Global conformal gauge in string theory

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Notation

$\mathbb{R}^2$ - a plane with coordinates $x = (x^\alpha) := (x^0, x^1) = (\tau, \sigma), \quad \alpha = 0,1$

$\mathbb{R}^{1,D-1}$ - D-dimensional Minkowskian space-time with

Cartesian coordinates $X = (X^A), \; A = 0,1,\dots,D-1, \; D \geq 2$

and Lorentz metric $\eta_{AB} := \mathrm{diag}(+-\dots-)$

Sufficiently smooth embedding: $\mathbb{R}^2 \supset \overline{\mathbb{U}} \ni \quad (\tau,\sigma) \mapsto \left(X^A(\tau,\sigma)\right) \quad \in \mathbb{R}^{1,D-1}$

String worldsheet $X(\overline{\mathbb{U}})$

The induced symmetric quadratic form: $h_{\alpha\beta} := \partial_\alpha X^A \partial_\beta X^B \eta_{AB} = \partial_\alpha X^A \partial_\beta X_A$

Properties of the embedding: $\left(\partial_0 X\right)^2 := \dot{X}^2 := \dot{X}^A \dot{X}^B \eta_{AB} > 0$

$$\left(\partial_1 X\right)^2 := X'^2 := X'^A X'^B \eta_{AB} < 0$$

$$\Rightarrow \quad h := \det h_{\alpha\beta} = \dot{X}^2 X'^2 - \left(\dot{X}, X'\right)^2 < 0$$

The induced metric $h_{\alpha\beta}$ of Lorentzian signature

Definition

Open string is the embedding $X(\bar{\mathbb{U}})$ where $\bar{\mathbb{U}}: \quad -\infty < \tau < \infty, \; 0 \le \sigma \le \pi$

Closed string is the embedding $X(\bar{\mathbb{U}})$ where $\bar{\mathbb{U}}: \quad -\infty < \tau < \infty, \; -\pi \le \sigma \le \pi$

and $\partial_1^k X^A |_{\sigma=-\pi} = \partial_1^k X^A |_{\sigma=\pi}, \quad \forall A, \forall \tau, \; k = 0,1,2,...$

Dynamics of the Nambu-Goto string

$$S_{\mathrm{NG}} := -\int_{\bar{\mathbb{U}}} dx \sqrt{|h|} = -\int_{\bar{\mathbb{U}}} d\tau d\sigma \sqrt{(\dot{X}, X')^2 - \dot{X}^2 X'^2}$$ - the action

$$\frac{1}{\sqrt{|h|}} \frac{\delta S_{\mathrm{NG}}}{\delta X_A} = \Box_{(h)} X^A = \frac{1}{\sqrt{|h|}} \partial_\alpha \left(\sqrt{|h|} h^{\alpha\beta} \partial_\beta X^A \right) = 0$$ - the Euler-Lagrange equations

$$s^\beta \partial_\beta X^A |_{\sigma=0,\pi} = 0$$ - boundary conditions for open string

s^β - components of a spacelike vector perpendicular to the boundary

The conformal gauge: $$h_{\alpha\beta} = e^{2\phi(x)} \eta_{\alpha\beta}, \quad \eta_{\alpha\beta} = \mathrm{diag}(+-)$$

$$\Box_{(h)} X^A = \partial^2_{\tau\tau} X^A - \partial^2_{\sigma\sigma} X^A = 0$$ - the Euler-Lagrange equations

The idea of the proof

$t = t^\alpha \partial_\alpha$ and $s = s^\alpha \partial_\alpha$ are timelike and spacelike vector fields on $\mathbb{U}$.

with properties: 1) $(t,s) = 0, \quad t^2 + s^2 = 0, \quad t^2 > 0, \quad \forall x \in \mathbb{U}$

where $(t,s) := t^\alpha s^\beta h_{\alpha\beta}, \quad t^2 := (t,t), \quad s^2 := (s,s)$

2) commutativity $[t,s] = 0$

Two families of integral curves $x^\alpha(\tilde{\tau}, \tilde{\sigma})$:

$$\frac{\partial x^\alpha}{\partial \tilde{\tau}} = t^\alpha, \quad \frac{\partial x^\alpha}{\partial \tilde{\sigma}} = s^\alpha$$

$$\frac{\partial s^\alpha}{\partial \tilde{\tau}} - \frac{\partial t^\alpha}{\partial \tilde{\sigma}} = t^\beta \partial_\beta s^\alpha - s^\beta \partial_\beta t^\alpha = [t,s]^\alpha = 0$$

There exists the coordinate transformation $(\tau, \sigma) \mapsto (\tilde{\tau}, \tilde{\sigma})$ on $\mathbb{U}$

In new coordinates

$$\tilde{h}_{00} = h_{\alpha\beta} \frac{\partial x^\alpha}{\partial \tilde{\tau}} \frac{\partial x^\beta}{\partial \tilde{\tau}} = t^2$$

$$\tilde{h}_{01} = h_{\alpha\beta} \frac{\partial x^\alpha}{\partial \tilde{\tau}} \frac{\partial x^\beta}{\partial \tilde{\sigma}} = (t,s) = 0$$

$$\tilde{h}_{11} = h_{\alpha\beta} \frac{\partial x^\alpha}{\partial \tilde{\sigma}} \frac{\partial x^\beta}{\partial \tilde{\sigma}} = s^2 = -t^2$$

- the conformal gauge

The final step is the analysis of domains of definitions of new coordinates $\tilde{\tau}, \tilde{\sigma}$.

Infinite string $(\tau,\sigma)\in\mathbb{U}=\mathbb{R}^2$

Two arbitrary timelike and spacelike tangent vector fields T and S to the string worldsheet can be decomposed on $\dot{X}$ and X':

$$T = A\left(\cosh\varphi\,\dot{X}+\sinh\varphi X'\right)$$

$$S = B\left(\sinh\psi\,\dot{X}+\cosh\psi X'\right) \qquad (*)$$

where $A(x)>0,\ B(x)>0,\ \varphi(x),\ \psi(x)$ are some functions.

Lemma 1. Vector fields T and S on $\mathbb{U}$ satisfy equalities

$$(T,S)=0,\quad T^2+S^2=0$$

if and only if vector field S is given by Eq. $(*)$ with arbitrary functions $B>0$ and $\psi\in\mathbb{R}$, and the second vector field T has the form

$$T=-\frac{B}{\sqrt{|h|}}\left[\cosh\psi\,X'^2+\sinh\psi(\dot{X},X')\right]\dot{X}+$$

$$+\frac{B}{\sqrt{|h|}}\left[\sinh\psi\,\dot{X}^2+\cosh\psi(\dot{X},X')\right]X'$$

Proof

$$\frac{(T,S)}{\cosh\varphi\cosh\psi} = \tanh\psi\,\dot{X}^2 + (1+\tanh\varphi\tanh\psi)(\dot{X},X') + \tanh\varphi\,X'^2 = 0$$

$$\Downarrow$$

$$\tanh\varphi = -\frac{\tanh\psi\,\dot{X}^2 + (\dot{X},X')}{X'^2 + \tanh\psi(\dot{X},X')}$$

$$\Downarrow$$

$$T = -\tilde{A}\left[X'^2 + \tanh\psi(\dot{X},X')\right]\dot{X} + \tilde{A}\left[\tanh\psi X'^2 + (\dot{X},X')\right]X'$$

where $A := \tilde{A}\sqrt{\left[\tanh\psi\,\dot{X}^2 + (\dot{X},X')\right]^2 + \left[X'^2 + \tanh\psi(\dot{X},X')\right]^2}.$

Algebraic equation $T^2 + S^2 = 0$ has the unique solution $\tilde{A} = \frac{B}{\sqrt{|h|}}\cosh\psi$

The differential of the embedding:

$$\mathbb{T}(\mathbb{U}) \ni \quad t = t^\alpha \partial_\alpha \mapsto T = t^\alpha \partial_\alpha X^A \partial_A \quad \in \mathbb{T}\left(\mathbb{R}^{1,D-1}\right)$$

Properties: $T^2 = t^2, \quad S^2 = s^2, \quad (T,S) = (t,s)$

$$t = -\frac{B}{\sqrt{|h|}}\left[\cosh\psi\, X'^2 + \sinh\psi (\dot{X}, X')\right]\partial_0 + \\ + \frac{B}{\sqrt{|h|}}\left[\sinh\psi\, \dot{X}^2 + \cosh\psi (\dot{X}, X')\right]\partial_1,$$

$$s = B\sinh\psi\, \partial_0 + B\cosh\psi\, \partial_1$$

Another form $$t^\alpha = \varepsilon^{\alpha\beta} s_\beta \quad \Leftrightarrow \quad s_\alpha = \varepsilon_{\alpha\beta} t^\beta \qquad (1)$$

$$\varepsilon^{\alpha\beta} = -\varepsilon^{\beta\alpha}, \quad \varepsilon^{01} = -1/\sqrt{|h|}$$ - the totally antisymmetric second rank tensor

Lemma 2. Vector fields t and s on $\mathbb{U}$ related by equalities (1) commute if and only if

$$t_\alpha = \frac{\partial_\alpha \chi}{\partial\chi^2}, \quad \partial\chi^2 := h^{\alpha\beta}\partial_\alpha\chi\,\partial_\beta\chi > 0 \qquad (2)$$

where χ is a nontrivial solution of the wave equation

$$\Box_{(h)}\chi := h^{\alpha\beta}\nabla_\alpha\nabla_\beta\chi = 0$$

For any nontrivial solution of Eq.(2) vector fields t and s commute.

Proof

$$[t, s] = 0$$

$$\Downarrow$$

$$s_\alpha \nabla_\beta s^\beta - \frac{1}{2}\nabla_\alpha s^2 - s^\beta \nabla_\beta s_\alpha = 0 \quad \Leftrightarrow \quad t_\alpha \nabla_\beta t^\beta - \frac{1}{2}\nabla_\alpha t^2 - t^\beta \nabla_\beta t_\alpha = 0$$

$$\Downarrow$$

$$\nabla_\alpha \left(\frac{s^\alpha}{s^2} \right) = 0 \quad \Leftrightarrow \quad \varepsilon^{\alpha\beta} \nabla_\alpha \left(\frac{t_\beta}{t^2} \right) = 0$$

Poincare lemma:

$$\frac{t_\alpha}{t^2} = \partial_\alpha \chi \quad \Rightarrow \quad t^2 = \frac{1}{\partial\chi^2} \quad \Rightarrow \quad t_\alpha = \frac{\partial_\alpha \chi}{\partial\chi^2}, \quad \partial\chi^2 > 0$$

$$\frac{\partial_\alpha \chi \, \Box_{(h)} \chi}{\partial\chi^2} = 0 \quad \Rightarrow \quad \Box_{(h)} \chi = 0$$

Global conformal gauge for infinite string

$$0<\varepsilon\le \lim_{\tau^2+\sigma^2\to\infty} |\det h_{\alpha\beta}|\le M<\infty \quad\Rightarrow\quad 0<\varepsilon_1\le t^2\le M_1<\infty \qquad (3)$$

A solution of $\Box_{(h)}\chi=0$ does exist on the whole plane $\mathbb{U}=\mathbb{R}^2$

J. Hadamard, 1932

Solutions are parameterized by the Cauchy data.
There are such Cauchy data that $\partial\chi^2>0$.

$$\frac{\partial\tau}{\partial\tilde\tau}=t^0 \quad\Rightarrow\quad \tilde\tau\sim\int^{\infty}\frac{d\tau}{t^0}$$ diverges because $t^0\le M_2$

The same is true for $\tilde\sigma$ ⇨ $(\tilde\tau,\tilde\sigma)\in\mathbb{R}^2$

Theorem 1. Let an arbitrary metric $h_{\alpha\beta}$ of Lorentzian signature be given on the whole plane $\mathbb{R}^2$. Let it be nondegenerate at infinity (3). Then there exists a surjective diffeomorphism

$$\mathbb{R}^2\ni \quad (x^\alpha)\mapsto\left(\tilde x^\alpha(x)\right) \quad \in\mathbb{R}^2 \qquad (4)$$

such that metric $h_{\alpha\beta}$ in new coordinate system has conformally flat form

$$\tilde h_{\alpha\beta}:=h_{\gamma\delta}\frac{\partial x^\gamma}{\partial\tilde x^\alpha}\frac{\partial x^\delta}{\partial\tilde x^\beta}=e^{2\phi}\eta_{\alpha\beta},$$

where $\phi(\tilde x)$ is some function on $\mathbb{R}^2$ separated from $\pm\infty$ at infinity $\tilde\tau^2+\tilde\sigma^2\to\infty$.

Corollary. Let

$$\tilde{\mathbb{U}}_0 := \left\{(\tilde{\tau},\tilde{\sigma}) \in \mathbb{R}^2 : \quad \tilde{\sigma} \in [\tilde{\sigma}_1, \tilde{\sigma}_2],\ \tilde{\tau} \in \mathbb{R}\right\}$$

be a closed vertical strip with vertical boundaries on the plane of new coordinates $\tilde{\tau},\tilde{\sigma}$ and assumptions of theorem 1 hold. Then there exists diffeomorphism (4) of a closed domain $(\tau,\sigma) \in \overline{\mathbb{U}} \subset \mathbb{R}^2$ bounded by integral curves $x(\tilde{\tau},\tilde{\sigma}_{1,2})$:

$$\frac{\partial x(\tilde{\tau},\tilde{\sigma}_{1,2})}{\partial \tilde{\tau}} = t_{1,2}$$

where $t_{1,2}$ are inverse images of vector fields $\partial / \partial \tilde{\tau}$ on the boundaries of $\tilde{\mathbb{U}}_0$.

Example

Consider $h_{\alpha\beta}dx^\alpha dx^\beta = e^{2\phi}(d\tau^2 - d\sigma^2) = e^{2\phi}d\xi d\eta, \quad \phi = \phi(x)$

where $\xi := \tau + \sigma, \quad \eta := \tau - \sigma$ are light cone coordinates.

Then $\quad \square_{(h)}\chi = (\partial_0^2 - \partial_1^2)\chi = 0 \quad \Rightarrow \quad \chi = F(\xi) + G(\eta)$

$$\partial\chi^2 = 4e^{-2\phi}F'G' > 0 \quad \Rightarrow \quad F'G' > 0$$

New metric $\quad \tilde{h}_{\alpha\beta}d\tilde{x}^\alpha d\tilde{x}^\beta = \dfrac{e^{2\phi}}{4F'G'}d\tilde{\xi}d\tilde{\eta}$

$\tilde{\xi} := 2F(\xi), \quad \tilde{\eta} := 2G(\eta) \quad$ - conformal transformation

Open string

Consider an open string which worldsheet is an infinite strip with left γ_{L} and right γ_{R} boundaries. First, we assume that the induced metric is nondegenerate on the boundaries. The problem is that after diffeomorphism $\tau,\sigma \to \tilde{\tau},\tilde{\sigma}$ the boundaries may be not straight vertical lines.

$$\gamma_{\mathrm{L}}: \quad \tilde{\eta} = \tilde{\eta}_{\mathrm{L}}(\tilde{\xi}), \qquad \gamma_{\mathrm{R}}: \quad \tilde{\eta} = \tilde{\eta}_{\mathrm{R}}(\tilde{\xi}), \quad \tilde{\xi} \in \mathbb{R}$$

Assume $$\tilde{\eta}_{\mathrm{L}} > \tilde{\eta}_{\mathrm{R}}, \qquad 0 < \varepsilon \le \frac{d\tilde{\eta}_{\mathrm{L,R}}}{d\tilde{\xi}} \le M < \infty, \quad \varepsilon, M \in \mathbb{R}$$

Theorem 2. The conformal transformation

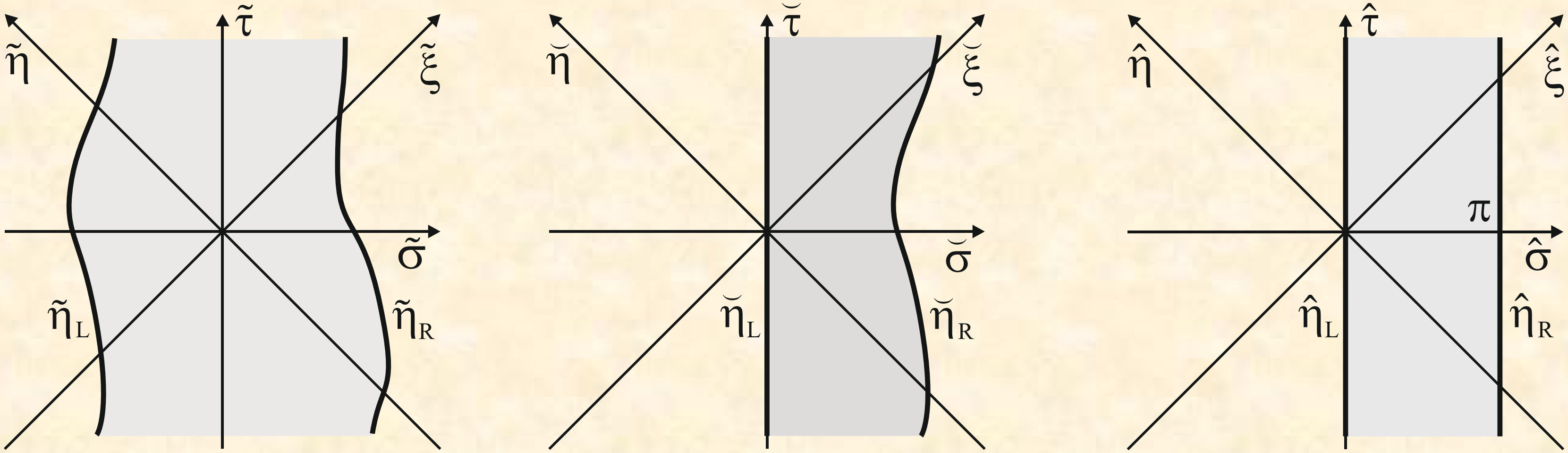
$$\hat{\xi} = F(\tilde{\xi}), \qquad \hat{\eta} = G(\tilde{\eta}), \qquad F, G \in C^1(\mathbb{R})$$

such that the boundaries of an open string become straight vertical lines

$$\gamma_{\mathrm{L}}: \quad \hat{\eta} = \hat{\xi}, \qquad \gamma_{\mathrm{R}}: \quad \hat{\eta} = \hat{\xi} - 2\pi, \quad \hat{\xi} \in \mathbb{R}$$

on the plane $\hat{\xi},\hat{\eta} \in \mathbb{R}^2$ exists.

Proof



First strengthen the left boundary by conformal transformation $\breve{\xi} = F(\tilde{\xi}), \quad \breve{\eta} = G(\tilde{\eta})$

The equation $\breve{\sigma} = 0 \quad \Rightarrow \quad F(\tilde{\xi}) = G\left(\tilde{\eta}_{\mathrm{L}}(\tilde{\xi})\right), \quad \forall \tilde{\xi} \in \mathbb{R}$ uniquely defines $G(\tilde{\eta})$

After strengthening the left boundary we are left with conformal transformations

$$\hat{\xi} = F(\breve{\xi}), \quad \hat{\eta} = F(\breve{\eta})$$

$$\hat{\sigma} = \pi \quad \Rightarrow \quad F\left(\breve{\eta}_{\mathrm{R}}(\breve{\xi})\right) = F(\breve{\xi}) - 2\pi, \quad \forall \breve{\xi} \in \mathbb{R} \qquad (**)$$

$f: \ \mathbb{R} \ni \ \breve{\xi} \mapsto \breve{\eta}_{\mathrm{R}}(\breve{\xi}) \ \in \mathbb{R}$ - bijective map

$\{f^k, \ k \in \mathbb{Z}\}$ - cyclic group

$\left[\breve{\eta}_{\mathrm{R}}(0), 0\right]$ - fundamental domain

$$\frac{d(**)}{d\breve{\xi}}: \quad F'\left(\breve{\eta}_{\mathrm{R}}(\breve{\xi})\right)\frac{d\breve{\eta}_{\mathrm{R}}}{d\breve{\xi}} = F'(\breve{\xi}) \qquad \frac{d\breve{\eta}_{\mathrm{R}}}{d\breve{\xi}} \geq \varepsilon > 0$$

Arbitrary function $F'(\breve{\xi})$ on fundamental domain $\left[\breve{\eta}_{\mathrm{R}}(0), 0\right]$

$F \in C^1(\mathbb{R})$

Open string (boundary conditions)

Parameterization of the metric: $h_{\alpha\beta} := \rho^2 k_{\alpha\beta}, \quad \det k_{\alpha\beta} := -1$

$k_{\alpha\beta}$ - unimodular metric (tensor density of degree 1).

The induced and unimodular metrics are conformally flat simultaneously.

The boundary condition $\quad n^0 \dot{X}^A + n^1 X'^A = 0 \;\Rightarrow\; X'^A = -\frac{n^0}{n^1}\dot{X}^A$

$$\det h_{\alpha\beta} = -\rho^4 = (\dot{X}, X')^2 - \dot{X}^2 X'^2 \to 0$$

Construct vector fields t and s for the unimodular metric $k_{\alpha\beta}$.

$\Box_{(h)} X^A = \frac{1}{\rho^2}\Box_{(k)} X^A = 0$ - equations of motion are the same on $\mathbb{U}$

$$\left.\begin{aligned} &\frac{\partial_\alpha \chi \Box_{(h)} \chi}{\partial_{(h)}\chi^2} = \frac{\partial_\alpha \chi \Box_{(k)} \chi}{\partial_{(k)}\chi^2} = 0 \\ &t = \frac{h^{\alpha\beta}\partial_\beta \chi}{\partial\chi^2}\partial_\alpha, \qquad s = \frac{\varepsilon^{\alpha\beta}\partial_\beta \chi}{\partial\chi^2}\partial_\alpha \end{aligned}\right\}$$

- do not depend on ρ.

If needed, one can compute ρ using equation $e^{2\phi} = \rho^2 k_{\alpha\beta} t^\alpha t^\beta$.

Closed string

In $\hat{\tau},\hat{\sigma}$ coordinates

$$\left.\frac{\partial^l X^A}{\partial\hat{\sigma}^l}\right|_{\hat{\sigma}=-\pi}=\left.\frac{\partial^l X^A}{\partial\hat{\sigma}^l}\right|_{\hat{\sigma}=\pi}, \quad \forall A,\ \forall\hat{\tau},\ l=0,1,2,\ldots$$

In the original coordinate system $\left.\nabla_s^l X^A\right|_{\hat{\sigma}=-\pi}=\left.\nabla_s^l X^A\right|_{\hat{\sigma}=\pi}$

The Euclidean signature

$t = t^\alpha \partial_\alpha$ and $s = s^\alpha \partial_\alpha$ are two nonzero vector fields on $\mathbb{U}$, related by $t^\alpha = \varepsilon^{\alpha\beta} s_\beta$

with properties: 1) $(t,s) = 0, \quad t^2 - s^2 = 0, \quad t^2 > 0, \quad \forall x \in \mathbb{U}$

where $(t,s) := t^\alpha s^\beta h_{\alpha\beta}, \quad t^2 := (t,t), \quad s^2 := (s,s)$

2) commutativity $[t,s] = 0$

Two families of integral curves $x^\alpha(\tilde{\tau},\tilde{\sigma})$:

$$\frac{\partial x^\alpha}{\partial \tilde{\tau}} = t^\alpha, \quad \frac{\partial x^\alpha}{\partial \tilde{\sigma}} = s^\alpha$$

$$\frac{\partial s^\alpha}{\partial \tilde{\tau}} - \frac{\partial t^\alpha}{\partial \tilde{\sigma}} = t^\beta \partial_\beta s^\alpha - s^\beta \partial_\beta t^\alpha = [t,s]^\alpha = 0$$

There exists the coordinate transformation $(\tau,\sigma) \mapsto (\tilde{\tau},\tilde{\sigma})$ on $\mathbb{U}$

In new coordinates

$$\tilde{h}_{00} = h_{\alpha\beta} \frac{\partial x^\alpha}{\partial \tilde{\tau}} \frac{\partial x^\beta}{\partial \tilde{\tau}} = t^2$$

$$\tilde{h}_{01} = h_{\alpha\beta} \frac{\partial x^\alpha}{\partial \tilde{\tau}} \frac{\partial x^\beta}{\partial \tilde{\sigma}} = (t,s) = 0$$

$$\tilde{h}_{11} = h_{\alpha\beta} \frac{\partial x^\alpha}{\partial \tilde{\sigma}} \frac{\partial x^\beta}{\partial \tilde{\sigma}} = s^2 = t^2$$

- the conformal gauge

Lemma 3. Vector fields t and s on $\mathbb{U}$ related by equalities (1) commute if and only if

$$t_\alpha = \frac{\partial_\alpha \chi}{\partial \chi^2}, \quad \partial\chi^2 := h^{\alpha\beta}\partial_\alpha \chi \partial_\beta \chi > 0 \qquad (2)$$

where χ is a nontrivial solution of the Laplace-Beltrami equation

$$\Delta_{(h)}\chi := h^{\alpha\beta}\nabla_\alpha \nabla_\beta \chi = 0$$

For any nontrivial solution of Eq.(2) vector fields t and s commute.

χ - is a harmonic function on a whole plain $\mathbb{R}^2$.

Example

Consider $h_{\alpha\beta}dx^\alpha dx^\beta = e^{2\phi}(d\tau^2 + d\sigma^2) = e^{2\phi}dz d\bar{z}, \quad \phi = \phi(x)$

where $z := x + iy$ is the complex coordinate.

Then $\Delta_{(h)}\chi = \partial_z \partial_{\bar{z}})\chi = 0 \quad \Rightarrow \quad \chi = w(z) + \bar{w}(\bar{z})$ - real valued solution

$$\partial\chi^2 = e^{-2\phi}\partial_z w \partial_{\bar{z}} \bar{w} > 0$$

New metric $\tilde{h}_{\alpha\beta}d\tilde{x}^\alpha d\tilde{x}^\beta = e^{2\phi}\partial_w z \partial_{\bar{w}} \bar{z}\, dw d\bar{w}$

$z \to w(z)$ - conformal transformation

Theorem 3. Let an arbitrary metric $h_{\alpha\beta}$ of Euclidean signature be given on the whole plane $\mathbb{R}^2$. Let it be nondegenerate at infinity (3). Then there exists a surjective diffeomorphism

$$\mathbb{R}^2 \ni \quad (x^\alpha) \mapsto \left(\tilde{x}^\alpha(x)\right) \quad \in \mathbb{R}^2 \qquad (4)$$

such that metric $h_{\alpha\beta}$ in new coordinate system has conformally flat form

$$\tilde{h}_{\alpha\beta} := h_{\gamma\delta} \frac{\partial x^\gamma}{\partial \tilde{x}^\alpha} \frac{\partial x^\delta}{\partial \tilde{x}^\beta} = e^{2\phi} \delta_{\alpha\beta},$$

where $\phi(\tilde{x})$ is some function on $\mathbb{R}^2$ separated from $\pm\infty$ at infinity $\tilde{\tau}^2 + \tilde{\sigma}^2 \to \infty$.

Conclusion

1) Let a sufficiently smooth Lorentzian metric $h_{\alpha\beta}(x)$ be given on the whole plane $\mathbb{R}^2$.
Then there exists the coordinate system where this metric is conformally flat

$$ds^2 = h_{\alpha\beta}(x)dx^\alpha dx^\beta = e^{2\phi(\tilde{x})}(d\tilde{\tau}^2 - d\tilde{\sigma}^2), \qquad (\tilde{\tau},\tilde{\sigma}) \in \mathbb{R}^2.$$

2) The conformal gauge for open and closed string does exist on the whole string worldsheet which is represented by a strip with straight boundaries.
3) Let a sufficiently smooth Euclidean metric $h_{\alpha\beta}(x)$ be given on the whole plane $\mathbb{R}^2$.
Then there exists the coordinate system where this metric is conformally flat

$$ds^2 = h_{\alpha\beta}(x)dx^\alpha dx^\beta = e^{2\phi(\tilde{x})}(d\tilde{\tau}^2 + d\tilde{\sigma}^2), \qquad (\tilde{\tau},\tilde{\sigma}) \in \mathbb{R}^2.$$