mu)}{\lambda+2\mu} + d_c\frac{\mu^2}{6}$$

$$\beta_\lambda = -2\varepsilon\lambda + \frac{10\lambda\mu(\lambda+\mu)}{\lambda+2\mu} + d_c\frac{6(\lambda+\mu)+\mu^2}{6}$$

FP	μ^*	λ^*	η
P_1	0	0	0
P_2	0	$\frac{2}{d_c}\varepsilon$	0
P_3	$\frac{12}{20+d_c}\varepsilon$	$-\frac{6}{20+d_c}\varepsilon$	$\frac{20}{20+d_c}\varepsilon$
P_4	$\frac{12}{24+d_c}\varepsilon$	$-\frac{4}{24+d_c}\varepsilon$	$\frac{24}{24+d_c}\varepsilon$

- Gaussian IR repulsive P_1
- IR unstable P_2, P_3
- IR attractive P_4

Results for critical exponents: FP3 and FP4

$$\eta_3 = \frac{20\varepsilon}{[20+d_c]} + \left(\frac{2800}{[20+d_c]^3} + \frac{1060}{3[20+d_c]^2} - \frac{74}{3[20+d_c]}\right)\varepsilon^2$$
$$+ \left(\frac{784000}{[20+d_c]^5} - \frac{40(615553-591624\zeta_3)}{27[20+d_c]^4} + \frac{2(1024193-1006344\zeta_3)}{27[20+d_c]^3} - \frac{2(17105-20736\zeta_3)}{27[20+d_c]^2} - \frac{155}{9[20+d_c]}\right)\varepsilon^3$$
$$+ \left(\frac{274400000}{[20+d_c]^7} - \frac{28000(648943-591624\zeta_3)}{27[20+d_c]^6} - \frac{40(63897618439+174575927736\zeta_3-263951628480\zeta_5)}{243[20+d_c]^5}\right.$$
$$+ \frac{4(226859519881+611469803304\zeta_3+239607720\zeta_4-927893517120\zeta_5)}{729[20+d_c]^4}$$
$$- \frac{2(15308397193+40857079644\zeta_3+40756932\zeta_4-62189551440\zeta_5)}{729[20+d_c]^3}$$
$$\left.+ \frac{24880019+65141136\zeta_3+186624\zeta_4-99921600\zeta_5}{81[20+d_c]^2} - \frac{769-2160\zeta_3}{54[20+d_c]}\right)\varepsilon^4 + \mathcal{O}(\varepsilon^5)$$

$$\eta_4 = \frac{24\varepsilon}{[24+d_c]} + \left(\frac{2880}{[24+d_c]^3} + \frac{456}{[24+d_c]^2} - \frac{24}{24+d_c}\right)\varepsilon^2$$
$$+ \left(\frac{691200}{[24+d_c]^5} - \frac{576(234137-192096\zeta_3)}{125[24+d_c]^4} + \frac{8(1031777-923616\zeta_3)}{125[24+d_c]^3} - \frac{4(39029-86832\zeta_3)}{375[24+d_c]^2} - \frac{64}{3[24+d_c]}\right)\varepsilon^3$$
$$+ \left(\frac{207360000}{[24+d_c]^7} - \frac{165888(20501-16008\zeta_3)}{5[24+d_c]^6} - \frac{32(1174399340197+3188610294336\zeta_3-4827670269120\zeta_5)}{1875[24+d_c]^5}\right.$$
$$+ \frac{4(2761899037843+7430870367648\zeta_3+1867173120\zeta_4-11277698973120\zeta_5)}{5625[24+d_c]^4}$$
$$- \frac{2(52639017319+140359656168\zeta_3+83125440\zeta_4-213590260800\zeta_5)}{1875[24+d_c]^3}$$
$$\left.+ \frac{2(1074978101+2807145072\zeta_3+3907440\zeta_4-4302849600\zeta_5)}{5625[24+d_c]^2} - \frac{16(13-27\zeta_3)}{9[24+d_c]}\right)\varepsilon^4 + \mathcal{O}(\varepsilon^5)$$

Checks from the large d_c expansion

- Leading in $1/d_c$ result, fixed point P_4: [LeDoussal,Radzihovsky'92]

$$\eta(D, d_c) = \frac{8}{d_c} \frac{D-1}{D+2} \frac{\Gamma(D)}{\Gamma(D/2)^3 \Gamma(2-D/2)} + \mathcal{O}\left(\frac{1}{d_c^2}\right)$$

- Agreement with $1/d_c$ expansion of the four-loop result

$$\eta_4 = \frac{1}{d_c}\left(24\varepsilon - 24\varepsilon^2 - \frac{64}{3}\varepsilon^3 - \frac{16}{9}(13 - 27\zeta_3)\varepsilon^4 + \mathcal{O}(\varepsilon^5)\right) + \mathcal{O}\left(\frac{1}{d_c^2}\right)$$

- For the P_3 FP: $\eta\left(D, \frac{D(D-1)}{(D-2)(D+1)} d_c\right)$ [LeDoussal,Radzihovsky'18]

$$\eta_3 = \frac{1}{d_c}\left(20\varepsilon - \frac{74}{3}\varepsilon^2 - \frac{155}{9}\varepsilon^3 - \frac{1}{54}(769 - 2160\zeta_3)\varepsilon^4 + \mathcal{O}(\varepsilon^5)\right) + \mathcal{O}\left(\frac{1}{d_c^2}\right)$$

Numerical results

- For the case $d_c = 1$ our result reads:

$$\eta_3 = 0.952\varepsilon - 0.071\varepsilon^2 - 0.069\varepsilon^3 - 0.075\varepsilon^4 + \mathcal{O}(\varepsilon^5)$$
$$\eta_4 = 0.96\varepsilon - 0.0461\varepsilon^2 - 0.0267\varepsilon^3 - 0.02\varepsilon^4 + \mathcal{O}(\varepsilon^5)$$

- Naive $\varepsilon = 1$ perturbative results:

	1-loop	2-loop	3-loop	4-loop
η_3	0.9524	0.8813	0.8116	0.7368
η_4	0.96	0.9139	0.8872	0.8670

- Resummation, Pade $[2/2]$:

$$\eta_4^{[2/2]} = 0.806$$

- Other methods:

	SCSA	NPRG	MC
η_4	0.821	0.849	0.795(10)

Conclusion

- From statistical mechanics side:
  - Performed calculations in the two-field model of polymerized membranes
  - Four-loop critical exponent η approaching other methods predictions
  - Three-loop part in agreement with previous calculations

- From field theory side:
  - No need for 3-pt and 4-pt functions for beta-function calculation
  - Example of apparent convergent series, reliable result without resummation

- From multi-loop calculations side:
  - Problem reduced to calculation of four-loop massless propagators divergencies
  - Highly nontrivial calculation due to integrals with numerators and higher powers of denominators

Thank you for attention!