Holographic Wilson loops in $\mathcal{N}=4$ SYM on $R\times\mathbb{S}^3$ at finite temperature

based on work with

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   - Holographic Wilson loops

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The AdS/CFT conjecture

The strongest version of the conjecture

$4d\ \mathcal{N}=4$ SYM with $SU(N)$ is dynamically equivalent to type IIB superstring theory(contains strings and D-branes) on $AdS_5 \times S^5$ with a string length $\ell_s = \sqrt{\alpha'}$ and coupling constant g_s with the radius L and N units of $F_{(5)}$ flux on S^5.

$$g_{YM}^2 = 2\pi g_s, \quad 2g_{YM}^2 N = \frac{L^4}{\alpha'^2}, \quad \lambda = g_{YM}^2 N.$$

Forms of the $\mathrm{AdS}_5/\mathrm{CFT}_4$ correspondence

	$\mathcal{N}=4$ SYM	IIB theory on $AdS_5 \times S^5$
Strongest form	any N and λ	Quantum string theory, $g_s \neq 0$, $\alpha'/L^2 \neq 0$
Strong form	$N \to \infty$,λ fixed but arbitrary	Classical string theory, $g_s \to 0$,$\alpha'/L^2 \neq 0$
Weak form	$N \to \infty$,λ large	Classical supergravity, $g_s \to 0$,$\alpha'/L^2 \to 0$

Holography at finite temperature

- Pure $AdS_5 \Leftrightarrow T = 0$ 4d $\mathcal{N} = 4$ SYM at strong coupling with $SU(N)$ (Maldacena'97)
  - the isometry group $SO(2, 4)$ of AdS_5 is a symmetry group of the dual CFT
  - field theory "lives" on the boundary of the gravity background
  - flat boundary $\Leftrightarrow$ CFT on R^4; spherical boundary $\Leftrightarrow$ CFT on cylinder $R \times \mathbb{S}^3$

  Example: global AdS_5

$$ds^2 = -(1 + y^2\ell^2)dT^2 + y^2(d\Theta^2 + \sin^2\Theta d\Phi^2 + \cos^2\Theta d\Psi^2) + \frac{dy^2}{1 + y^2\ell^2}.$$

  Boundary: $y \to \infty$, $R \times \mathbb{S}^3$: $ds^2 = -\ell^2 dT^2 + d\Theta^2 + \sin^2\Theta d\Phi^2 + \cos^2\Theta d\Psi^2$.
- AdS_5 BH $\Leftrightarrow$ thermal ensemble of $\mathcal{N} = 4$ SYM $SU(N)$ at strong coupling (Witten'98)
  T of CFT is identified with the Hawking temperature T_H of black hole

Sundborg'00: free $\mathcal{N} = 4$ SYM on $R \times \mathbb{S}^3$ at $T \neq 0$ has a phase transition at the Hagedorn temperature

Harmark et al.'18'20 the Hagedorn temperature at any value of the 't Hooft coupling

Holographic Wilson loops

- $d = 4$ $\mathcal{N} = 4$ SYM with $SU(N)$

$$W(\mathcal{C}) = \frac{1}{N}\mathrm{Tr}\mathcal{P}\exp\left(\oint ds A_\mu \dot{x}^\mu + |\dot{x}^i|\Phi_i\theta^i\right)$$

- The AdS/CFT duality (Maldacena'98): NG action of an open string in AdS_5

$$\langle W(\mathcal{C})\rangle = e^{-S_{NG,\min}-S_0}$$

Nambu-Goto action

$$S_{NG} = \frac{1}{2\pi\alpha'}\int \mathrm{d}\sigma\mathrm{d}\tau\sqrt{-\det(g_{\alpha\beta})},$$

where $(h_{\alpha\beta})$ is the induced metric on the string worldsheet

$$g_{\alpha\beta} = G_{MN}\partial_\alpha X^M\partial_N X^N,$$

G_{MN}– spacetime metric, X^M – embedding coordinates, α, β – WS indices.
Zarembo et al.'98; Gross et.al.'98: $\langle W\rangle|_{\lambda\to\infty} \sim e^{\sqrt{\lambda}}$
Sonnerschein et al.'98; Theisen'98: finite T holographic WL for "planar" AdS BH

Outline

Holographic model

$$\boxed{\mathcal{S} = \frac{1}{2\kappa^2} \int d^5x \sqrt{-g}\,(R_5 - 2\Lambda)\,.}$$

Solutions to Einstein equations with S^3-symmetry:

- Anti-de Sitter-Schwarzschild black hole (M)

$$ds^2 = -r^2 f(r) dt^2 + \frac{dr^2}{r^2 f(r)} + r^2 d\Omega_3^2, \quad f = \ell^2 + \frac{1}{r^2} - \frac{2M}{r^4}$$

- Kerr-Anti-de Sitter black hole (M, J)

$$\begin{aligned} ds^2 \quad &\simeq \quad -(1+y^2)dT^2 + \frac{dy^2}{1+y^2-\frac{2M}{\Delta^2 y^2}} + y^2(d\Theta^2 + \sin^2\Theta d\Phi^2 + \cos^2\Theta d\Psi^2) \\ &+ \quad \frac{2M}{\Delta^3 y^2} dT^2 + \frac{2Ma^2 \sin^4\Theta}{\Delta^3 y^2} d\Phi^2 + \frac{2Mb^2\cos^4\Theta}{\Delta^3 y^2} d\Psi^2 - \\ &- \quad \frac{4Ma\sin^2\Theta}{\Delta^3 y^2} dT d\Phi - \frac{4Mb\cos^2\Theta}{\Delta^3 y^2} dT d\Psi + \frac{4Mab\sin^2\Theta\cos^2\Theta}{\Delta^3 y^2} d\Phi d\Psi, \end{aligned}$$

The conformal boundary of 5d AdS BH is 4d $R \times \mathbb{S}^3$ at $r \to \infty$ ($y \to \infty$):

$$ds^2 = -dT^2 + d\Theta^2 + \sin^2\Theta d\Phi^2 + \cos^2\Theta d\Psi^2.$$

THE DEPENDENCE OF F ON T

Aref'eva, AG, Gourgoulghon'JHEP 4 (2021)

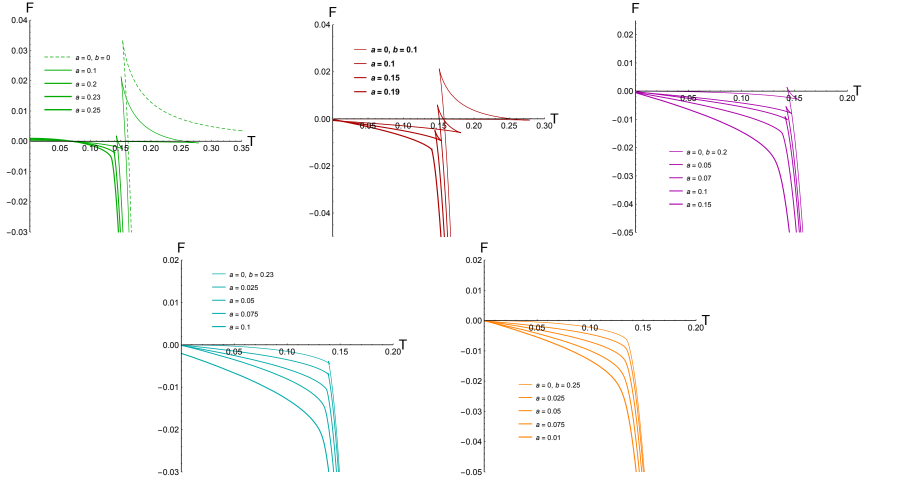


FIGURE: A logarithmic dependence of free energy on T_H on various values of a and b.

Wilson loop in Schwarzschild-AdS_5 black hole

Schwarzschild-AdS_5 black hole:

$$ds^2 = -\frac{f(r)}{r^2}\mathrm{d}t^2 + \frac{r^2}{f(r)}\mathrm{d}r^2 + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2 + \cos^2\theta\mathrm{d}\psi^2\right),$$

with

$$f(r) = r^2 + \ell^{-2}r^4 - 2M, \quad r_h = \frac{\ell\sqrt{\sqrt{8\ell^{-2}M+1}-1}}{\sqrt{2}}, \quad T_H = \frac{2r_h^2+\ell^2}{2\pi r_h\ell^2}$$

Nambu-Goto action of an open string

$$S_{NG} = \frac{1}{2\pi\alpha'}\int \mathrm{d}\sigma\mathrm{d}\tau\sqrt{-\det(g_{\alpha\beta})}, \quad g_{\alpha\beta} = G_{MN}\partial_\alpha X^M\partial_N X^N,$$

Parametrizing the static string coordinates by :

$$\boxed{\tau = t, \qquad \sigma = \phi, \qquad \phi\in[0, 2\pi L_\Phi], \quad r = r(\phi).}$$

Non-zero components of the induced metric are

$$g_{\tau\tau} = G_{tt} = -\frac{f(r)}{r^2}, \quad g_{\sigma\sigma} = G_{\phi\phi} + r'^2G_{rr} = r^2\left(\sin^2\theta + \frac{r'^2}{f(r)}\right), \quad r' \equiv \mathrm{d}r/\mathrm{d}\phi$$

The boundary conditions for endpoints $r\left(\phi = -\frac{L_\phi}{2}\right) = r\left(\phi = \frac{L_\phi}{2}\right) = \infty.$

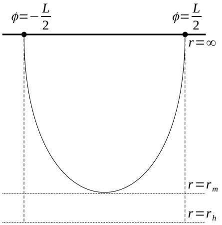


FIGURE: The string endpoints at $\phi=-\frac{L}{2}$ and $\phi=\frac{L}{2}$ and static straight strings(dashed lines)

$$S_{NG}=\frac{T}{2\pi\alpha'}\int_{-\frac{L_\phi}{2}}^{\frac{L_\phi}{2}} d\phi\sqrt{f(r)\sin^2\theta+r'^2}.$$

The integral of motion

$$\mathcal{H}=-\frac{\sin^2\theta\sqrt{f(r)}}{\sqrt{\sin^2\theta+\frac{r'^2}{f(r)}}}.$$

The string has a turning point: $r'|_{\phi_m}=0$,

$-\frac{\sin^2\theta\sqrt{f(r)}}{\sqrt{\sin^2\theta+\frac{r'^2}{f(r)}}}=-\sin\theta\sqrt{f(r)}|_{\phi_m}=-\frac{\ell}{C}$ where $C=\frac{\ell}{\sin\theta\sqrt{f(r)}}|_{r=r_m}$, .

$r_m=r(\phi_m)$.

Coming to the integration in terms of r we obtain

$$S_{NG} = \frac{T}{\pi\alpha'} \int\limits_{r_m}^{\infty} dr \, \frac{C \sin\theta \sqrt{f(r)}}{\sqrt{C^2 \sin^2\theta f(r) - \ell^2}}.$$

The distance between quarks L_ϕ:

$$\frac{L_\phi}{2} = \frac{\ell}{\sin\theta} \int\limits_{r_m}^{\infty} dr \, \frac{1}{\sqrt{f(r)}\sqrt{C^2 \sin^2\theta f(r) - \ell^2}}.$$

The renormalization is a subtraction of the action of two free quarks which corresponds to the straight lines configuration from the horizon r_h up to $r = \infty$:

$$\boxed{S_{NG}^{ren} = S_{NG} - S_0 = \frac{T}{\pi\alpha'} \left(\int\limits_{r_m}^{\infty} dr \, \left(\frac{C \sin\theta \sqrt{f(r)}}{\sqrt{C^2 \sin^2\theta f(r) - \ell^2}} - 1 \right) - r_m + r_h \right).}$$

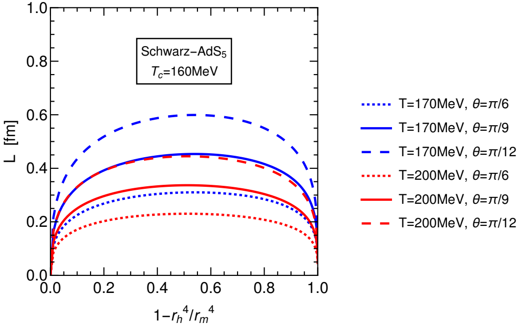


FIGURE: The distance L between quark and antiquark, depending on the string turning point r_m

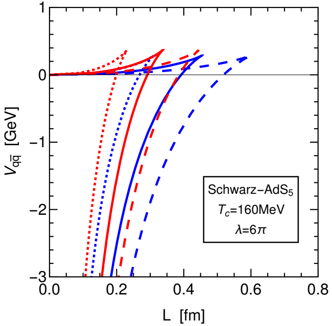


FIGURE: Numerical results for the dependence of V_{qq} on the distance between them L

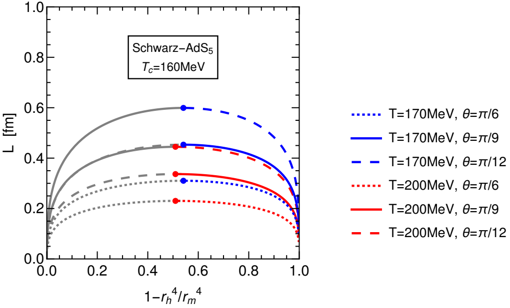


FIGURE: The distance between quark-antiquark L, depending on the string turning point r_m. The maximum distances – *the screening lengths* – are depicted by dots

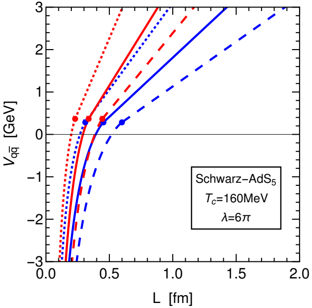


FIGURE: Numerical results of the heavy quark-antiquark potential V_{qq} dependence on the distance between them L

One can try to estimate the relation between S^{ren}_{NG} and L_ϕ. Denote

$$S^{ren}_{NG} = \frac{T}{\pi\alpha'} I_1(r_m, C), \quad L_\phi = 2 I_2(r_m, C).$$

We find that derivatives of these quantities with respect to C are related

$$\frac{\partial I_2(r_m, C)}{\partial C} = \frac{C}{\ell} \frac{\partial I_1(r_m, C)}{\partial C}.$$

It leads to

$$S^{reg}_{NG} = \frac{T}{\pi\alpha'} \frac{\ell}{C} \left(\frac{L_\phi}{2} + I_3(r_m, C) \right),$$

$$I_3(r_m, C) = \int_{r_m}^{\infty} dr \left(\frac{\sqrt{C^2 \sin^2\theta f(r) - \ell^2}}{\ell \sin\theta \sqrt{f(r)}} - C \right) - \frac{C}{\ell}(r_m - r_h)$$

We get the following relation for the quark-antiquark potential

$$V_{q\bar{q}} = \frac{\ell}{\pi\alpha' C} \left(\frac{L_\phi}{2} + I_3(r_m, C) \right).$$

Wilson loop in Kerr-AdS_5 black hole

Kerr-AdS_5 black hole

$$
\begin{aligned}
ds^2 \quad &\simeq \quad -(1+y^2)dT^2 + \frac{dy^2}{1+y^2-\frac{2M}{\Delta^2 y^2}} + y^2(d\Theta^2 + \sin^2\Theta d\Phi^2 + \cos^2\Theta d\Psi^2) \\
&+ \quad \frac{2M}{\Delta^3 y^2}dT^2 + \frac{2Ma^2\sin^4\Theta}{\Delta^3 y^2}d\Phi^2 + \frac{2Mb^2\cos^4\Theta}{\Delta^3 y^2}d\Psi^2 - \\
&- \quad \frac{4Ma\sin^2\Theta}{\Delta^3 y^2}dTd\Phi - \frac{4Mb\cos^2\Theta}{\Delta^3 y^2}dTd\Psi + \frac{4Mab\sin^2\Theta\cos^2\Theta}{\Delta^3 y^2}d\Phi d\Psi,
\end{aligned}
$$

where $\Delta = 1 - a^2\ell^{-2}\sin^2\Theta - b^2\ell^{-2}\cos^2\Theta$.

String worldsheet parametrization

$$
\tau = T, \qquad \sigma = \Phi, \qquad y = y(\Phi), \qquad \Phi \in [0, 2\pi L_\Phi].
$$

The boundary conditions $y\left(-\frac{L_\Phi}{2}\right) = y\left(\frac{L_\Phi}{2}\right) = 0$.

Wilson loop in Kerr-AdS_5 black hole

The Nambu-Goto action is

$$S_{NG} = \frac{T}{2\pi\alpha'} \int_{-\frac{L_\Phi}{2}}^{\frac{L_\Phi}{2}} d\Phi \sqrt{y'^2 \frac{f_{\Delta^3}(y)}{f_{\Delta^2}(y)} + y^2 F_{\Delta^3}(y) \sin^2\Theta},$$

where we redefine

$$f_{\Delta^2}(y) \equiv 1 + y^2\ell^{-2} - \frac{2M}{\Delta^2 y^2}, \qquad f_{\Delta^3}(y) \equiv 1 + y^2\ell^{-2} - \frac{2M}{\Delta^3 y^2}$$

$$F_{\Delta^3}(y) = f_{\Delta^3}(y) + \frac{2Ma^2\sin^2\Theta}{y^4\Delta^3}\left(1 + y^2\ell^{-2}\right).$$

The integral of motion

$$\mathcal{H} = -\frac{y^2 F_{\Delta^3}(y)\sin^2\Theta}{\sqrt{y'^2 \frac{f_{\Delta^3}(y)}{f_{\Delta^2}(y)} + y^2 F_{\Delta^3}(y)\sin^2\Theta}}.$$

The turning point is defined by $y' = 0$, so

$$y \sin\Theta \sqrt{F_{\Delta^3}(y)}\Big|_{y=y_m} = \frac{1}{\ell C}, \quad y_m = y(\Phi_m)$$

The equation of motion is

$$y'^2 = y^2 F_{\Delta^3}(y) \left[C^2 \ell^2 \sin^2\Theta y^2 F_{\Delta^3}(y) - 1\right] \frac{f_{\Delta^2}(y)}{f_{\Delta^3}(y)} \sin^2\Theta.$$

The renormalized NG action

$$S_{NG} = \frac{T}{\pi\alpha'}\left[\int_{y_m}^{\infty} dy \sqrt{\frac{f_{\Delta^3}(y)}{f_{\Delta^2}(y)}} \left(\frac{C\ell \sin\Theta y \sqrt{F_{\Delta^3}(y)}}{\sqrt{C^2\ell^2 \sin^2\Theta y^2 F_{\Delta^3}(y) - 1}} - 1\right) - \int_{y_+}^{y_m} dy \sqrt{\frac{f_{\Delta^3}(y)}{f_{\Delta^2}(y)}}\right].$$

The distance between quarks L_Φ:

$$\frac{L_\Phi}{2} = \int_{y_m}^{\infty} dy \frac{1}{\sin\Theta y \sqrt{F_{\Delta^3}(y)} \sqrt{C^2\ell^2 \sin^2\Theta y^2 F_{\Delta^3}(y) - 1}} \sqrt{\frac{f_{\Delta^3}(y)}{f_{\Delta^2}(y)}}.$$

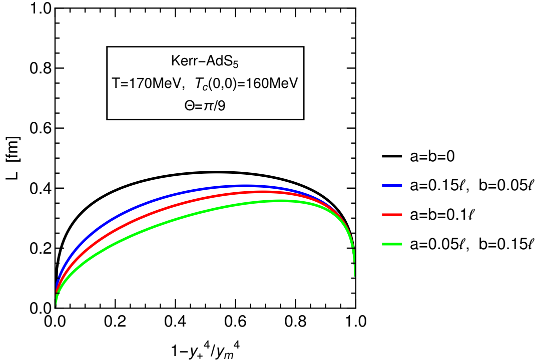


FIGURE: The distance between quark-antiquark L, depending on the string turning point y_m. The maximum distances – *the screening lengths* – are depicted by dots

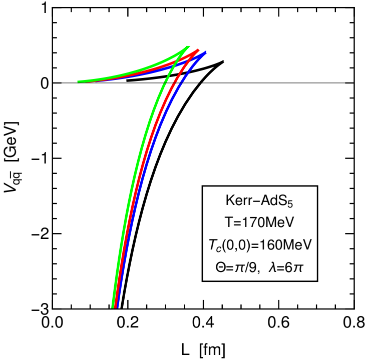
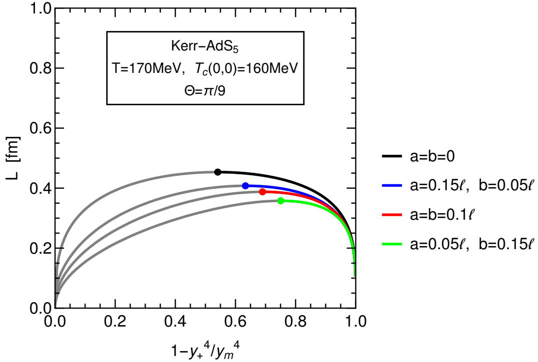


FIGURE: Numerical results of the heavy quark-antiquark potential V_{qq} dependence on the distance L



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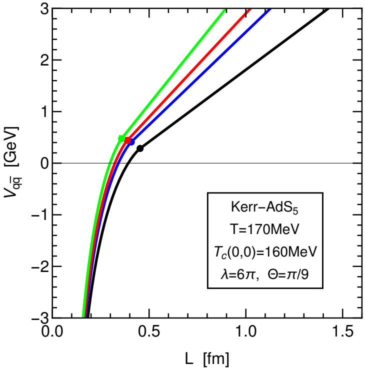

FIGURE: Numerical results of the heavy quark-antiquark potential V_{qq} dependence on the distance between them L

The relation between the string action and the quark-antiquark distance

$$S_{NG} = \frac{T}{\pi\alpha'} I_1(y_m, C), \quad \frac{L_\Phi}{2} = I_2(y_m, C).$$

We have the following relation :

$$\frac{\partial I_2(y_m, C)}{\partial C} = C\ell \frac{\partial I_1(y_m, C)}{\partial C}.$$

$$\frac{L_\Phi}{2} = \ell C \frac{\pi\alpha'}{T} S_{NG} + I_3(y_m, C),$$

where

$$I_3(y_m, C) = \int_{y_m}^{\infty} dy \sqrt{\frac{f_{\Delta^3}(y)}{f_{\Delta^2}(y)}} \left(\frac{\sqrt{\ell^2 C^2 \sin^2\Theta y^2 F_{\Delta^3}(y) - 1}}{y \sin\Theta \sqrt{F_{\Delta^3}(y)}} - C \right) - \ell C \int_{y_+}^{y_m} dy \sqrt{\frac{f_{\Delta^3}(y)}{f_{\Delta^2}(y)}}.$$

We find for the quark-antiquark potential

$$V_{q\bar{q}} = \frac{\ell C}{\pi\alpha'} \left(\frac{L_\Phi}{2} + I_3(y_m, C) \right).$$

LIGHT-LIKE WILSON LOOP IN SCHWARZSCHILD-AdS_5

"Light-cone" coordinates

$$dx^+ = \ell^2(dt - \ell d\phi), \quad dx^- = \ell^2(dt + \ell d\phi).$$

The string parametrization

$$\tau = x^-, \quad \sigma = \psi, \quad x^\mu = x^\mu(\sigma), \quad \theta(\sigma) = const, \quad x^+(\sigma) = const.$$

The Nambu-Goto action is

$$S = \frac{L^-}{2\pi\alpha'} \int_{-L/2}^{L/2} d\psi \, \frac{r}{2\ell^2} \sqrt{\left(\frac{f(r)}{r^2} - \ell^{-2} r^2 \sin^2\theta\right)\left(\cos^2\theta + \frac{r'^2}{f(r)}\right)}, \quad r' \equiv \partial r/\partial\psi$$

The first integral is given by

$$\mathcal{H} = -\frac{\cos^2\theta\sqrt{f(r) - r^4\ell^{-2}\sin^2\theta}}{2\ell^2\sqrt{\cos^2\theta + \frac{r'^2}{f(r)}}}.$$

The equation for $r(\sigma)$

$$r'^2 = \frac{f(r)\cos^2\theta}{4C^2\ell^6}[\cos^2\theta(f(r)\ell^2 - r^4\sin^2\theta) - 4C^2\ell^6], \quad C = const$$

The regularizied string action

$$S^{reg} = \frac{L^-}{\pi\alpha'} \int_{r_H+\epsilon}^{\infty} dr \, \frac{\sqrt{f(r)\ell^2 - r^4 \sin^2\theta}}{2\ell^3\sqrt{f(r)}} \left(\frac{\cos\theta\sqrt{f(r)\ell^2 - r^4\sin^2\theta}}{\sqrt{\cos^2\theta(f(r)\ell^2 - r^4\sin^2\theta) - 4C^2\ell^6}} - 1 \right).$$

Expanding for small C (in the low energy limit)

$$S^{reg} = -\frac{L^-}{\pi\alpha'}\frac{i\ell^2C^2}{\cos^2\theta}\mathcal{I}, \quad \mathcal{I} = \int_{r_m}^{\infty} \frac{dr}{\sqrt{f(r)}\sqrt{r^4\ell^{-2}\sin^2\theta - f(r)}}$$

and r_m is defined as a positive real solution to the equation

$$r^2 + r^4\ell^{-2}\cos^2\theta - 2M = 0.$$

To find the relation between L and C we remember that $r(\pm L/2) = \infty$

$$\frac{L}{2} = \int_{r_H}^{\infty} \frac{dr}{r'} = \frac{2C\ell^3}{\cos\theta} \int_{r_H}^{\infty} \frac{dr}{\sqrt{f(r)}\sqrt{\cos^2\theta(r^4\sin^2\theta - f(r)\ell^2) - 4C^2\ell^6}}.$$

For small C we have

$$\frac{L}{2} = \frac{2\ell^2 C}{\cos^2\theta}\mathcal{I}.$$

Then we come to

$$S^{reg} = \frac{L^-}{\pi\alpha'}\frac{L^2\cos^2\theta}{16\ell^2\int_{r_m}^{\infty}\frac{dr}{\sqrt{f(r)}\sqrt{f(r)-r^4\ell^{-2}\sin^2\theta}}}.$$

$r_m \geq r_H$ and r_H coincides with r_H only for $\theta = 0$, and in this case we need to shift the turning point to regularize the divergence near r_H, i.e. $r_m|_\theta = r_H + \epsilon$.

Wilson loop and jet-quenching parameter (Rajagopal'06)

$$\langle W^A(\mathcal{C})\rangle \approx \exp\Big[-\frac{1}{4\sqrt{2}}\hat{q}L^- L^2\Big].$$

The jet-quenching parameter is

$$\hat{q} = \frac{\ell^2\sqrt{\lambda}}{\sqrt{2}\pi\cos^2\theta\int_{r_m}^{\infty}\frac{dr}{\sqrt{f(r)}\sqrt{f(r)-r^4\ell^{-2}\sin^2\theta}}},$$

$$\hat{q} = \frac{\pi^2\sqrt{\lambda}T^3}{\beta} = \frac{\pi^{3/2}\Gamma(\frac{3}{4})}{\Gamma(\frac{5}{4})}\sqrt{\lambda}T^3.$$

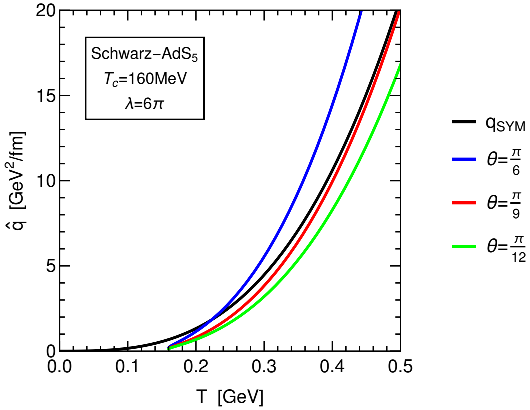

FIGURE: $\hat{q}$ on T

Thank you for attention!