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Non-abelian Fermionic T-duality in Supergravity

Based on:
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Radial symmetry of closed string

Consider the closed bosonic string in space $\mathcal{S}^1 \times \mathcal{R}^{1,24}$ (KK compactification on radius R) and find it's energy spectrum. One can show that the masses of the quantum string states take the values

$$M^2 = \frac{m^2}{R^2} + \frac{n^2 R^2}{\alpha'^2} + \frac{2}{\alpha'}(N + \tilde{N} - 2),$$

where N and $\tilde{N}$ are the number operators for right- and left-moving oscillation modes of the string.

Immediately notice that mass squared M^2 is invariant under

$$m \leftrightarrow n, \quad R \leftrightarrow \frac{\alpha'}{R}.$$

Conclusions:

- Two strings compactified on the circles with T-dual radii R and $\frac{\alpha'}{R}$ have identical spectra (for $m \leftrightarrow n$)
- Spectra of the T-dual theories coincide at any order of the string perturbation theory

Busher's procedure

Consider the Polyakov action for bosonic string in conformal gauge

$$S = \int d^2z \, [g_{mn}(x) + b_{mn}(x)] \, \partial x^m \bar{\partial} x^n. \tag{1}$$

it is written in terms of complex worldsheet coordinates.

Choose the coordinates $\{x_1, x_i\}$, $i > 1$ in such a way that the direction alongside x_1 is an isometry, so fields g and b do not depend on x_1. The dual background fields are related to the original ones by:

$$S' = \int d^2z \left[g_{11} A\bar{A} + l_{1i} A \bar{\partial} x^i + l_{i1} \partial x^i \bar{A} + l_{ij} \partial x^i \bar{\partial} x^j + \tilde{x}^1 (\partial \bar{A} - \bar{\partial} A) \right], \tag{2}$$

where $l_{mn} = g_{mn} + b_{mn}$.

Here we make a substitution

$$(\partial x^1, \bar{\partial} x^1) \rightarrow (A, \bar{A}).$$

The last term in (2) imposes the constraint $F = dA = 0$ via the Lagrange multiplier $\tilde{x}^1$.

Busher's procedure

Exclude the field A by using its equations of motion

$$A = g_{11}^{-1}\left(\partial\tilde{x}^1 - l_{i1}\partial x^i\right),$$
$$\bar{A} = -g_{11}^{-1}\left(\bar{\partial}\tilde{x}^1 + l_{1i}\bar{\partial}x^i\right),$$

then we obtain the dual theory, which action

$$S'' = \int d^2z\left[\tilde{g}_{mn}(x) + \tilde{b}_{mn}(x)\right]\partial y^m\bar{\partial}y^n$$

is written in coordinates $y_m = \{\tilde{x}_1, x_i\}$. The Lagrange multiplier in (2) acts as a dual coordinate, and the dual theory is again isometric in the $\tilde{x}_1$ direction. he dual background fields are related to the original ones by:

$$\tilde{g}_{11} = (g_{11})^{-1}, \quad \tilde{g}_{1i} = (g_{11})^{-1} b_{1i}, \quad \tilde{b}_{1i} = (g_{11})^{-1} g_{1i},$$

$$\tilde{g}_{ij} = g_{ij} - (g_{11})^{-1}\left(g_{i1}g_{1j} + b_{i1}b_{1j}\right), \quad \tilde{b}_{ij} = b_{ij} - (g_{11})^{-1}\left(g_{i1}b_{1j} + b_{i1}g_{1j}\right).$$

At the quantum level adding the dilaton in the action this manipulation carried at the same manner. Consider the path integral:

$$\int \mathcal{D}A\mathcal{D}\bar{A}\mathcal{D}x^i\mathcal{D}\tilde{x}^1 e^{-S'[\tilde{x},x,A]}. \tag{3}$$

Integrating out A brings in a Jacobian factor in the path integral and results to the dilaton shift:

$$\phi' = \phi - \frac{1}{2}\log g_{11}. \tag{4}$$

Pure spinor formalism

Consider the action in pure spinor formalism:

$$S = \frac{1}{2\pi\alpha'} \int d^2z \Big[L_{MN}(Z)\partial Z^M \bar{\partial} Z^N + P^{\alpha\hat{\beta}}(Z) d_\alpha \hat{d}_{\hat{\beta}} + E_M^\alpha(Z) d_\alpha \bar{\partial} Z^M$$
$$+ E_M^{\hat{\alpha}}(Z)\partial Z^M \hat{d}_{\hat{\alpha}} + \Omega_{M\alpha}^{\beta}(Z)\lambda^\alpha w_\beta \bar{\partial} Z^M + \hat{\Omega}_{M\hat{\alpha}}^{\hat{\beta}}(Z)\partial Z^M \hat{\lambda}^{\hat{\alpha}} \hat{w}_{\hat{\beta}}$$
$$+ C_\alpha^{\beta\hat{\gamma}}(Z)\lambda^\alpha w_\beta \hat{d}_{\hat{\gamma}} + \hat{C}_{\hat{\alpha}}^{\hat{\beta}\gamma}(Z) d_\gamma \hat{\lambda}^{\hat{\alpha}} \hat{w}_{\hat{\beta}} + S_{\alpha\hat{\gamma}}^{\beta\hat{\delta}} \lambda^\alpha w_\beta \hat{\lambda} \hat{\gamma} \hat{w}_{\hat{\delta}} + w_\alpha \bar{\partial}\lambda^\alpha + \hat{w}_{\hat{\alpha}} \partial \hat{\lambda}^{\hat{\alpha}} \Big]$$
$$+ \frac{1}{4\pi} \int d^2z \Phi(Z)\mathcal{R}.$$

Superfield $P_{\alpha\hat{\beta}}$ consist of RR-fields:

$$P^{\alpha\hat{\beta}}|_{\theta=\hat{\theta}=0} = \frac{i}{16} e^\phi F^{\alpha\hat{\beta}}, \tag{5}$$

$$F_{IIA}^{\alpha\hat{\beta}} = m + \frac{1}{2}\left(\gamma^{m_1m_2}\right)^{\alpha\beta} F_{m_1m_2} + \frac{1}{4!}\left(\gamma^{m_1\dots m_4}\right)^{\alpha\beta} F_{m_1\dots m_4}, \tag{6}$$

$$F_{IIB}^{\alpha\hat{\beta}} = \left(\gamma^m\right)^{\alpha\beta} F_m + \frac{1}{3!}\left(\gamma^{m_1m_2m_3}\right)^{\alpha\beta} F_{m_1m_2m_3} + \frac{1}{2}\frac{1}{5!}\left(\gamma^{m_1\dots m_5}\right)^{\alpha\beta} F_{m_1\dots m_5}. \tag{7}$$

E_M^α and $E_M^{\hat{\alpha}}$ are the parts of supervielbein, consist of ordinary vielbein and gravitini ψ_m^α and $\psi_m^{\hat{\alpha}}$. Lowest $\theta = \hat{\theta} = 0$ order components of Ω, C, and S are spin connection mixed with NSNS 3-form $H = db$, gravitino field strength tensor , and Riemann tensor also mixed with H, correspondingly.

Fermionic T-duality

We can carry out the Buscher's procedure for the Berkovitz action. Obtain the new superfields:

$$
\begin{gathered}
P'^{\alpha\hat{\beta}} = P^{\alpha\hat{\beta}} - (B_{11})^{-1} E_1^{\alpha} E_1^{\hat{\beta}}, \\
E_1'^{\alpha} = (B_{11})^{-1} E_1^{\alpha}, \quad E_1'^{\hat{\alpha}} = (B_{11})^{-1} E_1^{\hat{\alpha}}, \\
E_M'^{\alpha} = E_M^{\alpha} - (B_{11})^{-1} L_{1M} E_1^{\alpha}, \quad E_M'^{\hat{\alpha}} = E_M^{\hat{\alpha}} - (B_{11})^{-1} E_1^{\hat{\alpha}} L_{M1}, \\
\phi' = \phi + \tfrac{1}{2} \log (B_{11})\big|_{\theta=0}.
\end{gathered} \tag{8}
$$

The supervielbein index 1 in these formulae is spinorial, corresponding to the isometry coordinate θ_1. Taking the $\theta = \hat{\theta} = 0$ components one can establish that fermionic T-duality transformation leaves invariant the NSNS tensor fields g_{mn} and b_{mn}. What does transform are the RR fluxes and the dilaton:

$$
\frac{i}{16} e^{\phi'} F'^{\alpha\hat{\beta}} = \frac{i}{16} e^{\phi} F^{\alpha\hat{\beta}} - \epsilon^{\alpha} \hat{\epsilon}^{\hat{\beta}} C^{-1}, \quad \phi' = \phi + \frac{1}{2} \log C, \tag{9}
$$

where we denote

$$
C = B_{11}\big|_{\theta=\hat{\theta}=0}, \quad \left(\epsilon^{\alpha}, \hat{\epsilon}^{\hat{\alpha}}\right) = \left(E_1^{\alpha}, E_1^{\hat{\alpha}}\right)\Big|_{\theta=\hat{\theta}=0}. \tag{10}
$$

Fermionic T-duality

The superspace torsion constraints help us to find an expression for C in terms of $(\epsilon^{\alpha}, \hat{\epsilon}^{\hat{\alpha}})$:

$$\partial_m C = i\left(\bar{\epsilon}\Gamma_m\epsilon - \bar{\hat{\epsilon}}\Gamma_m\hat{\epsilon}\right) = \begin{cases} i\left(\epsilon\bar{\gamma}_m\epsilon + \hat{\epsilon}\gamma_m\hat{\epsilon}\right) & \text{(IIA)}\ , \\ i\left(\epsilon\bar{\gamma}_m\epsilon - \hat{\epsilon}\gamma_m\hat{\epsilon}\right) & \text{(IIB)}\ . \end{cases} \tag{11}$$

So, we set the spinors $(\epsilon, \hat{\epsilon})$, find the function C, and then we can explicitly find dual fields in the following way:

$$\frac{i}{16}e^{\phi'}F'^{\alpha\hat{\beta}} = \frac{i}{16}e^{\phi}F^{\alpha\hat{\beta}} - \epsilon^{\alpha}\hat{\epsilon}^{\hat{\beta}}C^{-1},$$

$$\phi' = \phi + \frac{1}{2}\log C.$$

Non-abelian Fermionic T-duality

Anticommutation constraint for the Killing spinors is given by the vanishing of the Killing vector field

$$\tilde{K}^m = \begin{Bmatrix} \epsilon\bar{\gamma}^m\epsilon - \hat{\epsilon}\gamma^m\hat{\epsilon} & \text{(IIA)} \\ \epsilon\bar{\gamma}^m\epsilon + \hat{\epsilon}\bar{\gamma}^m\hat{\epsilon} & \text{(IIB)} \end{Bmatrix} \stackrel{!}{=} 0 \qquad \text{abelian constraint.} \tag{12}$$

Similarly to the previous expression introduce

$$\partial_m C = iK_m = \begin{cases} i\left(\epsilon\bar{\gamma}_m\epsilon + \hat{\epsilon}\gamma_m\hat{\epsilon}\right) & \text{(IIA)}\ , \\ i\left(\epsilon\bar{\gamma}_m\epsilon - \hat{\epsilon}\bar{\gamma}_m\hat{\epsilon}\right) & \text{(IIB)}\ . \end{cases}$$

One can show that $\tilde{K}^m K_m = 0$ from Fierz identities for chiral $d = 10$ spinors ϵ and $\hat{\epsilon}$.

Next, using the Killing equations, one can obtain $\nabla_m \tilde{K}^m = 0$.

These observations suggest that the non-abelian fermionic T-dual background can be defined using the same transformation rules, but with the modified prescription for the scalar parameter C:

$$\begin{cases} \partial_m C = iK_m - ib_{mn}\tilde{K}^n, \\ \tilde{\partial}^m C = i\tilde{K}^m, \end{cases}$$

where $\tilde{\partial}^m$ denotes derivative with respect to the dual coordinate $\tilde{x}_m$ of double field theory, and b_{mn} term is added in order to make the two equations consistent. Also the constraints on C from double field theory for such choice of K_m and $\tilde{K}^m$ are satisfied:

$$\partial_m C \tilde{\partial}^m C = 0, \quad \partial_m \tilde{\partial}^m C = 0.$$

Double field theory

This approach introduces usual coordinates x^m together with dual coordinates $\tilde{x}_m$ combined into $\mathbb{X}^M = (x^m, \tilde{x}_m)$ and also covariant constraint

$$\eta^{MN}\partial_M \bullet \partial_N \bullet = 0, \quad \eta^{MN} = \begin{bmatrix} 0 & \delta_m{}^n \\ \delta_n{}^m & 0 \end{bmatrix}. \tag{13}$$

This section constraint efficiently eliminates half of the coordinates ensures closure of the algebra of local coordinate transformations.

The action of ten-dimensional supergravity on such doubled space can be made manifestly covariant under the global $O(d,d;\mathcal{R})$ T-duality rotations as well as the local generalized diffeomorphisms:

$$S = S_{NSNS} + S_{RR} = \int d^{10}x\, d^{10}\tilde{x} \left(e^{-2d}\mathcal{R}(\mathcal{H}, d) + \frac{1}{4}(\partial\!\!\!/\chi)^\dagger S\, \partial\!\!\!/\chi \right), \tag{14}$$

where the NSNS degrees of freedom are encoded by the invariant dilaton d and the generalized metric $\mathcal{H}_{MN}$ with its spin representative $S \in \mathrm{Spin}(d,d)$, while the RR field strengths are contained in the spinorial variable χ.

The invariant dilaton d is simply

$$d = \phi - \frac{1}{4}\log g, \tag{15}$$

where $g = \det g_{mn}$. The generalized metric of DFT is an element of the coset space $\mathrm{O}(d,d)/\mathrm{O}(d) \times \mathrm{O}(d)$ and in terms of the background fields is defined as follows

$$\mathcal{H}_{MN} = \begin{bmatrix} g_{mn} - b_{mp}g^{pq}b_{qn} & b_{mp}g^{pl} \\ -g^{kp}b_{pn} & g^{kl} \end{bmatrix}. \tag{16}$$

Examples

Geometric example

Consider Minkowski flat space in IIB theory. This is maximally supersymmetric supergravity solution, thus there are 16 ϵ and 16 $\hat{\epsilon}$ constant Killing spinors. They form 32d vector spinor space $\mathcal{N} = (2,0)$ in $d = 1+9$, where we choose basis $\{\epsilon_i, \hat{\epsilon}_i\}$, $i \in \{1, \dots, 16\}$ as follows

$$(\epsilon_i)^\alpha = \delta_i{}^\alpha, \quad (\hat{\epsilon}_i)^{\hat{\alpha}} = \delta_i{}^{\hat{\alpha}}.$$

As an example consider the fermionic T-duality in the direction set up by the spinors

$$\epsilon = \epsilon_1 - i\hat{\epsilon}_9, \quad \hat{\epsilon} \quad = -\hat{\epsilon}_1 - i\hat{\epsilon}_9.$$

We find function C:

$$C = 4(x^8 + i\tilde{x}_9).$$

and RR-fields:

$$\begin{aligned}
F_0 &= -2\,i\,C^{-3/2}, \\
F_{089} &= F_{127} = -F_{134} = -F_{156} = F_{235} = -F_{246} = F_{367} = F_{457} = -2\,C^{-3/2}, \\
F_{01236} &= F_{01245} = -F_{01357} = F_{01467} = -F_{02347} = -F_{02567} = F_{03456} = \\
F_{12789} &= -F_{13489} = -F_{15689} = F_{23589} = -F_{24689} = F_{36789} = F_{45789} = 2\,i\,C^{-3/2}.
\end{aligned}$$

Examples

Non-geometric example

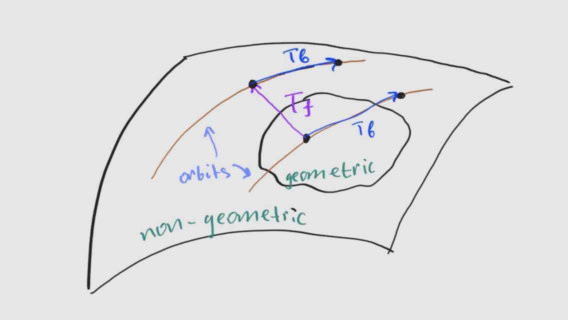
Next, consider fermionic T-duality generated by only one spinor:

$$\epsilon = \frac{1}{\sqrt{2}}(\epsilon_1 + i\epsilon_9), \quad \hat{\epsilon} = 0.$$

Hence

$$C = -x^8 - \tilde{x}_8 + i(x^9 + \tilde{x}_9)$$

so our dual background has vanishing $F_{(p)} = 0$ and cannot be bosonically T-dualized into some geometric background.



Examples

D-brane

Supergravity solution IIB Dp-brane as a solitonic background, $p < 7$, has a metric

$$g_{\mu\nu} = \left(H_{D_p}^{-\frac{1}{2}} \eta_{ij}, H_{D_p}^{\frac{1}{2}} \delta_{mn} \right), \quad H_{D_p} = 1 + \frac{Q}{(\delta_{mn} x^m x^n)^{\frac{7-p}{2}}},$$

where i, j and m, n denote brane coordinates and transverse coordinates correspondingly.

From BPS condition there are only 16 independent Killing spinors, parameterized by the constant ϵ_0:

$$\epsilon = H_{D_p}^{-\frac{1}{8}} \epsilon_0, \quad \hat{\epsilon} = -\gamma^{0\bar{1}..p} \epsilon = -H_{D_p}^{-\frac{1}{8}} \gamma^{0\bar{1}..p} \epsilon_0.$$

One can obtain that for the Dp-brane we can choose certain ϵ_0 to consider C in the following way:

$$C = 2(x_m + i\tilde{x}_j), \tag{17}$$

where m can be only from $p + 1$ to 10 and j can be only from 0 to $p + 1$, i.e. C cannot depend on coordinates dual to the transverse directions.

Examples

D3-brane

For concreteness consider D3-brane, choose the constant spinor

$$\hat{\epsilon}_0^\alpha = \frac{1}{2\sqrt{2}} e^{\frac{i\pi}{4}} \left(-\delta_1^\alpha + i\delta_2^\alpha + \delta_{15}^\alpha + i\delta_{16}^\alpha\right)$$

Next,

$$C = x^4 + i\hat{x}_1,$$

and RR-fields:

$$F_{(1)} = -\frac{e^{-\phi_0}}{2C^{3/2}} dx^6,$$

$$F_{(3)} = \frac{ie^{-\phi_0}}{2C^{3/2}} \Big[dx^0 \big(H^{-1} dx^{23} + dx^{58} - dx^{79}\big) - dx^{146} +$$

$$+ i dx^2 \big(dx^{57} + dx^{89}\big) + i dx^3 \big(dx^{59} + dx^{78}\big)\Big],$$

$$F_{(5)} = -\frac{e^{-\phi_0}}{2C^{3/2}} \Big[\sum_{k=4}^{9} \frac{1}{H} \big(\delta_k^4 + \frac{2C}{H} \partial_k H\big) dx^{0123k} +$$

$$+ dx^{014} \big(dx^{58} - dx^{79}\big) - i dx^{06} \Big(dx^2 \big(dx^{59} + dx^{78}\big) +$$

$$+ dx^3 \big(dx^{57} + dx^{89}\big)\Big)\Big].$$

Examples

Fundamental string

Consider the simplest background with non-vanishing Kalb-Ramond field b_{mn}. Proceed with the background of the Type II fundamental string, given by

$$\begin{aligned} ds^2 &= H^{-1}(-dt^2 + dy^2) + dx^2_{(8)}, \\ B_{ty} &= H^{-1} - 1, \quad e^{-2\phi} = He^{-2\phi_0}, \\ H &= 1 + \frac{h}{|x_{(8)}|^6}. \end{aligned} \tag{18}$$

This background preserves half of the total supersymmetry and the corresponding Killing spinors are defined by

$$\begin{aligned} &\begin{pmatrix} \epsilon \\ \hat{\epsilon} \end{pmatrix} = H^{-\frac{1}{4}} \begin{pmatrix} \epsilon_0 \\ \hat{\epsilon}_0 \end{pmatrix}, \quad (1 + \Gamma^{01}\mathcal{O}) \begin{pmatrix} \epsilon_0 \\ \hat{\epsilon}_0 \end{pmatrix} = 0, \\ &\mathcal{O} = \begin{cases} \Gamma_{11}, & IIA, \\ \sigma^3, & IIB. \end{cases} \end{aligned} \tag{19}$$

The general expression for the function C:

$$C = \frac{1}{2}(A + B)(x^1 + \tilde{x}_0) + \frac{1}{2}(A - B)(x^0 - \tilde{x}_1), \tag{20}$$

where A, B are the sums of squared Killing spinors components. C depends only on string coordinates.

Examples

Type IIA fundamental string

Choose such Killing spinors, that $A = B = 1$, so

$$C = x^1 + \tilde{x}_0, \tag{21}$$

and obtain the T-duals:

$$\begin{gathered}
e^{-2\phi} = \frac{He^{-2\phi_0}}{x^1 + \tilde{x}_0}, \\
m = 0, \\
F_{(2)} = -\frac{e^{-\phi_0}}{2C^{3/2}}\Big[dx^{67} + dx^{38} + dx^{49} - dx^{25}\Big], \\
F_{(4)} = \frac{e^{-\phi_0}}{2C^{3/2}}\Big[\frac{1}{H}dx^{01}(dx^{67} - dx^{25} + dx^{38} + dx^{49}) + \\
+(dx^{89} - dx^{34})(dx^{26} + dx^{57}) + (dx^{39} - dx^{48})(dx^{27} - dx^{56})\Big].
\end{gathered}$$

In this case we obtain formally real background by the virtue of dual time. This example is noteworthy with only possibility Roman's mass to be independent on dual coordinate.

Generalized SUGRA appearance

Now consider fundamental Type IIB string with the following function C $(A = -B = 1)$:

$$C = x^0 - \tilde{x}_1. \tag{22}$$

Make bosonic T-duality along x_1 for this fermionic T-dual IIB background example.

After bosonic T-duality NSNS-fields and dilaton are:

$$\begin{aligned} ds^2 &= -(2-H)dt^2 + H dy^2 + 2(1-H)dtdy + dx_{(8)}^2, \\ B &= 0, \quad e^{-2\phi'} = \frac{e^{-2\phi_0}}{x^0 - x^1}, \\ H &= 1 + \frac{h}{|x_{(8)}|^6}. \end{aligned} \tag{23}$$

From the rule $\epsilon^{\phi'} F' = \sqrt{g_{11}} e^{\phi} F \cdot \gamma_1$ we can find the RR-fields:

$$m = 0,$$

$$F_{(2)} = \frac{ie^{-\phi_0}}{2C^{3/2}} dx^4(dx^1 - dx^0),$$

$$F_{(4)} = \frac{ie^{-\phi_0}}{2C^{3/2}} \Big[(dx^1 - dx^0)(dx^{356} + dx^{327} - dx^{268} - dx^{578} + dx^{259} - dx^{679} - dx^{389})\Big].$$

Should we obtain some IIA supergravity theory? The answer is surprising.

Generalized SUGRA appearance

Check the following generalised IIA SUGRA equations for the dualized fields on the previous slide:

$$R_{mn} - \frac{1}{4}H_{mkl}H_n{}^{kl} - T_{mn} + D_m X_n + D_n X_m = 0, \tag{24}$$

$$\frac{1}{2}D^k H_{kmn} + \frac{1}{2}mF_{mn} + \frac{1}{8}F_{mnpq}F^{pq} = X^k H_{kmn} + D_m X_n - D_n X_m = 0, \tag{25}$$

$$R - \frac{1}{12}H^2 + 4D_m X^m - 4X_m X^m = 0, \tag{26}$$

where $X_m = \mathcal{I}_m + \partial_m \phi' - B_{mn}\mathcal{I}^m$ and $\mathcal{I}^m$ satisfies

$$\mathcal{I}^m \partial_m \phi' = 0 \tag{27}$$

and

$$D_m \mathcal{I}_n + D_n \mathcal{I}_m = 0. \tag{28}$$

It appears that these equations become the equations on Killing vector $\mathcal{I}^m$ only with the following solution with an arbitrary smooth function f:

$$\mathcal{I}^0 = \mathcal{I}^1 = f(x_0 - x_1), \quad \mathcal{I}^2 = .. = \mathcal{I}^9 = 0. \tag{29}$$

Is it feature of the initial B-field? Will we obtain the generalized supergravity within this scheme in general?

Results and discussion

- The mechanism of non-abelian fermionic T-duality takes us out of the ordinary supergravity solutions. What is the general DFT formulation of NAFTD?
- There is connection between SUGRA and generalized SUGRA through the combination of two dualities. Is it general? Is there any connection between genuinely non-geometric backgrounds and generalized supergravity?
- Does NAFTD have any connection with fermionic TsT-deformation?
- What if we take two different Killing spinors, can we obtain the true real background?

Thank you for attention!