Transverse momentum distributions of hadrons in the nonextensive statistics

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International Workshop on Nuclear and Particles Physics
Almaty, Kazakhstan, April 24-30, 2022

Transverse momentum distributions of hadrons in pp and AA collisions

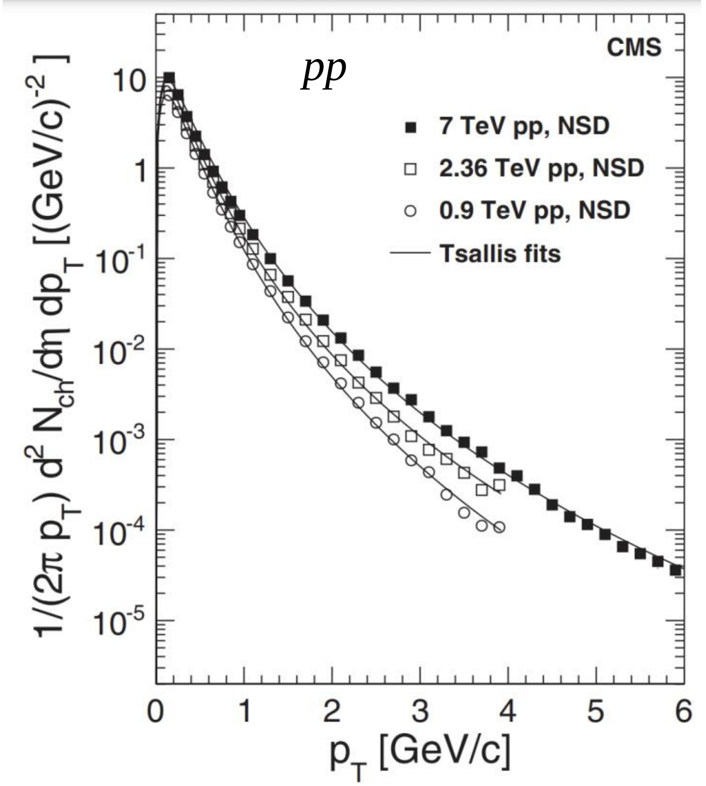
Tsallis-like distributions

$$E\frac{d^3N_{ch}}{d^3p} = C\frac{dN_{ch}}{dy}\left(1+\frac{E_T}{nT}\right)^{-n}, \qquad n = \frac{1}{q-1}$$

Phenomenological Tsallis distribution

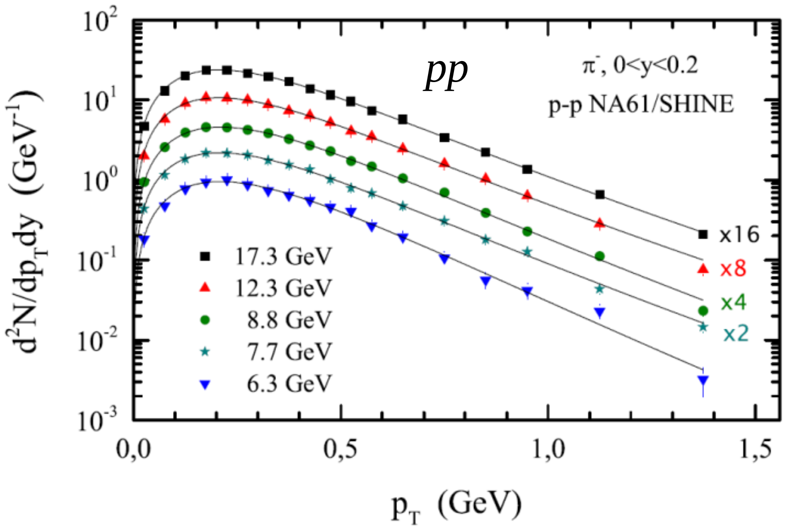
$$\frac{d^2N}{dp_T dy} = \frac{gV}{(2\pi)^2} p_T m_T \cosh y \left[1-(1-q)\frac{m_T \cosh y - \mu}{T}\right]^{\frac{q}{1-q}}$$

Charged-hadron yield in the range $|\eta| < 2.4$



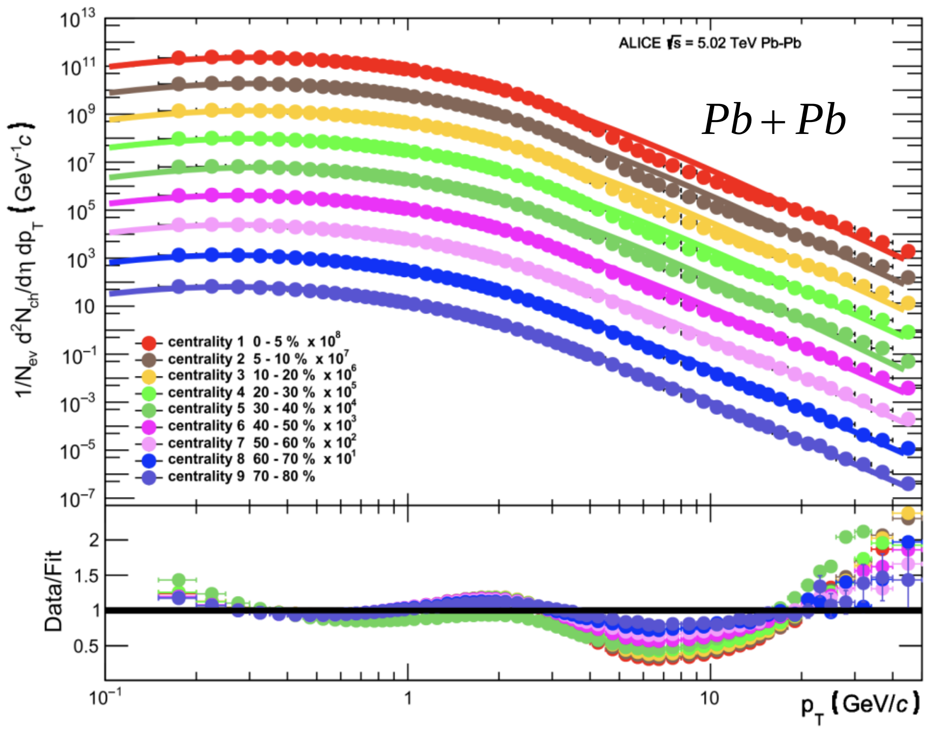
V. Khachatryan et al., PRL 105 (2010) 022002

Identified hadrons



A.S.P., O.V.Teryaev, J.Cleymans
Eur. Phys. J. A 53 (2017) 102

Charged-hadron yields



M.D. Azmi, T. Bhattacharyya, J. Cleymans, M. Paradza
J. Phys. G: Nucl. Part. Phys. 47 (2020) 045001

Tsallis statistical mechanics

Tsallis entropy: $S = -\sum_i \frac{p_i - p_i^q}{1-q}$ $\quad 0 < q < \infty$ $\quad \xrightarrow{q \to 1} \quad S_G = -\sum_i p_i \ln p_i$ - Boltzmann-Gibbs entropy

Norm equations for microstates: $\mathrm{Tr}[\hat{\varrho}_{eq}] = \sum_i p_i = 1$

Mean value of the quantum operator A: $\langle \hat{A} \rangle = \mathrm{Tr}[\hat{A}\hat{\varrho}_{eq}] = \sum_i p_i A_i$ - liniar averages

C. Tsallis, J. Stat. Phys. 52 (1988) 479

Probabilities of microstates – eigenvalues of statistical operator: $\hat{\varrho}_{eq} = \sum_i p_i |\Psi_i\rangle\langle\Psi_i|$

$\hat{\varrho}_{eq}$ - statistical operator is unknown

The statistical operator is determined from the principle of thermodynamic equilibrium :

$$[d(Y - \lambda\varphi)]_{X^1,\dots,X^n} = 0$$

$X = (X^1, \dots, X^n)$ - thermodynamic variables of state

Thermodynamic potential: $Y = Y(\{\hat{\varrho}_{eq}\}, X) = \mathrm{Tr}[\hat{\varrho}_{eq}\hat{Y}(\{\hat{\varrho}_{eq}\}, X)]$

Additional equation: $\varphi(\{\hat{\varrho}_{eq}\}) = \mathrm{Tr}[\hat{\varrho}_{eq}] - 1 = 0$

The solution is the equilibrium statistical operator as a function of the thermodynamic state variables $X = (X^1, \dots, X^n)$:

$$\hat{\varrho}_{eq} = \hat{\varrho}_{eq}(\hat{H}, X, \Lambda(X)), \qquad \mathrm{Tr}[\hat{\varrho}_{eq}(\hat{H}, X, \Lambda(X))] = 1$$

Definitions of different variants of nonextensive statistics in grand canonical ensemble

$$S_G = -\sum_i p_i \ln p_i$$ - Boltzmann-Gibbs entropy

$q \to 1$

Tsallis-1 statistics

$$S = -\sum_i \frac{p_i - p_i^q}{1-q}$$ - Tsallis entropy

$$\sum_i p_i = 1$$

$$\langle A \rangle = \sum_i p_i A_i, \qquad \langle 1 \rangle = 1$$

$$p_i = \left[1 + \frac{q-1}{q}\frac{\Lambda - E_i + \mu N_i}{T}\right]^{\frac{1}{q-1}}$$

$$\sum_i \left[1 + \frac{q-1}{q}\frac{\Lambda - E_i + \mu N_i}{T}\right]^{\frac{1}{q-1}} = 1$$

C. Tsallis, J. Stat. Phys. 52 (1988) 479

Tsallis-2 statistics

$$S = -\sum_i \frac{p_i - p_i^q}{1-q}$$ - Tsallis entropy

$$\sum_i p_i = 1$$

$$\langle A \rangle = \sum_i p_i^q A_i, \qquad \langle 1 \rangle \neq 1$$

$$p_i = \frac{1}{Z}\left[1 - (1-q)\frac{E_i - \mu N_i}{T}\right]^{\frac{1}{1-q}}$$

$$Z = \sum_i \left[1 - (1-q)\frac{E_i - \mu N_i}{T}\right]^{\frac{1}{1-q}}$$

C. Tsallis et al., Physica A 261 (1998) 534

$q \to 1/q$

q-dual statistics

$$S = q\sum_i \frac{p_i^{1/q} - p_i}{q-1}$$ - q-dual entropy

$$\sum_i p_i = 1$$

$$\langle A \rangle = \sum_i p_i A_i, \qquad \langle 1 \rangle = 1$$

$$p_i = \left[1 + (1-q)\frac{\Lambda - E_i + \mu N_i}{T}\right]^{\frac{q}{1-q}}$$

$$\sum_i \left[1 + (1-q)\frac{\Lambda - E_i + \mu N_i}{T}\right]^{\frac{q}{1-q}} = 1$$

A.S.P., Eur. Phys. J. A 56 (2020) 4, 106

A.S.P., Eur. Phys. J. A 51 (2015) 108; A.S.P., Eur. Phys. J. A 53 (2017) 53

The Tsallis-2 statistics is inconsistent since the mean value of unity is not equal to unity $\langle 1 \rangle \neq 1$. The Tsallis-1 statistics and q-dual statistics are correctly defined and consistent.

Tsallis-1 statistics in the grand canonical ensemble: General formalism

The statistical averages of the operators in the Tsallis-1 statistics:

$$\langle A\rangle = \frac{1}{\Gamma\left(\frac{1}{1-q}\right)}\int_0^\infty t^{\frac{q}{1-q}} e^{-t\left[1+\frac{q-1}{q}\frac{\Lambda-\Omega_G(\beta')}{T}\right]}\langle A\rangle_G(\beta')dt = \sum_{n=0}^{\infty}\frac{1}{n!\Gamma\left(\frac{1}{1-q}\right)}\int_0^\infty t^{\frac{q}{1-q}} e^{-t\left[1+\frac{q-1}{q}\frac{\Lambda}{T}\right]}\left(-\beta'\Omega_G(\beta')\right)^n\langle A\rangle_G(\beta')dt \qquad q<1$$

$$\langle A\rangle = \Gamma\left(\frac{q}{q-1}\right)\frac{i}{2\pi}\oint_C (-t)^{\frac{q}{1-q}} e^{-t\left[1+\frac{q-1}{q}\frac{\Lambda-\Omega_G(\beta')}{T}\right]}\langle A\rangle_G(\beta')dt = \sum_{n=0}^{\infty}\frac{1}{n!}\Gamma\left(\frac{q}{q-1}\right)\frac{i}{2\pi}\oint_C (-t)^{\frac{q}{1-q}} e^{-t\left[1+\frac{q-1}{q}\frac{\Lambda}{T}\right]}\left(-\beta'\Omega_G(\beta')\right)^n\langle A\rangle_G(\beta')dt \qquad q>1$$

Series expansion:

$$e^{-\beta'\Omega_G(\beta')} = \sum_{n=0}^{\infty}\frac{1}{n!}\left(-\beta'\Omega_G(\beta')\right)^n$$

A.S.P., Eur. Phys. J. A 53 (2017) 53;
A.S.P., T.Bhattacharyya, Eur. Phys. J. A 56 (2020) 72

The Boltzmann-Gibbs quantities:

$$\Omega_G(\beta') = -\frac{1}{\beta'}\ln Z_G(\beta'), \quad Z_G(\beta') = \sum_i e^{-\beta'(E_i-\mu N_i)}, \quad \langle A\rangle_G(\beta') = \frac{1}{Z_G(\beta')}\sum_i A_i e^{-\beta'(E_i-\mu N_i)}, \quad \beta' = \frac{t(1-q)}{qT}$$

Norm function Λ in the Tsallis-1 statistics:

$$\frac{1}{\Gamma\left(\frac{1}{1-q}\right)}\int_0^\infty t^{\frac{q}{1-q}} e^{-t\left[1+\frac{q-1}{q}\frac{\Lambda-\Omega_G(\beta')}{T}\right]}dt = \sum_{n=0}^{\infty}\frac{1}{n!\Gamma\left(\frac{1}{1-q}\right)}\int_0^\infty t^{\frac{q}{1-q}} e^{-t\left[1+\frac{q-1}{q}\frac{\Lambda}{T}\right]}\left(-\beta'\Omega_G(\beta')\right)^n dt = 1 \qquad q<1$$

$$\Gamma\left(\frac{q}{q-1}\right)\frac{i}{2\pi}\oint_C (-t)^{\frac{q}{1-q}} e^{-t\left[1+\frac{q-1}{q}\frac{\Lambda-\Omega_G(\beta')}{T}\right]}dt = \sum_{n=0}^{\infty}\frac{1}{n!}\Gamma\left(\frac{q}{q-1}\right)\frac{i}{2\pi}\oint_C (-t)^{\frac{q}{1-q}} e^{-t\left[1+\frac{q-1}{q}\frac{\Lambda}{T}\right]}\left(-\beta'\Omega_G(\beta')\right)^n dt = 1 \qquad q>1$$

p_T - distribution in Tsallis-1 statistics: Relativistic ideal gas

The relativistic transverse momentum distribution for the Tsallis-1 statistics in grand canonical ensemble ($\eta=-1$ -B-E statistics, $\eta=1$ -F-D statistics, $\eta=0$ - M-B statistics)

$$\frac{d^2N}{dp_T dy}=\frac{gV}{(2\pi)^2}p_T m_T \cosh y\sum_{n=0}^{\infty}\frac{1}{n!\Gamma\left(\frac{1}{1-q}\right)}\int_0^{\infty} t^{\frac{q}{1-q}}e^{-t\left[1+\frac{q-1}{q}\frac{\Lambda}{T}\right]}\frac{\left(-\beta'\Omega_G(\beta')\right)^n}{e^{\beta'(m_T\cosh y-\mu)}+\eta}dt \qquad q<1$$

$$\frac{d^2N}{dp_T dy}=\frac{gV}{(2\pi)^2}p_T m_T \cosh y\sum_{n=0}^{\infty}\frac{\Gamma\left(\frac{q}{q-1}\right)}{n!}\frac{i}{2\pi}\oint_C (-t)^{\frac{q}{1-q}}e^{-t\left[1+\frac{q-1}{q}\frac{\Lambda}{T}\right]}\frac{\left(-\beta'\Omega_G(\beta')\right)^n}{e^{\beta'(m_T\cosh y-\mu)}+\eta}dt \qquad q>1$$

A.S.P., T.Bhattacharyya, Eur. Phys. J. A 56 (2020) 72

Norm function Λ:

$$\sum_{n=0}^{\infty}\frac{1}{n!\Gamma\left(\frac{1}{1-q}\right)}\int_0^{\infty} t^{\frac{q}{1-q}}e^{-t\left[1+\frac{q-1}{q}\frac{\Lambda}{T}\right]}\left(-\beta'\Omega_G(\beta')\right)^n dt=1 \qquad q<1$$

$$\sum_{n=0}^{\infty}\frac{1}{n!}\Gamma\left(\frac{q}{q-1}\right)\frac{i}{2\pi}\oint_C (-t)^{\frac{q}{1-q}}e^{-t\left[1+\frac{q-1}{q}\frac{\Lambda}{T}\right]}\left(-\beta'\Omega_G(\beta')\right)^n dt=1 \qquad q>1$$

$$-\beta'\Omega_G(\beta')=\sum_{\mathbf{p},\sigma}\ln\left[1+\eta e^{-\beta'(\varepsilon_{\mathbf{p}}-\mu)}\right]^{\frac{1}{\eta}}$$

$$\varepsilon_{\mathbf{p}}=\sqrt{\mathbf{p}^2+m^2}=m_T\cosh y$$

$$m_T=\sqrt{p_T^2+m^2}$$

$$\beta'=\frac{t(1-q)}{qT}$$

A.S.P, EPJ Web Conf. 138 (2017) 03008

p_T - distribution in Tsallis-1 statistics: Maxwell-Boltzmann statistics of particles $\eta = 0$

Relativistic transverse momentum distribution for the Tsallis-1 statistics in grand canonical ensemble for Maxwell-Boltzmann statistics of particles:

$$\frac{d^2N}{dp_T dy} = \frac{gV}{(2\pi)^2} p_T m_T \cosh y \sum_{n=0}^{\infty} \frac{\omega^n}{n!} \frac{1}{\Gamma\left(\frac{1}{1-q}\right)} \int_0^{\infty} t^{\frac{q}{1-q}-n} e^{-t\left[1+\frac{q-1}{q}\frac{\Lambda - m_T\cosh y + \mu(n+1)}{T}\right]} \left(K_2\left(\frac{t(1-q)m}{qT}\right)\right)^n dt \qquad q<1$$

$$\frac{d^2N}{dp_T dy} = \frac{gV}{(2\pi)^2} p_T m_T \cosh y \sum_{n=0}^{\infty} \frac{(-\omega)^n}{n!} \Gamma\left(\frac{q}{q-1}\right) \frac{i}{2\pi} \oint_C (-t)^{\frac{q}{1-q}-n} e^{-t\left[1+\frac{q-1}{q}\frac{\Lambda - m_T\cosh y + \mu(n+1)}{T}\right]} \left(K_2\left(\frac{t(1-q)m}{qT}\right)\right)^n dt \qquad q>1$$

$$\Omega_G(\beta') = -\frac{gV}{2\pi^2}\frac{m^2}{\beta'^2} e^{\beta'\mu} K_2(\beta' m), \qquad \omega = \frac{gVTm^2}{2\pi^2}\frac{q}{1-q}$$

Norm function Λ:

A.S.P., T.Bhattacharyya, Eur. Phys. J. A 56 (2020) 72

$$\sum_{n=0}^{\infty} \frac{\omega^n}{n!} \frac{1}{\Gamma\left(\frac{1}{1-q}\right)} \int_0^{\infty} t^{\frac{q}{1-q}-n} e^{-t\left[1+\frac{q-1}{q}\frac{\Lambda + \mu n}{T}\right]} \left(K_2\left(\frac{t(1-q)m}{qT}\right)\right)^n dt = 1 \qquad q<1$$

$$\sum_{n=0}^{\infty} \frac{(-\omega)^n}{n!} \Gamma\left(\frac{q}{q-1}\right) \frac{i}{2\pi} \oint_C (-t)^{\frac{q}{1-q}-n} e^{-t\left[1+\frac{q-1}{q}\frac{\Lambda + \mu n}{T}\right]} \left(K_2\left(\frac{t(1-q)m}{qT}\right)\right)^n dt = 1 \qquad q>1$$

p_T - distribution in Tsallis-1 statistics: Zeroth term approximation $(n=0)$

The relativistic transverse momentum distribution for the Tsallis-1 statistics in zeroth term approximation ($\eta=-1$ -B-E statistics, $\eta=1$ -F-D statistics, $\eta=0$ - M-B statistics)

$$\frac{d^2N}{dp_T dy} = \frac{gV}{(2\pi)^2} p_T m_T \cosh y \sum_{k=0}^{\infty} (-\eta)^k \left[1+(k+1)\frac{1-q}{q}\frac{m_T \cosh y - \mu}{T}\right]^{\frac{1}{q-1}} \qquad \eta=-1,0,1$$

A.S.P., T.Bhattacharyya, Eur. Phys. J. A 56 (2020) 72

The relativistic transverse momentum distribution for the Tsallis-1 statistics in zeroth term approximation $(n=0)$ for Maxwell-Boltzmann statistics of particles:

$$\frac{d^2N}{dp_T dy} = \frac{gV}{(2\pi)^2} p_T m_T \cosh y \left[1+\frac{1-q}{q}\frac{m_T \cosh y - \mu}{T}\right]^{\frac{1}{q-1}} \qquad \eta=0$$

A.S.P., Eur. Phys. J. A 53 (2017) 53;
A.S.P., Eur. Phys. J. A 52 (2016) 355;
A.S.P., T.Bhattacharyya, Eur. Phys. J. A 56 (2020) 72

The transverse momentum distribution for the Tsallis-1 statistics in the zeroth term approximation for Maxwell-Boltzmann statistics of particles exactly recovers the phenomenological Tsallis distribution [**J. Cleymans, D. Worku, Eur. Phys. J. A 48 (2012) 160**] under the transformation $(q \to 1/q)$

Zeroth term approximation for Tsallis statistics was introduced in: **A.S.P., Eur. Phys. J. A 53 (2017) 53; A.S.P., Eur. Phys. J. A 52 (2016) 355**

Ultrarelativistic p_T distribution in Tsallis-1 statistics for Maxwell-Boltzmann statistics of particles

Ultrarelativistic transverse momentum distribution for the Tsallis-1 statistics in grand canonical ensemble for Maxwell-Boltzmann statistics of particles: $(m=0)$

$$\frac{d^2N}{dp_T dy} = \frac{gV}{(2\pi)^2} p_T^2 \cosh y \sum_{N=0}^{N_0} \frac{\tilde{\omega}^N}{N!} h_0(0) \left[1 + \frac{q-1}{q} \frac{\Lambda - p_T \cosh y + \mu(N+1)}{T}\right]^{\frac{1}{q-1}+3N}$$

Norm function Λ:

$$\sum_{N=0}^{N_0} \frac{\tilde{\omega}^N}{N!} h_0(0) \left[1 + \frac{q-1}{q} \frac{\Lambda + \mu N}{T}\right]^{\frac{1}{q-1}+3N} = 1, \qquad \tilde{\omega} = \frac{gVT^3}{\pi^2}$$

A.S.P., Eur. Phys. J. A 53 (2017) 53;
A.S.P., Eur. Phys. J. A 52 (2016) 355

$$h_\eta(\xi) = \frac{\left(\frac{q}{1-q}\right)^{3(N+\eta)} \Gamma\left(\frac{1}{1-q} - \xi - 3(N+\eta)\right)}{\Gamma\left(\frac{1}{1-q} - \xi\right)} \quad q<1, \qquad h_\eta(\xi) = \frac{\left(\frac{q}{q-1}\right)^{3(N+\eta)} \Gamma\left(\frac{q}{q-1} + \xi\right)}{\Gamma\left(\frac{q}{q-1} + \xi + 3(N+\eta)\right)} \quad q>1$$

Ultrarelativistic zeroth term $(n=0)$ approximation:

$$\frac{d^2N}{dp_T dy} = \frac{gVp_T^2 \cosh y}{(2\pi)^2} \left[1 - \frac{q-1}{q} \frac{p_T \cosh y - \mu}{T}\right]^{\frac{1}{q-1}}, \qquad m_T = p_T, \qquad m=0$$

The ultrarelativistic transverse momentum distribution for the Tsallis-1 statistics in the zeroth term approximation for Maxwell-Boltzmann statistics of particles exactly recovers the ultrarelativistic phenomenological Tsallis transverse momentum distribution under the transformation $(q \to 1/q)$

Tsallis-2 statistics in the grand canonical ensemble: General formalism

The statistical averages of the operators in the Tsallis-2 statistics:

$$\langle A\rangle = \frac{1}{Z^q}\frac{1}{\Gamma\left(\frac{q}{q-1}\right)}\int_0^\infty t^{\frac{1}{q-1}} e^{-t\left[1-(1-q)\frac{\Omega_G(\beta')}{T}\right]}\langle A\rangle_G(\beta')\,dt = \sum_{n=0}^{\infty}\frac{1}{n!Z^q\Gamma\left(\frac{q}{q-1}\right)}\int_0^\infty t^{\frac{1}{q-1}} e^{-t}\left(-\beta'\Omega_G(\beta')\right)^n\langle A\rangle_G(\beta')\,dt \qquad q>1$$

$$\langle A\rangle = \frac{1}{Z^q}\Gamma\left(\frac{1}{1-q}\right)\frac{i}{2\pi}\oint_C (-t)^{\frac{1}{q-1}} e^{-t\left[1-(1-q)\frac{\Omega_G(\beta')}{T}\right]}\langle A\rangle_G(\beta')\,dt = \sum_{n=0}^{\infty}\frac{1}{n!Z^q}\Gamma\left(\frac{1}{1-q}\right)\frac{i}{2\pi}\oint_C (-t)^{\frac{1}{q-1}} e^{-t}\left(-\beta'\Omega_G(\beta')\right)^n\langle A\rangle_G(\beta')\,dt \qquad q<1$$

Series expansion:

$$e^{-\beta'\Omega_G(\beta')} = \sum_{n=0}^{\infty}\frac{1}{n!}\left(-\beta'\Omega_G(\beta')\right)^n$$

A.S.P., Eur. Phys. J. A 53 (2017) 53;
A.S.P., T.Bhattacharyya, Eur. Phys. J. A 56 (2020) 72

The Boltzmann-Gibbs quantities:

$$\Omega_G(\beta') = -\frac{1}{\beta'}\ln Z_G(\beta'),\quad Z_G(\beta') = \sum_i e^{-\beta'(E_i-\mu N_i)},\quad \langle A\rangle_G(\beta') = \frac{1}{Z_G(\beta')}\sum_i A_i e^{-\beta'(E_i-\mu N_i)},\quad \beta' = \frac{t(q-1)}{T}$$

Norm function Z in the Tsallis-2 statistics:

$$Z = \frac{1}{\Gamma\left(\frac{1}{q-1}\right)}\int_0^\infty t^{\frac{1}{q-1}-1} e^{-t\left[1-(1-q)\frac{\Omega_G(\beta')}{T}\right]}dt = \sum_{n=0}^{\infty}\frac{1}{n!\Gamma\left(\frac{1}{q-1}\right)}\int_0^\infty t^{\frac{1}{q-1}-1} e^{-t}\left(-\beta'\Omega_G(\beta')\right)^n dt \qquad q>1$$

$$Z = \Gamma\left(\frac{2-q}{1-q}\right)\frac{i}{2\pi}\oint_C (-t)^{\frac{1}{q-1}-1} e^{-t\left[1-(1-q)\frac{\Omega_G(\beta')}{T}\right]}dt = \sum_{n=0}^{\infty}\frac{1}{n!}\Gamma\left(\frac{2-q}{1-q}\right)\frac{i}{2\pi}\oint_C (-t)^{\frac{1}{q-1}-1} e^{-t}\left(-\beta'\Omega_G(\beta')\right)^n dt \qquad q<1$$

p_T - distribution in Tsallis-2 statistics: Relativistic ideal gas

The relativistic transverse momentum distribution for the Tsallis-2 statistics in grand canonical ensemble ($\eta=-1$ -B-E statistics, $\eta=1$ -F-D statistics, $\eta=0$ - M-B statistics)

$$\frac{d^2N}{dp_T dy} = \frac{gV}{(2\pi)^2} p_T m_T \cosh y \sum_{n=0}^{\infty} \frac{1}{n! Z^q \Gamma\left(\frac{q}{q-1}\right)} \int_0^{\infty} t^{\frac{1}{q-1}} e^{-t} \frac{\left(-\beta'\Omega_G(\beta')\right)^n}{e^{\beta'(m_T \cosh y - \mu)} + \eta} dt \qquad q>1$$

$$\frac{d^2N}{dp_T dy} = \frac{gV}{(2\pi)^2} p_T m_T \cosh y \sum_{n=0}^{\infty} \frac{1}{n! Z^q} \Gamma\left(\frac{1}{1-q}\right) \frac{i}{2\pi} \oint_C (-t)^{\frac{1}{q-1}} e^{-t} \frac{\left(-\beta'\Omega_G(\beta')\right)^n}{e^{\beta'(m_T \cosh y - \mu)} + \eta} dt \qquad q<1$$

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Norm function Z in the Tsallis-2 statistics:

$$Z = \sum_{n=0}^{\infty} \frac{1}{n! \Gamma\left(\frac{1}{q-1}\right)} \int_0^{\infty} t^{\frac{1}{q-1}-1} e^{-t} \left(-\beta'\Omega_G(\beta')\right)^n dt \qquad q>1$$

$$Z = \sum_{n=0}^{\infty} \frac{1}{n!} \Gamma\left(\frac{2-q}{1-q}\right) \frac{i}{2\pi} \oint_C (-t)^{\frac{1}{q-1}-1} e^{-t} \left(-\beta'\Omega_G(\beta')\right)^n dt \qquad q<1$$

$$-\beta'\Omega_G(\beta') = \sum_{\mathbf{p},\sigma} \ln\left[1+\eta e^{-\beta'(\varepsilon_{\mathbf{p}}-\mu)}\right]^{\frac{1}{\eta}}$$

$$\varepsilon_{\mathbf{p}} = \sqrt{\mathbf{p}^2+m^2} = m_T \cosh y$$

$$m_T = \sqrt{p_T^2+m^2}$$

$$\beta' = \frac{t(q-1)}{T}$$

A.S.P, EPJ Web Conf. 138 (2017) 03008

p_T - distribution in Tsallis-2 statistics: Maxwell-Boltzmann statistics of particles $\eta = 0$

Relativistic transverse momentum distribution for the Tsallis-2 statistics in grand canonical ensemble for Maxwell-Boltzmann statistics of particles:

$$\frac{d^2N}{dp_T dy} = \frac{gV}{(2\pi)^2} p_T m_T \cosh y \sum_{n=0}^{\infty} \frac{\omega^n}{n!} \frac{1}{Z^q} \frac{1}{\Gamma\left(\frac{q}{q-1}\right)} \int_0^{\infty} t^{\frac{1}{q-1}-n} e^{-t\left[1-(1-q)\frac{m_T \cosh y - \mu(n+1)}{T}\right]} \left(K_2\left(\frac{t(q-1)m}{T}\right)\right)^n dt \qquad q>1$$

$$\frac{d^2N}{dp_T dy} = \frac{gV}{(2\pi)^2} p_T m_T \cosh y \sum_{n=0}^{\infty} \frac{(-\omega)^n}{n!} \frac{1}{Z^q} \Gamma\left(\frac{1}{1-q}\right) \frac{i}{2\pi} \oint_C (-t)^{\frac{1}{q-1}-n} e^{-t\left[1-(1-q)\frac{m_T \cosh y - \mu(n+1)}{T}\right]} \left(K_2\left(\frac{t(q-1)m}{T}\right)\right)^n dt \qquad q<1$$

$$\Omega_G(\beta') = -\frac{gV}{2\pi^2} \frac{m^2}{\beta'^2} e^{\beta'\mu} K_2(\beta' m), \qquad \omega = \frac{gVTm^2}{2\pi^2} \frac{1}{q-1}$$

Norm function Z in the Tsallis-2 statistics:

A.S.P., T.Bhattacharyya, Eur. Phys. J. A 56 (2020) 72

$$Z = \sum_{n=0}^{\infty} \frac{\omega^n}{n!} \frac{1}{\Gamma\left(\frac{1}{q-1}\right)} \int_0^{\infty} t^{\frac{1}{q-1}-1-n} e^{-t\left[1+(1-q)\frac{\mu n}{T}\right]} \left(K_2\left(\frac{t(q-1)m}{T}\right)\right)^n dt \qquad q>1$$

$$Z = \sum_{n=0}^{\infty} \frac{(-\omega)^n}{n!} \Gamma\left(\frac{2-q}{1-q}\right) \frac{i}{2\pi} \oint_C (-t)^{\frac{1}{q-1}-1-n} e^{-t\left[1+(1-q)\frac{\mu n}{T}\right]} \left(K_2\left(\frac{t(q-1)m}{T}\right)\right)^n dt \qquad q<1$$

p_T - distribution in Tsallis-2 statistics: Zeroth term approximation $(n=0)$

The relativistic transverse momentum distribution for the Tsallis-2 statistics in zeroth term approximation ($\eta=-1$ -B-E statistics, $\eta=1$ -F-D statistics, $\eta=0$ - M-B statistics)

$$\frac{d^2N}{dp_T dy} = \frac{gV}{(2\pi)^2} p_T m_T \cosh y \sum_{k=0}^{\infty} (-\eta)^k \left[1+(k+1)(q-1)\frac{m_T \cosh y - \mu}{T}\right]^{\frac{q}{1-q}} \qquad \eta=-1,0,1$$

A.S.P., T.Bhattacharyya, Eur. Phys. J. A 56 (2020) 72

The relativistic transverse momentum distribution for the Tsallis-2 statistics in zeroth term approximation $(n=0)$ for Maxwell-Boltzmann statistics of particles:

$$\frac{d^2N}{dp_T dy} = \frac{gV}{(2\pi)^2} p_T m_T \cosh y \left[1+(q-1)\frac{m_T \cosh y - \mu}{T}\right]^{\frac{q}{1-q}} \qquad \eta=0$$

A.S.P., Eur. Phys. J. A 53 (2017) 53;
A.S.P., Eur. Phys. J. A 52 (2016) 355;
A.S.P., T.Bhattacharyya, Eur. Phys. J. A 56 (2020) 72

The phenomenological Tsallis distribution [**J. Cleymans, D. Worku, Eur. Phys. J. A 48 (2012) 160**] coincides exactly with the transverse momentum distribution for the Tsallis-2 statistics in the zeroth term approximation for Maxwell-Boltzmann statistics of particles.

Thus, the phenomenological Tsallis distribution is inconsistent in the case of the Tsallis statistics since the Tsallis-2 statistics is incorrect, $\langle 1 \rangle \neq 1$!

Ultrarelativistic p_T - distribution in Tsallis-2 statistics: Maxwell-Boltzmann statistics of particles

Ultrarelativistic transverse momentum distribution for the Tsallis-2 statistics in grand canonical ensemble for Maxwell-Boltzmann statistics of particles: $(m=0)$

$$\frac{d^2N}{dp_T dy} = \frac{gV}{(2\pi)^2} p_T^2 \cosh y \frac{1}{Z^q} \sum_{N=0}^{N_0} \frac{\tilde{\omega}^N}{N!} a_0(1) \left[1+(q-1)\frac{p_T \cosh y - \mu(N+1)}{T}\right]^{\frac{q}{1-q}+3N} \qquad q>1$$

Norm function Z:

$$Z = \sum_{N=0}^{N_0} \frac{\tilde{\omega}^N}{N!} a_0(0) \left[1+(1-q)\frac{\mu N}{T}\right]^{\frac{1}{1-q}+3N}, \qquad \tilde{\omega} = \frac{gVT^3}{\pi^2}$$

A.S.P., Eur. Phys. J. A 53 (2017) 53;
A.S.P., Eur. Phys. J. A 52 (2016) 355

$$a_\eta(\xi) = \frac{\Gamma\left(\frac{1}{q-1}+\xi-3(N+\eta)\right)}{(q-1)^{3(N+\eta)}\Gamma\left(\frac{1}{q-1}+\xi\right)} \qquad q>1$$

Ultrarelativistic zeroth term $(n=0)$ approximation:

$$\frac{d^2N}{dp_T dy} = \frac{gVp_T^2 \cosh y}{(2\pi)^2}\left[1+(q-1)\frac{p_T \cosh y - \mu}{T}\right]^{\frac{q}{1-q}}, \qquad m_T = p_T, \qquad m=0$$

The ultrarelativistic phenomenological Tsallis distribution [**J. Cleymans, D. Worku, Eur. Phys. J. A 48 (2012) 160**] coincides exactly with the ultrarelativistic transverse momentum distribution for the Tsallis-2 statistics in the zeroth term approximation for Maxwell-Boltzmann statistics of particles.

Thus, the ultrarelativistic phenomenological Tsallis distribution is inconsistent in the case of the Tsallis statistics since the Tsallis-2 statistics is incorrect, $\langle 1 \rangle \neq 1$!

q-dual statistics in the grand canonical ensemble: General formalism $q>1$

The statistical averages of the operators in the q-dual statistics:

$$\langle A\rangle = \frac{1}{\Gamma\left(\frac{q}{q-1}\right)}\int_0^\infty t^{\frac{1}{q-1}} e^{-t\left[1+(1-q)\frac{\Lambda-\Omega_G(\beta')}{T}\right]}\langle A\rangle_G(\beta')\,dt = \sum_{n=0}^{\infty}\frac{1}{n!\,\Gamma\left(\frac{q}{q-1}\right)}\int_0^\infty t^{\frac{1}{q-1}} e^{-t\left[1+(1-q)\frac{\Lambda}{T}\right]}\left(-\beta'\Omega_G(\beta')\right)^n\langle A\rangle_G(\beta')\,dt$$

Series expansion:

$$e^{-\beta'\Omega_G(\beta')} = \sum_{n=0}^{\infty}\frac{1}{n!}\left(-\beta'\Omega_G(\beta')\right)^n$$

A.S.P., Eur. Phys. J. A 56 (2020) 106

The Boltzmann-Gibbs quantities:

$$\Omega_G(\beta') = -\frac{1}{\beta'}\ln Z_G(\beta'),\quad Z_G(\beta') = \sum_i e^{-\beta'(E_i-\mu N_i)},\quad \langle A\rangle_G(\beta') = \frac{1}{Z_G(\beta')}\sum_i A_i e^{-\beta'(E_i-\mu N_i)},\quad \beta' = \frac{t(q-1)}{T}$$

Norm function Λ in the q-dual statistics:

$$\frac{1}{\Gamma\left(\frac{q}{q-1}\right)}\int_0^\infty t^{\frac{1}{q-1}} e^{-t\left[1+(1-q)\frac{\Lambda-\Omega_G(\beta')}{T}\right]}dt = \sum_{n=0}^{\infty}\frac{1}{n!\,\Gamma\left(\frac{q}{q-1}\right)}\int_0^\infty t^{\frac{1}{q-1}} e^{-t\left[1+(1-q)\frac{\Lambda}{T}\right]}\left(-\beta'\Omega_G(\beta')\right)^n dt = 1$$

p_T - distribution in q-dual statistics: Relativistic ideal gas $q>1$

The relativistic transverse momentum distribution for the q-dual statistics in grand canonical ensemble ($\eta=-1$ -B-E statistics, $\eta=1$ -F-D statistics, $\eta=0$ - M-B statistics)

$$\frac{d^2N}{dp_T dy} = \frac{gV}{(2\pi)^2} p_T m_T \cosh y \sum_{n=0}^{\infty} \frac{1}{n!\Gamma\left(\frac{q}{q-1}\right)} \int_0^{\infty} t^{\frac{1}{q-1}} e^{-t\left[1+(1-q)\frac{\Lambda}{T}\right]} \frac{\left(-\beta'\Omega_G(\beta')\right)^n}{e^{\beta'(m_T \cosh y-\mu)}+\eta} dt$$

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Norm function Λ in the q-dual statistics:

$$\sum_{n=0}^{\infty} \frac{1}{n!\Gamma\left(\frac{q}{q-1}\right)} \int_0^{\infty} t^{\frac{1}{q-1}} e^{-t\left[1+(1-q)\frac{\Lambda}{T}\right]} \left(-\beta'\Omega_G(\beta')\right)^n dt = 1$$

$$-\beta'\Omega_G(\beta') = \sum_{\mathbf{p},\sigma} \ln\left[1+\eta e^{-\beta'(\varepsilon_{\mathbf{p}}-\mu)}\right]^{\frac{1}{\eta}}$$

$$\varepsilon_{\mathbf{p}} = \sqrt{\mathbf{p}^2+m^2} = m_T \cosh y$$

$$m_T = \sqrt{p_T^2+m^2}$$

$$\beta' = \frac{t(q-1)}{T}$$

p_T - distribution in q-dual statistics: Maxwell-Boltzmann statistics of particles $\eta = 0$

Relativistic transverse momentum distribution for the Tsallis-1 statistics in grand canonical ensemble for Maxwell-Boltzmann statistics of particles: $q > 1$

$$\frac{d^2N}{dp_T dy} = \frac{gV}{(2\pi)^2} p_T m_T \cosh y \sum_{n=0}^{\infty} \frac{\omega^n}{n!} \frac{1}{\Gamma\left(\frac{q}{q-1}\right)} \int_0^{\infty} t^{\frac{1}{q-1}-n} e^{-t\left[1+(1-q)\frac{\Lambda - m_T \cosh y + \mu(n+1)}{T}\right]} \left(K_2\left(\frac{t(q-1)m}{T}\right)\right)^n dt$$

$$\Omega_G(\beta') = -\frac{gV}{2\pi^2} \frac{m^2}{\beta'^2} e^{\beta'\mu} K_2(\beta' m), \qquad \omega = \frac{gVTm^2}{2\pi^2} \frac{1}{q-1}$$

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Norm function Λ:

$$\sum_{n=0}^{\infty} \frac{\omega^n}{n!} \frac{1}{\Gamma\left(\frac{q}{q-1}\right)} \int_0^{\infty} t^{\frac{1}{q-1}-n} e^{-t\left[1+(1-q)\frac{\Lambda + \mu n}{T}\right]} \left(K_2\left(\frac{t(q-1)m}{T}\right)\right)^n dt = 1$$

p_T - distribution in *q*-dual statistics: Zeroth term approximation $(n=0)$

The relativistic transverse momentum distribution for the q-dual statistics in zeroth term approximation ($\eta=-1$ -B-E statistics, $\eta=1$-F-D statistics, $\eta=0$ - M-B statistics)

$$\frac{d^2N}{dp_T dy} = \frac{gV}{(2\pi)^2} p_T m_T \cosh y \sum_{k=0}^{\infty} (-\eta)^k \left[1-(k+1)(1-q)\frac{m_T \cosh y - \mu}{T}\right]^{\frac{q}{1-q}} \qquad \eta=-1,0,1$$

A.S.P., Eur. Phys. J. A 56 (2020) 106

The relativistic transverse momentum distribution for the q-dual statistics in zeroth term approximation $(n=0)$ for Maxwell-Boltzmann statistics of particles:

$$\frac{d^2N}{dp_T dy} = \frac{gV}{(2\pi)^2} p_T m_T \cosh y \left[1-(1-q)\frac{m_T \cosh y - \mu}{T}\right]^{\frac{q}{1-q}} \qquad \eta=0$$

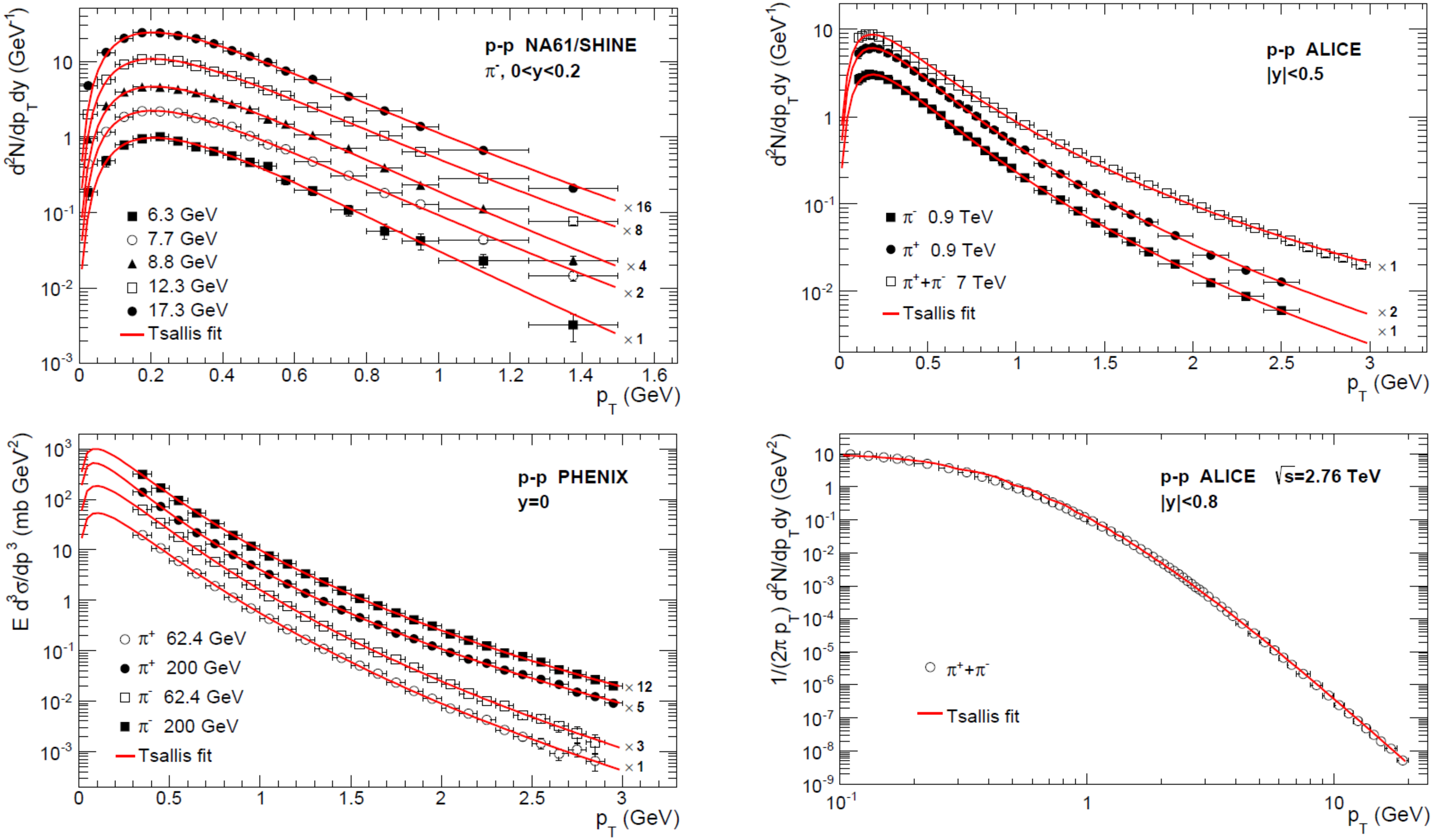
<u>The phenomenological Tsallis distribution</u> [**J. Cleymans, D. Worku, Eur. Phys. J. A 48 (2012) 160**] coincides exactly with the transverse momentum distribution of the q-dual statistics in the zeroth term approximation for Maxwell-Boltzmann statistics of particles.

Thus, the phenomenological Tsallis distribution is consistent in the case of the q-dual statistics since the q-dual statistics is correctly defined, $\langle 1\rangle = 1$!

Comparison of exact p_T - distribution of the Tsallis-1 statistics with experimental data

Charged pions (*pp* collisions): Ultrarelativistic transverse momentum distribution

A.S.P., Eur. Phys. J. A 52 (2016) 355



- Transverse momentum distributions of charged pions produced in *pp* collisions at SPS, RHIC and LHC energies
- The yields were integrated in the experimental rapidity interval $y_0 \le y \le y_1$
- The solid curves are the fits of the experimental data to the **ultrarelativistic (*m*=0) transverse momentum distributions** of Tsallis-1 statistics
- The curves are the same for all statistics but only the parameters are different.

Experimental Data: **NA61/SHINE, EPJC 74 (2014) 2794; PHENIX, PRC 83 (2011) 064903; ALICE, EPJC 71 (2011) 1655; ALICE, EPJC 75 (2015) 226; ALICE, PLB 736 (2014) 196**

Exact Tsallis-1 distribution

Table 1. Parameters of the Tsallis statistics fit for the pions produced in pp collisions at different energies.

Collaboration	Type	$\sqrt{s}$, GeV	T, MeV	R, fm	q	χ^2/ndf
NA61/SHINE	π^-	6.3	85.78 ± 10.79	4.047 ± 0.235	0.9623 ± 0.0142	2.821/15
NA61/SHINE	π^-	7.7	79.05 ± 8.01	4.304 ± 0.204	0.9505 ± 0.0107	1.472/15
NA61/SHINE	π^-	8.8	82.01 ± 9.28	4.294 ± 0.212	0.9542 ± 0.0123	1.821/15
NA61/SHINE	π^-	12.3	75.47 ± 7.41	4.627 ± 0.253	0.9451 ± 0.0083	1.152/15
NA61/SHINE	π^-	17.3	95.83 ± 6.38	4.798 ± 0.246	0.9326 ± 0.0166	0.865/15
PHENIX	π^+	62.4	97.62 ± 11.92	3.744 ± 0.648	0.9197 ± 0.0093	1.654/23
PHENIX	π^-	62.4	93.76 ± 11.69	3.971 ± 0.716	0.9184 ± 0.0091	0.878/23
PHENIX	π^+	200.0	79.89 ± 11.80	4.247 ± 0.899	0.8894 ± 0.0082	0.987/24
PHENIX	π^-	200.0	87.20 ± 11.48	3.823 ± 0.714	0.8965 ± 0.0081	0.691/24
ALICE	π^+	900.0	82.72 ± 2.01	3.965 ± 0.069	0.8766 ± 0.0037	3.609/30
ALICE	π^-	900.0	83.92 ± 2.02	3.918 ± 0.068	0.8790 ± 0.0036	1.610/30
ALICE	$\pi^+ + \pi^-$	2760.0	90.61 ± 1.45	3.496 ± 0.057	0.8726 ± 0.0012	12.18/60
ALICE	$\pi^+ + \pi^-$	7000.0	78.75 ± 1.86	4.606 ± 0.093	0.8533 ± 0.0024	9.775/38

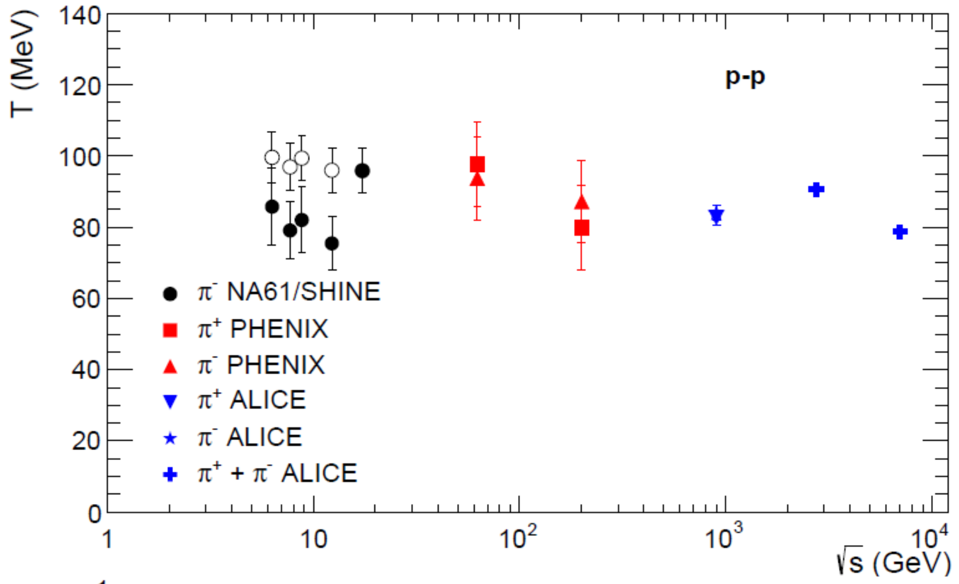
Table 2. Parameters of the fit by the distribution function of the Tsallis-factorized statistics for the charged pions produced in pp collisions at different energies.

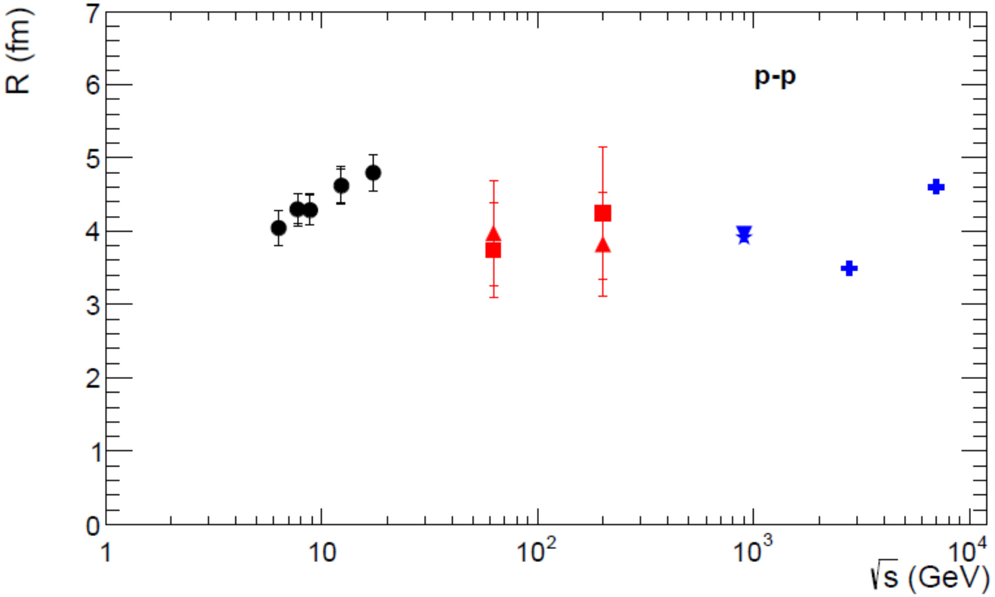
Phenomenoligical Tsallis distribution

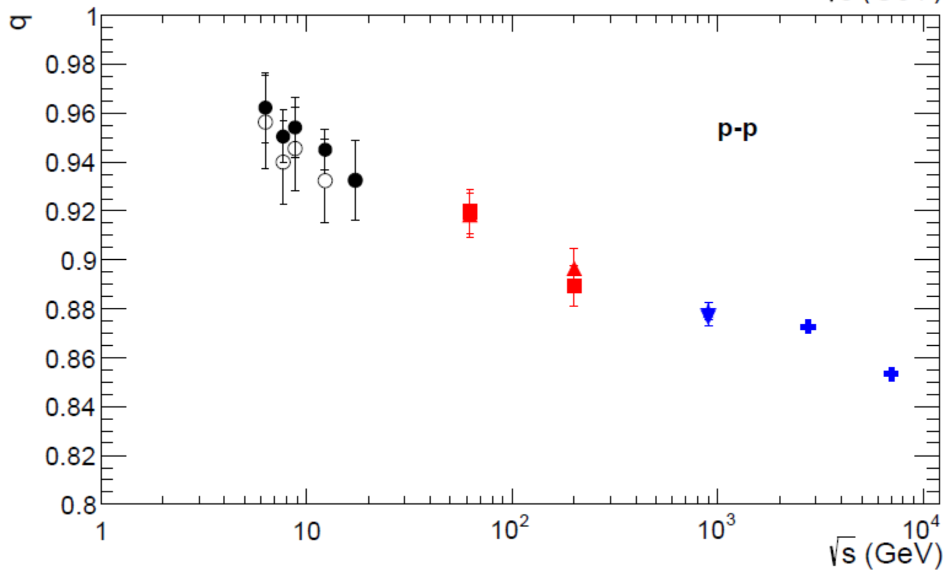
Collaboration	Type	$\sqrt{s}$, GeV	T, MeV	R, fm	$q = 1/q_c$	q_c	χ^2/ndf
NA61/SHINE	π^-	6.3	99.59 ± 7.32	4.045 ± 0.234	0.9563 ± 0.0190	1.0457 ± 0.0208	2.825/15
NA61/SHINE	π^-	7.7	96.93 ± 6.49	4.300 ± 0.222	0.9400 ± 0.0171	1.0638 ± 0.0194	1.481/15
NA61/SHINE	π^-	8.8	99.37 ± 6.29	4.290 ± 0.204	0.9455 ± 0.0172	1.0576 ± 0.0193	1.838/15
NA61/SHINE	π^-	12.3	95.92 ± 6.29	4.619 ± 0.228	0.9324 ± 0.0170	1.0725 ± 0.0196	1.175/15
NA61/SHINE	π^-	17.3	95.83 ± 6.38	4.798 ± 0.246	0.9326 ± 0.0166	1.0722 ± 0.0191	0.865/15
PHENIX	π^+	62.4	97.62 ± 11.92	3.744 ± 0.648	0.9197 ± 0.0093	1.0874 ± 0.0110	1.654/23
PHENIX	π^-	62.4	93.76 ± 11.69	3.971 ± 0.715	0.9184 ± 0.0091	1.0888 ± 0.0108	0.878/23
PHENIX	π^+	200.0	79.89 ± 11.81	4.247 ± 0.899	0.8894 ± 0.0082	1.1244 ± 0.0104	0.987/24
PHENIX	π^-	200.0	87.20 ± 11.49	3.823 ± 0.714	0.8965 ± 0.0081	1.1155 ± 0.0101	0.691/24
ALICE	π^+	900.0	82.72 ± 2.01	3.965 ± 0.069	0.8766 ± 0.0037	1.1408 ± 0.0048	3.609/30
ALICE	π^-	900.0	83.92 ± 2.02	3.918 ± 0.068	0.8790 ± 0.0036	1.1376 ± 0.0047	1.610/30
ALICE	$\pi^+ + \pi^-$	2760.0	90.61 ± 1.45	3.496 ± 0.057	0.8726 ± 0.0012	1.1460 ± 0.0016	12.18/60
ALICE	$\pi^+ + \pi^-$	7000.0	78.75 ± 1.86	4.606 ± 0.093	0.8533 ± 0.0024	1.1719 ± 0.0032	9.775/38

Thermodynamic parameters of exact p_T - distribution of the Tsallis-1 statistics

Charged pions (*pp* collisions): Ultrarelativistic transverse momentum distribution **A.S.P., Eur. Phys. J. A 52 (2016) 355**





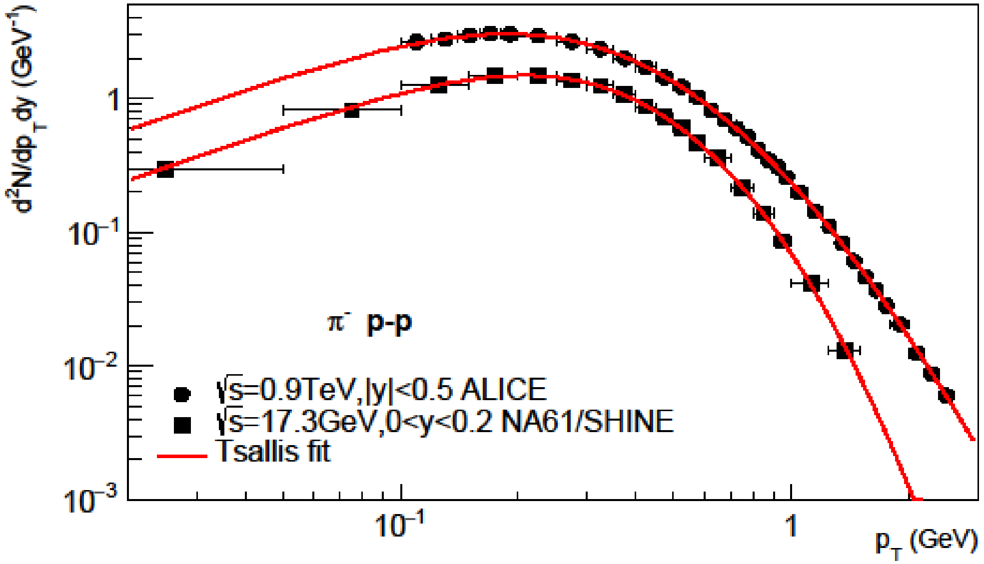


- Solid points are the results of the fit by ultrarelativistic pT-distribution of the Tsallis-1 statistics
- Open symbols are the results of the fit by the phenomenological Tsallis distribution (pT-distribution of the q-dual statistics in the zeroth term approximation)

Comparison of exact p_T - distribution of the Tsallis-1 statistics with experimental data

Charged pions (*pp* collisions): Relativistic transverse momentum distribution $(m \neq 0)$

A.S.P., T.Bhattacharyya, Eur. Phys. J. A 56 (2020) 2, 72



- Transverse momentum distributions of charged pions produced in *pp* collisions at SPS and LHC energies
- The yields were calculated at rapidity $y = 0$

- The solid curves are the fits of the experimental data to the **exact transverse momentum distribution** of Tsallis-1 statistics considering the mass of particles at different values of the cut-off parameter n_0
- The curves are the same for all values of the cut-off parameter but only the thermodynamic parameters are different.

n_0	T(MeV)	R(fm)	q	χ^2/ndf
0	71.837±2.423	4.743±0.134	0.873±0.004	2.214/30
1	50.526±1.439	5.029±0.112	0.911±0.001	1.363/30
2	42.245±0.233	5.553±0.032	0.9321±0.0003	1.827/30
3	33.750±0.035	6.859±0.015	0.9442±0.0001	5.553/30

TABLE I. Parameters of the Tsallis-1 statistics fit with different cut-off n_0 for the π^- particles produced in the p-p collisions as obtained by the ALICE Collaboration at $\sqrt{s}$=0.9 TeV [4].

n_0	T(MeV)	R(fm)	q	χ^2/ndf
0	90.574±5.092	3.140±0.148	0.931±0.013	0.459/15
1	79.224±8.610	3.145±0.225	0.946±0.011	0.453/15
2	76.835±8.048	3.138±0.217	0.957±0.007	0.4503/15
3	77.527±2.293	3.116±0.063	0.964±0.002	0.4575/15

TABLE II. Parameters of the Tsallis-1 statistics fit with different cut-off n_0 for the π^- particles produced in the p-p collisions as obtained by the NA61/SHINE collaboration at $\sqrt{s}$=17.3 GeV[46].

Experimental Data:

NA61/SHINE, Eur. Phys. J. C 74 (2014) 2794;
ALICE, Eur. Phys. J. C 71 (2011) 1655

Inconsistent Tsallis-like distributions

Tsallis-like distributions obtained in ref.[**1**]:

$$\langle n_{\mathbf{p}\sigma}\rangle = \left[1-(1-q)\frac{\varepsilon_{\mathbf{p}}-\mu}{T}\right]^{\frac{1}{1-q}}$$ - M-B statistics,

$$\langle n_{\mathbf{p}\sigma}\rangle = \frac{1}{\left[1-(1-q)\frac{\varepsilon_{\mathbf{p}}-\mu}{T}\right]^{\frac{1}{q-1}} \pm 1} \qquad (1)$$ - F-D and B-E statistics

The Tsallis-like distributions (1) are erroneous for the Tsallis statistics since they were derived inconsistently in ref. [**1**].

The derivation scheme of F. Buyukkilic et al. in ref.[1]

Tsallis-1 statistics

$$p_i^{(1)} = \left[1+\frac{q-1}{q}\frac{\Lambda - E_i + \mu N_i}{T}\right]^{\frac{1}{q-1}}$$

$$\langle A\rangle^{(1)} = \sum_i p_i^{(1)} A_i$$

Tsallis-2 statistics

$$p_i^{(2)} = \frac{1}{Z}\left[1-(1-q)\frac{E_i - \mu N_i}{T}\right]^{\frac{1}{1-q}}$$

$$\langle A\rangle^{(2)} = \sum_i \left(p_i^{(2)}\right)^q A_i$$

Buyukkilic et al.

$$\langle A\rangle^{(1)} = \sum_i p_i^{(2)} A_i$$ - contradicts the maximum entropy principle

This scheme is incorrect because we have

$$p_i^{(2)} \leftrightarrow \langle A\rangle^{(2)} \quad \text{and} \quad p_i^{(1)} \leftrightarrow \langle A\rangle^{(1)}$$ - from maximum entropy principle

Thus, the calculations of Buyukkilic et al. in ref.[1] lead to the incorrect results for the Tsallis statistics.

Therefore, the Tsallis-like distributions (1) obtained in ref.[**1**] are inconsistent for the Tsallis statistics!

A.S.P., T.Bhattacharyya, J. Phys. A 54 (2021) 325004

[1] F. Buyukkilic et al., Phys. Lett. A 197, 209 (1995)

Derivation of the Tsallis-like distributions in ref. [1]

Calculation of the mean occupation numbers of Tsallis-1 statistics using the probability distribution of the Tsallis-2 statistics:

$$\langle n_{\mathbf{p}\sigma}\rangle = \langle n_{\mathbf{p}\sigma}\rangle^{(1)} = \sum_{\{n_{\mathbf{p}\sigma}\}} n_{\mathbf{p}\sigma} G\{n_{\mathbf{p}\sigma}\} p^{(2)}\{n_{\mathbf{p}\sigma}\} = \sum_{n_{\mathbf{p}\sigma}=0}^{K} n_{\mathbf{p}\sigma} f^{(2)}(n_{\mathbf{p}\sigma})$$

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$K = 1$ - F-D statistics

$K = \infty$ - B-E and M-B stat

$G\{n_{\mathbf{p}\sigma}\} = 1$ - F-D and B-E stat

$G\{n_{\mathbf{p}\sigma}\} = \frac{1}{\prod_{\mathbf{p},\sigma} n_{\mathbf{p}\sigma}!}$ - M-B statistics

$$p^{(2)}\{n_{\mathbf{p}\sigma}\} = \frac{1}{Z}\left[1-(1-q)\frac{\sum_{\mathbf{p},\sigma} n_{\mathbf{p}\sigma}(\varepsilon_{\mathbf{p}}-\mu)}{T}\right]^{\frac{1}{1-q}}, \quad f^{(2)}(n_{\mathbf{p}\sigma}) = \frac{1}{Z}\sum_{\{n_{\mathbf{p}\sigma}\}'} G\{n_{\mathbf{p}\sigma}\}\left[1-(1-q)\frac{\sum_{\mathbf{p},\sigma} n_{\mathbf{p}\sigma}(\varepsilon_{\mathbf{p}}-\mu)}{T}\right]^{\frac{1}{1-q}}$$

1.) Factorization approximation adopted in ref. [1]:

$$\left[1-(1-q)\frac{\sum_{\mathbf{p},\sigma} n_{\mathbf{p}\sigma}(\varepsilon_{\mathbf{p}}-\mu)}{T}\right]^{\xi} \simeq \prod_{\mathbf{p},\sigma}\left[1-(1-q)\frac{n_{\mathbf{p}\sigma}(\varepsilon_{\mathbf{p}}-\mu)}{T}\right]^{\xi}$$

2.) Additional factorization approximation:

$$\left[1-(1-q)\frac{n_{\mathbf{p}\sigma}(\varepsilon_{\mathbf{p}}-\mu)}{T}\right]^{\xi} \simeq \left[1-(1-q)\frac{\varepsilon_{\mathbf{p}}-\mu}{T}\right]^{\xi n_{\mathbf{p}\sigma}}$$

Tsallis-like transverse momentum distributions:

$$\frac{d^2N}{dp_T dy} = \frac{gV}{(2\pi)^2} p_T \varepsilon_{\mathbf{p}}\left[1-(1-q)\frac{\varepsilon_{\mathbf{p}}-\mu}{T}\right]^{\frac{1}{1-q}}$$ - M-B statistics,

$$\frac{d^2N}{dp_T dy} = \frac{gV}{(2\pi)^2} p_T \varepsilon_{\mathbf{p}} \frac{1}{\left[1-(1-q)\frac{\varepsilon_{\mathbf{p}}-\mu}{T}\right]^{\frac{1}{q-1}} \pm 1}$$ - F-D and B-E statistics

[1] F. Buyukkilic et al., Phys. Lett. A 197, 209 (1995)

These Tsallis-like transverse momentum distributions are not consistent with the Tsallis statistics!

Conclusions

1) The exact transverse momentum distributions for the Tsallis statistics and *q*-dual statistics in the grand canonical ensemble for the Fermi-Dirac, Bose-Einstein and Maxwell-Boltzmann statistics of particles have been found.

2) The exact p_T-distributions of the Tsallis statistics have been used to describe the experimental data for transverse momentum distributions of identified hadrons produced in pp collisions at LHC and RHIC energies.

3) The *q*-dual statistical mechanics based on the *q*-dual entropy obtained from the Tsallis entropy by the multiplicative transformation $q \to 1/q$ was introduced.

4) It was demonstrated that the phenomenological Tsallis p_T-distribution is not consistent with the Tsallis statistics because it is equivalent to the p_T-distribution of the Tsallis-2 statistics in the zeroth term approximation. The Tsallis-2 statistic is incorrect since it contains $\langle 1 \rangle \neq 1$.

5) It was proved that the phenomenological Tsallis p_T-distribution is consistent with the *q*-dual statistics because it is equivalent to the p_T-distribution of the *q*-dual statistics in the zeroth term approximation. The *q*-dual statistics is well defined and consistent.

6) It was proved that the Tsallis-like p_T-distributions are not consistent with the Tsallis statistics.

Thank you for your attention