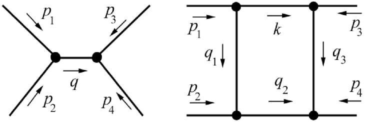
Hypergeometric approach in Feynman integral calculation

V. V. Bytev

Joint Institute for Nuclear Research, Dubna, Russia.

The International Workshop on Nuclear and Particles Physics, April 2022

Feynman Diagrams: Basic Definitions



- Quantum field theory amplitudes are represented as a sum of Feynman Diagrams, graphs for which each line and vertex is represented by a factor in a term of the quantum amplitude.
- Integrating over all unconstrained momenta gives rise to a Feynman Integral, FI. For L loops and n internal lines, and allowing the propagators to be raised to powers ν_j,

$$J(m_1^2, ...m_n^2; p_1^2, ...p_k^2, \alpha_1, ..., \alpha_n) = \int \frac{d^d k_1 .. d^d k_L}{(i\pi^{d/2})^L} \Pi_{i=1}^n \frac{1}{(q_i^2 - m_i^2)^{\alpha_i}}.$$

$$F(s, t, a_1, a_2, a_3, a_4; d) = \int \frac{d^d k}{(k^2)^{a_1}((k+p_3)^2)^{a_2}((k+p_3+p_4)^2)^{a_3}((k-p_1)^2)^{a_4}}.$$

- Our final goal is to find a solution of FI as a series in dimensional regularization parameter ε where coefficients are expressed in terms of some special functions with well-established properties.

IBP Relations, Master Integrals and Differential Equations

- **Integration by parts** leads to a set of recurrence relations among diagrams of a given topology but different powers of the propagators. K.G. Chetyrkin, F.V. Tkachov, Nucl. Phys. B 192, 159 (1981)
- The full set of recurrence relations should be solved by finding how the integral with powers of propagators $(j_1 + j_2 + \cdots + j_k)$ reduced to integrals with powers $(j_1 + j_2 + \cdots + j_k - 1)$
- The method involves taking derivatives of each integral with respect to momenta and reducing it to the original integral.
- The relations found permit a **reduction** to a basis set of **master integrals** in terms of which the diagrams of this class may be expressed.
- The diffrential equation system fo FI is obtained by taking some derivatives of a given master integral with respect to kinematical invariants and masses.
  A.V. Kotikov, Phys. Lett. B 254, 158 (1991)
  Then the result is written in terms of Feynman integrals of the given family and, according to the known reduction, in terms of the master integrals.
- Finally, one obtains a system of differential equations for the master integrals which can be solved with appropriate boundary conditions.

Questions to answer

- to find some independent way for calculation of recurrence relation among FI
- establish the class of function in terms of which FI could be expressed
- find a way to construct the differential equation for given FI (master integral) for arbitrary value of propagator powers, masses and impulses
- establish the exact number of master integrals for FI

Hypergeometric function definition

- Gauss hypergeometric function
  first used by John Wallis in 1655
  systematic treatment was given by Carl Friedrich Gauss (1813)

$$ {}_2F_1\left(\begin{matrix} a, b \\ c \end{matrix} \middle| z \right) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{x^n}{n!} $$

  Pochhammer symbol:

$$ (a)_n = \frac{\Gamma(a+n)}{\Gamma(a)}, \qquad n\Gamma(n) = \Gamma(n+1) $$

  - differential equation (Fuchsian equation):

$$ z(z-1)\frac{d^2u}{dz^2} + (c-(a+b+1)z)\frac{du}{dz} - abu = 0 $$

  - ratio of series expansion coefficients:

$$ {}_2F_1\left(\begin{matrix} a, b \\ c \end{matrix} \middle| z \right) = \sum_{n=0}^{\infty} C_n x^n, \qquad \frac{c_{n+1}}{c_n} = \frac{(a+n)(b+n)}{(c+n)(n+1)} $$

- how to obtain new hypergeometric function:

$$ \sum_{n=0}^{\infty} \frac{(a)_n (b)_n (b_2)_{m+n}}{(c)_n (e)_{2m-k+l}} \frac{x^n}{n!} \frac{y^m}{m!} \frac{z^l}{l!} $$

generalized Lauricella series

- Appell hypergeometric function:

$$F_c(a, b; c_1, c_2; z_1, z_2) = \sum_{k_1, k_2} \frac{(a)_{k_1+k_2}(b)_{k_1+k_2}}{(c_1)_{k_1}(c_2)_{k_2}} \frac{z_1^{k_1} z_2^{k_2}}{k_1! k_2!}$$

- To express the Feynman integral we need hypergeometric function called generalized Lauricella series:

$$\sum_{m_1,\dots,m_l}^{\infty} \prod_{i,j} \frac{(a_j)_{\sum_k^l q_k m_k}}{(b_i)_{\sum_k^l q_k m_k}} \prod_{n=1}^{l} \frac{x_n^{m_n}}{m_n!}, \quad q_k \in \mathbb{Z},$$

-

$$F^{A:B^{(1)};\dots;B^{(n)}}_{C:D^{(1)};\dots;D^{(n)}} \left(\begin{matrix} [(a):\theta^{(1)},\dots,\theta^{(n)}]:[(b^1):\phi^{(1)}];\dots;[(b^n):\phi^{(n)}] \\ [(c):\psi^{(1)},\dots,\psi^{(n)}]:[(d^1):\delta^{(1)}];\dots;[(d^n):\delta^{(n)}] \end{matrix} \; x_1,\dots,x_n \right)$$

$$= \sum_{s_1,\dots,s_n=0}^{\infty} \Omega(s_1,\dots,s_n) \frac{x_1^{s_1}}{s_1!} \frac{x_n^{s_n}}{s_n!},$$

$$\Omega(s_1,\dots,s_n) = \frac{\prod_{j=1}^{A}(a_j)_{s_1\theta_j^{(1)}+\dots+s_n\theta_j^{(n)}} \prod_{j=1}^{B^{(1)}}(b_j^{(1)})_{s_1\phi_j^{(1)}} \cdots \prod_{j=1}^{B^{(n)}}(b_j^{(n)})_{s_n\phi_j^{(n)}}}{\prod_{j=1}^{C}(c_j)_{s_1\psi_j^{(1)}+\dots+s_n\psi_j^{(n)}} \prod_{j=1}^{D^{(1)}}(d_j^{(1)})_{s_1\delta_j^{(1)}} \cdots \prod_{j=1}^{D^{(n)}}(d_j^{(n)})_{s_n\delta_j^{(n)}}},$$

The Mellin-Barnes Representation

The Mellin-Barnes representation relies on the identity

$$\frac{1}{(A+B)^{\lambda}} = \frac{1}{\Gamma(\lambda)}\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+\infty} dz\,\Gamma(-z)\,\Gamma(\lambda+z)\,\frac{B^{z}}{A^{\lambda+z}}.$$

The contour is chosen to separate the poles in $\Gamma(-z)$ from the poles in $\Gamma(\lambda+z)$.
This relation is applied to the denominator in the Feynman Parametrization to break it up into monomials in the Feynman parameters x_i. The integration over the Feynman parameters can then be easily performed in terms of Γ functions,

$$J_l(m_1^2,\ldots.m_n^2;p_1^2,\ldots p_k^2,\alpha_1,\ldots,\alpha_n) = C\int_{-i\infty}^{+i\infty}\prod_{j,l} du_l\,\frac{\Gamma(\sum_i a_{ij}u_i+b_j)}{\Gamma(\sum_i c_{ij}u_i+d_j)}\,z_l^{\sum f_{kl}u_k}$$

Upon application of Cauchy's theorem, the Feynman integral can be converted into a linear combination of multiple series:

$$\Phi(n,\vec{x}\,)\sim\sum_{k_1,\cdots,k_{r+m}=0}^{\infty}\prod_{a,b}\frac{\Gamma(\sum_{i=1}^{m}A_{ai}k_i+B_a)}{\Gamma(\sum_{j=1}^{r}C_{bj}k_j+D_b)}x_1^{k_1}\cdots x_{r+m}^{k_{r+m}}$$

The Mellin-Barnes Representation II

- Mellin-Barnes representation of FI:

$$J_l(m_1^2,....m_n^2;p_1^2,...p_k^2,\alpha_1,...,\alpha_n) = C\int_{-i\infty}^{+i\infty}\prod_{j,l}du_l\,\frac{\Gamma(\sum_i a_{ij}u_i+b_j)}{\Gamma(\sum_i c_{ij}u_i+d_j)}z_l^{\sum f_{kl}u_k}$$

- An FI answer we could write down in terms of hypergeometric function of Horn type

$$\Phi(n,\vec{x}\,)\sim\sum_{k_1,\cdots,k_{r+m}=0}^{\infty}\prod_{a,b}\frac{\Gamma(\sum_{i=1}^{m}A_{ai}k_i+B_a)}{\Gamma(\sum_{j=1}^{r}C_{bj}k_j+D_b)}x_1^{k_1}\cdots x_{r+m}^{k_{r+m}}$$

- calculation of Mellin-Barnes representation in terms of series is not easy
- we need further expansion of FI in terms of dimensional regularization parameter.

The Mellin-Barnes Representation -what could be inferred from it?

- An FI could be written in terms of hypergeometric series of Horn type
- The expansion over dimesional regularization parameter (derivatives of Horn type hypergeometric function) could be written in terms of function of the same class
  V.B., B. Kniehl, Nucl.Phys.B 952 (2020) 114911
  - Derivatives of the generalized Lauricella series in one of their (upper or lower) parameters can be expressed as a finite sum of the generalized Lauricella series
  - the n-th term of the ε series can be expressed as a Horn-type hypergeometric function in $n+m$ variables, where m is the number of summations in the Horn-type representation of the Feynman integral
  - The region of convergence of any of these parameter derivatives, i.e., the coefficients in the ε expansion, and the initial FI are the same
- we could find differential contiguous relations
  - All special functions in coefficients of dimensional regularization expansion must inherit algebra of that differential operators (shuffle and stuffle algebra in MPL)
- we could establish differential equation for particular FI

Differential Contiguous Relations

- Shifting the integration contours in the Mellin-Barnes representation or equivalently shifting summation indices in hypergeometric representation we find differential contiguous relations.
- They can be expressed in terms of step-up $L^+_{b_j}$ and step-down $L^-_{d_j}$ operators which shift indices b_j, d_j by a unit:

$$H(\mathbf{a}, \vec{b} + \vec{e}_j, \mathbf{c}, \vec{d};\ \vec{z}) = L^+_{b_j} H(\mathbf{a}, \vec{b}, \mathbf{c}, \vec{d};\ \vec{z}) = \Big(\sum_i a_{ij}\theta_i + b_j\Big) H(\mathbf{a}, \vec{b}, \mathbf{c}, \vec{d};\ \vec{z}),$$

$$H(\mathbf{a}, \vec{b}, \mathbf{c}, \vec{d} - \vec{e}_j;\ \vec{z}) = L^-_{d_j} H(\mathbf{a}, \vec{b}, \mathbf{c}, \vec{d};\ \vec{z}) = \Big(\sum_i c_{ij}\theta_i + d_j\Big) H(\mathbf{a}, \vec{b}, \mathbf{c}, \vec{d};\ \vec{z}),$$

- The inverse operators $L^-_{b_j}$, $L^+_{d_j}$ can not be directly constructed form Mellins-Barnes representation
- Together operators $L^-_{b_j}$, $L^+_{d_j}$, $L^+_{b_j}$, $L^-_{d_j}$ helps one to change parameters of FI (Horn hypergeometric function) on integer number and find relation between the number of nontrivial master integrals found from IBP (which are not expressible in terms of Gamma functions) and the maximal power of derivatives generated by the $L^-_{b_j}$, $L^+_{d_j}$
- The inverse operators $L^-_{b_j}$, $L^+_{d_j}$ inherit information about simplification of hypergeometric function (lowering its order)

Differential Equation System Derivation

- From the differential continuous relations a dynamical symmetry algebra could be constructed, and from the Lie algebra the system of differential equations could be established

$$\left(\prod_{j=m^+{}_k} L^+_{b_j} - \frac{1}{z_k} \theta_k \prod_{j=m^+{}_k} L^-_{d_j} \right) H(\mathbf{a}, \vec{b}, \mathbf{c}, \vec{d};\ \vec{z}) = 0,$$

  set m^+_k consists of integers j defined from equation $\sum_i |a_{ij}| e_i \neq 0$ and m^-_k consists of integers j: $\sum_i |c_{ij}| e_i \neq 0$.
- variables z_i are independent (all mass and external momenta are different and not equal to zero), and the indices of propagators are real numbers.
- we consider the multivariable specialization of initial differential equation:

$$z_j = y_j(\vec{x}), \qquad j = 1, \ldots, n$$
$$\vec{x} = (x_1, \ldots, x_k), \qquad k < n\,.$$

- when multivariable specialization fall in singular locus of differential equation, the rank of new system will be lower
- The rank of differential system after multivariable specialization could be also lower in some specific combinations of parameters and variables.

Differential Equation System Derivation II

$$\left[\begin{array}{c|c} M_1 & \\ \hline B_1 & M_2 \end{array}\right] \left[\begin{array}{c} \frac{\partial^\chi H(\vec{z})}{\partial z_{j_1}\ldots\partial z_{j_\chi}} \\ \vdots \\ \frac{\partial^i H(\vec{z})}{\partial z_{j_1}\ldots\partial z_{j_i}} \\ \vdots \\ \frac{\partial H(\vec{z})}{\partial z_j} \\ \hline \frac{\partial^l H(\vec{x})}{\partial x_{j_1}\ldots\partial x_{j_l}} \\ \vdots \\ H(\vec{x}) \end{array}\right] = 0$$

- Applying the chain rule χ times we construct system of differential equation where various derivatives of order less or equal χ w.r.t. new variables $\vec{x}$ are expressed in terms of derivatives w.r.t. old $\vec{z}$ and derivatives of $y_i(\vec{x})$ functions.

One-Loop Two-point Diagram, different masses

- one-loop two-point diagram with different masses and arbitrary powers of propagators:

$$J(\alpha_1, \alpha_2, m_1, m_2) = \int \frac{\mathrm{d}^n k}{(k^2 - m_1^2)^{\alpha_1} \left((k-p)^2 - m_2^2\right)^{\alpha_2}} .$$

- By constructing step-up and step down operators we obtain the system of partial differential equations of second order with two variables for $J(\alpha, \beta, m_1, m_2)$:

$$\theta_1 \left(-\alpha_1 + \frac{n}{2} + \theta_1\right) - \frac{(2\alpha_1 + 2\alpha_2 - n - 2\theta_1 - 2\theta_2)(\alpha_1 + \alpha_2 - n - \theta_1 - \theta_2 + 1)}{2z_1} = 0 ,$$

$$\theta_2 \left(-\alpha_2 + \frac{n}{2} + \theta_2\right) - \frac{(2\alpha_1 + 2\alpha_2 - n - 2\theta_1 - 2\theta_2)(\alpha_1 + \alpha_2 - n - \theta_1 - \theta_2 + 1)}{2z_2} = 0 .$$

- it is equivalent to the equation of Appell hypergeometric function $F_4(a, b, c_1, c_2, z_1, z_2)$ and has 4 different solutions. The singular locus on $\mathbf{P}^2$ is $z_1 = 0$, $z_2 = 0$, the line at infinity, $z_1^2 + z_2^2 + 1 = 2z_1 z_2 + 2z_1 + 2z_2$

One-Loop Two-point Diagram, different masses

- we choose $\chi = 4$, the number of derivatives w.r.t. x. In this case we need $\chi - \eta = 2$ differentiations w.r.t. sets of variables z_1, z_2 and obtain an Fuchsian differential equation over one variable

$$L_4(x) J(\alpha_1, \alpha_2, m_1, m_2) = 0\,,$$

  where $L_4(x)$ is the differential operator of the fourth order.
- it has 4 singular points inherited from initial differential system, so the final answer could not be expressed in terms of hypergeometric function of one variable
- monodromy is reduced, is defined by $\{a, b, c_1 - a, c_1 - b, c_2 - a, c_2 - b, c_1 + c_2 - a, c_1 + c_2 - b\} \in Z$, and in our case we have $-b + c_1 + c_2 = 3$, so one solution of system degenerates to Puiseux-type and one-variable equation for F_4 must factorize by first-order differential operator
-

$$L_1(x) L_3(x) J(\alpha_1, \alpha_2, m_1, m_2) = 0$$

  By defining arbitrary constants, we could see that final answer for bubble FI with two different masses and arbitrary powers of propagators has only two F_4 terms for the variables $z_1 = p^2/m_2^2$, $z_2 = m_1^2/m_2^2$ and three terms in variables $z_1 = m_2^2/p^2$, $z_2 = m_2^2/p^2$

One-Loop Two-point Diagram, equal masses

- J is equivalent to a two-loop bubble FI with equal masses $m_1 = m_2$: $z_1 = y_1(x) = x$ and $z_2 = y_2(x) = x$.
- we could find an differential equation ofr case of equal masses by two different ways: consider the case $z_1 = z_2 = x$ and $z_1 = x, z_2 = const = x$.
- this univariate specialization does not belong to singular locus, the rank of new differential system should be the same.

$$\tilde{L}_4(x)F_4(x,x) = 0$$

- we have three distinct poles at points $0, 1/4, \infty$. Compare the singular points and local exponents with differential equation for hypergeometric function ${}_4F_3$, we came to the well-known result for univariate specialization of F_4

$$F_4\left(\begin{array}{c} a, b \\ c_1, c_2 \end{array} \middle| x, x \right) = {}_4F_3\left(\begin{array}{c} a, b, \frac{c_1+c_2}{2}, \frac{c_1+c_2-1}{2} \\ c_1, c_2, c_1 + c_2 - 1 \end{array} \middle| 4x \right).$$

One-Loop Two-point Diagram, equal masses II

- The monodromy group of initial differential equation for bubble is reduced due to equation for the parameters $-b + c_1 + c_2 = 3$, so we find the factorization of $\tilde{L}_4(x)$ if we substitute parameters:

$$L_1(x)L_3(x)J(\alpha_1,\alpha_2,m,m) = 0,$$

$$L_1(x) = \frac{\mathrm{d}}{dx} + \frac{((x-4)(-\alpha_1-\alpha_2+n)+3x-8)}{x(x-4)},$$

$$L_3(x) = \frac{\mathrm{d}^3}{dx^3} + \frac{-(x-8)(\alpha_1+\alpha_2-n-3)+2n+18}{(x-4)x}\frac{\mathrm{d}^2}{dx^2}$$
$$-\frac{4\left((\alpha_1+\alpha_2)(5(\alpha_1+\alpha_2)-8n+1)+3n^2\right)+x(2\alpha_1-n-2)(-2\alpha_2+n+2)}{4(x-4)x^2}\frac{\mathrm{d}}{dx}$$
$$+\frac{(\alpha_1+\alpha_2-n+1)(\alpha_1+\alpha_2-n+2)(2(\alpha_1+\alpha_2)-n)}{2(x-4)x^3},$$

- The final answer for bubble FI with equal masses could be expressed through hypergeometric function ${}_3F_2$ and polynomial expression.

Thank you for your Attention

thank you for your attention