Wave models
of Shapovalov type III spacetimes

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The Hamilton-Jacobi equation

The **Hamilton-Jacobi equation** for a test particle of mass m in a gravitational field with the metric tensor $g_{\alpha\beta}(x^\gamma)$ has the form:

$$g^{\alpha\beta} \frac{\partial S}{\partial x^\alpha} \frac{\partial S}{\partial x^\beta} = m^2 c^2, \tag{1}$$

$$\alpha, \beta, \gamma... = 0, ..., (n-1),$$

where the capital letter S denotes the test particle action function, n is the space dimension.

We will set the speed of light c to be equal to unity.

The Shapovalov wave spacetimes considered in this report admit the existence of "privileged" coordinate systems, where the complete integral of the Hamilton-Jacobi equation for test particle can be written in the "separated" form:

$$S = \phi_0(x^0, \lambda_k) + \phi_1(x^1, \lambda_k) + \phi_2(x^2, \lambda_k) + \phi_2(x^2, \lambda_k) + \phi_3(x^3, \lambda_k), \quad (2)$$

where λ_k are independent constant parameters.

Moreover, one of the separated variables is **null (wave) variable**, i.e. the space-time interval along this variable is zero.

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Thus, in the Shapovalov space-times, one can obtain the complete integral S of the Hamilton-Jacobi equation for test particles.

Since the test particles move in the gravitational field along geodesic lines, then we get a complete integral for the geodesics.

There are three classes of Shapovalov space-times according to the number of commuting Killing vectors they admit, or, which is the same, according to the number of variables on which the metric in the privileged coordinate system depends. For example, Shapovalov type III spacetimes admit 3 commuting Killing vectors in "complete set".

The particle trajectory equations

The test particle coordinates x^α are functions of proper time τ on the base geodesic line along which the test particle moves, and is given in the Hamilton-Jacobi formalism by the particle trajectory equations in the form:

$$\frac{\partial S(x^\alpha, \lambda_k)}{\partial \lambda_j} = \sigma_j, \qquad \tau = S(x^\alpha, \lambda_k)\Big|_{m=1}, \qquad (3)$$

$$\alpha, \beta, \gamma = 0, 1, 2, 3; \qquad i, j, k = 1, 2, 3;$$

where λ_k, σ_k are independent constant parameters determined by the initial (or boundary) data for the motion of a test particle along the base geodesic line, the variable τ is the proper time of the particle.

On the other hand, the presence of a complete integral for the equations of geodesics makes it possible to integrate **the equations of geodesic deviation**:

$$\frac{D^2\eta^\alpha}{d\tau^2} = R^\alpha{}_{\beta\gamma\delta}u^\beta u^\gamma \eta^\delta. \tag{4}$$

where τ is the proper time along the base geodesic line, $R^\alpha{}_{\beta\gamma\delta}$ is the Riemann curvature tensor, $u^\alpha = dx^\alpha/d\tau$ is the four-velocity vector along the base geodesic, and $\eta^\alpha(\tau)$ is the deviation vector of infinitely close geodesics.

Exact solution of the geodesic deviation equation for the model

We can construct the general solution of the geodesic deviation equation through the complete integral of the Hamilton-Jacobi equation.

The deviation vector η^α can be found as a solution of a linear algebraic system of equations on the trajectory of the "base" test particle

$$\eta^\alpha \frac{\partial u_\alpha(x^\beta, \lambda_j)}{\partial \lambda_i} + \rho_k \frac{\partial^2 S(x^\beta, \lambda_j)}{\partial \lambda_i \partial \lambda_k} = \vartheta_i, \tag{5}$$

$$u_\alpha(x^\beta, \lambda_i)\, \eta^\alpha = 0, \tag{6}$$

$$i, j, k = 1 \ldots 3; \qquad \alpha, \beta, \gamma = 0 \ldots 3,$$

where λ_i, ρ_i, ϑ_i are independent constant parameters.

Shapovalov type III wave space-time metric

Shapovalov type III wave space-time metric in a privileged coordinate system depends on one wave variable and can be reduced to the following form:

$$ds^2 = 2dx^0 dx^1 + g_{ab}(x^0)\Big(dx^a + g^a(x^0)\, dx^1\Big)\Big(dx^b + g^b(x^0)\, dx^1\Big), \quad (7)$$

where indices a, b run through the values 2, 3. Thus, in the general case, the metric includes 5 arbitrary functions of the wave variable x^0.
Einstein's equations with cosmological constant Λ in vacuum

$$R_{\alpha\beta} = \Lambda g_{\alpha\beta} \tag{8}$$

for the metric (7) lead to the following necessary conditions:

$$\Lambda = g^a = 0. \tag{9}$$

The space-time metric for a gravitational wave (7) has type N according to Petrov's classification.

The Shapovalov type III wave model

The Shapovalov type III wave model metric for vacuum Einstein's equations in a privileged coordinate system can be represented as:

$$g^{ij} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & g^{22}(x^0) & g^{23}(x^0) \\ 0 & 0 & g^{23}(x^0) & g^{33}(x^0) \end{pmatrix}, \qquad g^{22}g^{33} - (g^{23})^2 > 0, \quad (10)$$

where x^0 is the wave (null) variable and:

$$g^{33\prime\prime} = \frac{6(g^{23})^2 \left(g^{22\prime}g^{33\prime} + g^{23\prime 2}\right) - 12g^{22}g^{23}g^{23\prime}g^{33\prime} + 3g^{22^2}g^{33\prime 2}}{2g^{22}\left(g^{22}g^{33} - (g^{23})^2\right)}$$

$$+ g^{33}\frac{g^{22\prime\prime}(g^{23})^2 + 3g^{22}g^{23\prime 2} + g^{23}\left(2g^{23\prime\prime}g^{22} - 6g^{22\prime}g^{23\prime}\right)}{g^{22}\left(g^{22}g^{33} - (g^{23})^2\right)}$$

$$+ \frac{g^{33^2}\left(3g^{22\prime 2} - 2g^{22\prime\prime}g^{22}\right) - 4g^{23\prime\prime}g^{23^3}}{2g^{22}\left(g^{22}g^{33} - (g^{23})^2\right)}. \quad (11)$$

Symmetries of the Shapovalov type III wave model

The space with the metric (10) admits a covariantly constant vector K and, therefore, is a plane-wave space:

$$\nabla_\beta K_\alpha = 0 \quad \rightarrow \quad K_\alpha = \big(K_0, 0, 0, 0\big), \tag{12}$$

where K_0 is a constant.

The Shapovalov space with a metric (10) admits the Abelian group with Killing vectors $X_{(0)}$, $X_{(1)}$, and $X_{(2)}$, which can be chosen in the privileged coordinate system in the form

$$X_{(0)}^\alpha = \big(0, 1, 0, 0\big), \qquad X_{(1)}^\alpha = \big(0, 0, 1, 0\big), \qquad X_{(2)}^\alpha = \big(0, 0, 0, 1\big), \tag{13}$$

The Killing vector $X_{(0)}$ is a null vector, because $g_{\alpha\beta} X_{(0)}^\alpha X_{(0)}^\beta = 0$.

The complete integral of the Hamilton-Jacobi equation

We will look for the complete integral of the Hamilton-Jacobi equation in a separated form:

$$S(x^\alpha, \lambda_k) = \phi_0(x^0) + \sum_{k=1}^{3} \lambda_k x^k, \tag{14}$$

where the independent constant parameters λ_k are determined by the initial (or boundary) conditions for the motion of a test particle.
Then the Hamilton-Jacobi equation gives for the function $\phi_0(x^0)$ ($\lambda_1 \neq 0$):

$$\phi_0(x^0) = -\frac{\lambda_2{}^2 I_{22} + 2\lambda_2\lambda_3 I_{23} + \lambda_3{}^2 I_{33} - x^0}{2\lambda_1}, \tag{15}$$

$$I_{22} = \int g^{22}(x^0)\,dx^0, \quad I_{23} = \int g^{23}(x^0)\,dx^0, \quad I_{33} = \int g^{33}(x^0)\,dx^0. \tag{16}$$

We present the result of solving the trajectory equations (3) in the form of a test particle trajectory in a privileged coordinate system:

$$x^0(\tau) = \lambda_1 \tau, \tag{17}$$

$$x^1(\tau) = \frac{\tau}{2\lambda_1} - \left.\frac{\lambda_2{}^2 I_{22} + 2\lambda_2\lambda_3 I_{23} + \lambda_3{}^2 I_{33}}{2\lambda_1{}^2}\right|_{x^0=\lambda_1\tau}, \tag{18}$$

$$x^2(\tau) = \left.\frac{\lambda_2 I_{22} + \lambda_3 I_{23}}{\lambda_1}\right|_{x^0=\lambda_1\tau}, \tag{19}$$

$$x^3(\tau) = \left.\frac{\lambda_2 I_{23} + \lambda_3 I_{33}}{\lambda_1}\right|_{x^0=\lambda_1\tau}, \tag{20}$$

where τ is the proper time of the test particle.

The 4-velocity of the test particle in the privileged coordinate system

For the solutions obtained, we find the 4-velocity of the test particle in the privileged coordinate system $u^\alpha(\tau) = Dx^\alpha/d\tau$:

$$u^0 = \lambda_1, \qquad x^0 = \lambda_1\tau,$$

$$u^1 = -\left.\frac{\lambda_2{}^2 g^{22} + 2\lambda_2\lambda_3 g^{23} + \lambda_3{}^2 g^{33} - 1}{2\lambda_1}\right|_{x^0 = \lambda_1\tau},$$

$$u^2 = \lambda_2 g^{22} + \lambda_3 g^{23},$$

$$u^3 = \lambda_2 g^{23} + \lambda_3 g^{33},$$

We present the exact solution for the geodesic deviation vector $\eta^\alpha(\tau)$ in the privileged coordinate system (τ is the proper time of the particle):

$$\eta^0(\tau) = \rho_1\tau - \lambda_1\Omega, \qquad x^0 = \lambda_1\tau, \tag{21}$$

$$\eta^1(\tau) = -\frac{(\rho_1\tau - \lambda_1\Omega)\left(\lambda_2{}^2 g^{22} + 2\lambda_2\lambda_3 g^{23} + \lambda_3{}^2 g^{33} + 1\right)}{2\lambda_1{}^2}$$

$$-\frac{2I_{23}(\lambda_2 R_3 + \lambda_3 R_2) + 2\lambda_2 R_2 I_{22} + 2\lambda_3 R_3 I_{33}}{2\lambda_1{}^3} - \frac{\Omega}{\lambda_1} + \vartheta_1, \tag{22}$$

$$\eta^2(\tau) = \frac{\lambda_1\left(\rho_1\tau - \lambda_1\Omega\right)\left(\lambda_2 g^{22} + \lambda_3 g^{23}\right) + R_2 I_{22} + R_3 I_{23}}{\lambda_1{}^2} + \vartheta_2, \tag{23}$$

$$\eta^3(\tau) = \frac{\lambda_1\left(\rho_1\tau - \lambda_1\Omega\right)\left(\lambda_2 g^{23} + \lambda_3 g^{33}\right) + R_2 I_{23} + R_3 I_{33}}{\lambda_1{}^2} + \vartheta_3, \tag{24}$$

Where, for brevity, the auxiliary notation R_2, R_3 and Ω is introduced:

$$R_2 = \lambda_2\rho_1 - \lambda_1\rho_2, \qquad R_3 = \lambda_3\rho_1 - \lambda_1\rho_3, \tag{25}$$

$$\Omega = \lambda_1\vartheta_1 + \lambda_2\vartheta_2 + \lambda_3\vartheta_3, \tag{26}$$

Here the constants λ_k, ρ_k and ϑ_k are given by the initial (or boundary) conditions for the velocities and mutual positions of particles on neighboring geodesics.

The synchronous frame

The transformation from the privileged coordinate system x^α to the synchronous frame $\tilde{x}^\alpha$ has the form $x^\alpha \to \tilde{x}^\alpha = (\tau, \lambda_1, \lambda_2, \lambda_3)$:

$$x^0 = \tilde{x}^1 \tau, \tag{27}$$

$$x^1 = -\frac{(\tilde{x}^2)^2 I_{22} + 2\tilde{x}^2\tilde{x}^3 I_{23} + (\tilde{x}^3)^2 I_{33} - \tilde{x}^1 \tau}{2(\tilde{x}^1)^2}\Bigg|_{x^0 = \tilde{x}^1 \tau}, \tag{28}$$

$$x^2 = \frac{\tilde{x}^2 I_{22} + \tilde{x}^3 I_{23}}{\tilde{x}^1}\Bigg|_{x^0 = \tilde{x}^1 \tau}, \tag{29}$$

$$x^3 = \frac{\tilde{x}^2 I_{23} + \tilde{x}^3 I_{33}}{\tilde{x}^1}\Bigg|_{x^0 = \tilde{x}^1 \tau}. \tag{30}$$

We can write down the form of the gravitational wave metric in the synchronous frame of reference:

$$ds^2 = d\tau^2 - dl^2 = d\tau^2 + \tilde{g}_{ij}(\tau, \tilde{x}^k)\, d\tilde{x}^i d\tilde{x}^j, \qquad i, j, k = 1, 2, 3; \quad (31)$$

where τ is the time variable (the observer's proper time on the base geodesic), dl is the spatial distance element, $\tilde{x}^k$ are the spatial coordinates.

The components of the gravitational wave metric in the synchronous frame take the following form:

$$\tilde{g}^{00} = 1, \qquad \tilde{g}^{0k} = 0, \qquad (32)$$

$$\tilde{g}_{11}(\tau, \tilde{x}^k) = -\frac{\tau^2}{(\tilde{x}^1)^2} - \frac{I_{23}^2 \left((\tilde{x}^2)^2 g^{22} + \tilde{x}^3 \left(\tilde{x}^3 g^{33} - 2\tilde{x}^2 g^{23}\right)\right)}{(\tilde{x}^1)^4 \left((g^{23})^2 - g^{22}g^{33}\right)}$$

$$+ \frac{2I_{23} \left(\tilde{x}^3 I_{33} \left(\tilde{x}^3 g^{23} - \tilde{x}^2 g^{22}\right) + \tilde{x}^2 I_{22} \left(\tilde{x}^2 g^{23} - \tilde{x}^3 g^{33}\right)\right)}{(\tilde{x}^1)^4 \left((g^{23})^2 - g^{22}g^{33}\right)}$$

$$- \frac{(\tilde{x}^2)^2 I_{22}^2 g^{33} - 2\tilde{x}^2 \tilde{x}^3 I_{22} I_{33} g^{23} + (\tilde{x}^3)^2 I_{33}^2 g^{22}}{(\tilde{x}^1)^4 \left((g^{23})^2 - g^{22}g^{33}\right)}, \tag{33}$$

$$\tilde{g}_{12}(\tau, \tilde{x}^k) = \frac{\tilde{x}^2 \left(I_{23}^2 g^{22} - 2I_{22} I_{23} g^{23} + I_{22}^2 g^{33}\right)}{(\tilde{x}^1)^3 \left((g^{23})^2 - g^{22}g^{33}\right)}$$

$$+ \frac{\tilde{x}^3 \left(I_{23} I_{33} g^{22} - I_{22} I_{33} g^{23} + I_{22} I_{23} g^{33} - I_{23}^2 g^{23}\right)}{(\tilde{x}^1)^3 \left((g^{23})^2 - g^{22}g^{33}\right)}, \tag{34}$$

$$\tilde{g}_{13}(\tau, \tilde{x}^k) = \frac{\tilde{x}^2 \left(I_{23} I_{33} g^{22} - I_{22} I_{33} g^{23} + I_{22} I_{23} g^{33} - I_{23}^2 g^{23}\right)}{(\tilde{x}^1)^3 \left((g^{23})^2 - g^{22}g^{33}\right)}$$

$$+ \frac{\tilde{x}^3 \left(I_{33}^2 g^{22} - 2I_{23} I_{33} g^{23} + I_{23}^2 g^{33}\right)}{(\tilde{x}^1)^3 \left((g^{23})^2 - g^{22}g^{33}\right)}, \tag{35}$$

$$\tilde{g}_{22}\big(\tau, \tilde{x}^k\big) = -\frac{I_{23}{}^2 g^{22} - 2I_{22}I_{23}g^{23} + I_{22}{}^2 g^{33}}{(\tilde{x}^1)^2 \left((g^{23})^2 - g^{22}g^{33}\right)}, \tag{36}$$

$$\tilde{g}_{23}\big(\tau, \tilde{x}^k\big) = \frac{I_{22}I_{33}g^{23} + I_{23}{}^2 g^{23} - I_{23}\left(I_{33}g^{22} + I_{22}g^{33}\right)}{(\tilde{x}^1)^2 \left((g^{23})^2 - g^{22}g^{33}\right)}, \tag{37}$$

$$\tilde{g}_{33}\big(\tau, \tilde{x}^k\big) = -\frac{I_{33}{}^2 g^{22} - 2I_{23}I_{33}g^{23} + I_{23}{}^2 g^{33}}{(\tilde{x}^1)^2 \left((g^{23})^2 - g^{22}g^{33}\right)}, \tag{38}$$

where $x^0 \to \tilde{x}^1 \tau$.

The deviation vector $\eta^\alpha(\tau)$ in the synchronous frame:

$$\tilde{\eta}^0 = 0, \tag{39}$$

$$\tilde{\eta}^1(\tau) = \rho_1 - \frac{\lambda_1 \Omega}{\tau}, \tag{40}$$

$$\tilde{\eta}^2(\tau) = \rho_2 - \frac{\lambda_2 \Omega}{\tau} + \frac{\lambda_1 \vartheta_2 I_{33}}{I_{22}I_{33} - I_{23}{}^2} + \frac{\lambda_1 \vartheta_3 I_{23}}{I_{23}{}^2 - I_{22}I_{33}}, \tag{41}$$

$$\tilde{\eta}^3(\tau) = \rho_3 - \frac{\lambda_3 \Omega}{\tau} + \frac{\lambda_1 \vartheta_2 I_{23}}{I_{23}{}^2 - I_{22}I_{33}} + \frac{\lambda_1 \vartheta_3 I_{22}}{I_{22}I_{33} - I_{23}{}^2}, \tag{42}$$

Here the constants λ_k, ρ_k and ϑ_k are given by the initial (or boundary) conditions for the velocities and mutual positions of particles on neighboring geodesics.

The tidal accelerations in the synchronous frame

The tidal accelerations in the synchronous frame

$$\tilde{A}^{\alpha} = \frac{D^2 \tilde{\eta}^{\alpha}}{d^2 \tau}, \tag{43}$$

where

$$\tilde{A}^0 = 0, \qquad \tilde{A}^1 = 0, \tag{44}$$

$$\tilde{A}^2 \neq 0, \qquad \tilde{A}^3 \neq 0. \tag{45}$$

Then, in the synchronous reference frame used, the components of the gravitational wave tidal acceleration are not equal to zero only in the plane of variables $\tilde{x}^2$, $\tilde{x}^3$.

The gravitational wave propagates along the space variable $\tilde{x}^1$.

The Lienard-Wiechert potentials A^α for the Minkowski spacetime can be written as:

$$A^\alpha(\tau, \vec{r}) = e\, \frac{u^\alpha}{R_\beta u^\beta}, \qquad (46)$$

where e is the charge of the particle, u^α is the 4-velocity of the charge at time τ', $R^\beta = \{c(\tau - \tau'), \vec{r} - \vec{r}\,'\}$ is the difference of the observer's 4-vector at time τ and the 4-vector of the charge at the time τ', c is the speed of light, $\vec{r}$ is the spatial radius vector of the charge, and the time τ' and the components of the radius vector charges $\vec{r}\,'$ are related to τ and $\vec{r}$ by the following relation:

$$R_\alpha R^\alpha = 0. \qquad (47)$$

On retarded potentials generated by a charge in a gravitational wave

In the weak field approximation we can apply the Lienard-Wiechert potential formula for an approximate, qualitative study of the problem, but taking into account the specifics of the exact solution we obtained for the deviation vector η^α. The charge vector $x^\alpha(\tau)$ and the vector R^α in the synchronous reference frame, one can take the expressions

$$\tilde{x}^\alpha(\tau) = \left\{ c\tau,\ \vec{\eta}(\tau) + \vec{\lambda} \right\}, \qquad R^\alpha = \left\{ c(\tau - \tau'), -\vec{\eta}(\tau') \right\}, \quad (48)$$

$$\vec{\eta} = \left\{ \tilde{\eta}^1, \tilde{\eta}^2, \tilde{\eta}^3 \right\}, \qquad \vec{\lambda} = \left\{ \lambda_1, \lambda_2, \lambda_3 \right\}. \quad (49)$$

In the expression for the Lienard-Wiechert potentials $A^\alpha(\tau)$ the values on the right are taken at the moment time τ', which is related to the time τ in the synchronous reference frame by the relation

$$c^2(\tau - \tau')^2 + \tilde{g}^{kl}\, \tilde{\eta}_k \tilde{\eta}_l \Big|_{x^\alpha = \{c\tau', \vec{\lambda}\}} = 0, \quad (50)$$

where $k, l = 1, 2, 3$ and the gravitational wave metric $\tilde{g}^{\alpha\beta}$ in the synchronous frame (32)-(38).

- An **exact solution for the trajectories of test particles** in a gravitational wave is obtained for Shapovalov type III gravitational wave models in a privileged coordinate system;

- An **exact solution of the geodesic deviation equations** in a gravitational wave is obtained for Shapovalov type III models in a privileged coordinate system and in synchronous frame;

- An explicit form for **the wave metric in a synchronous frame** is obtained;

- An explicit form of **the tidal acceleration in a gravitational wave** is obtained for a privileged coordinate system and for a synchronous frame.

Acknowledgments

The exact models obtained describe the gravitational waves and can be used to calculate the secondary physical effects that arise during the passage of a wave.

The approach presented in the report can be used both in the general theory of relativity and in modified theories of gravity.