Schwarzians with extended supersymmetry

Nikolay Kozyrev

BLTP JINR, Dubna, Russia

in collaboration with S.Krivonos

Introduction

The Schwarzian derivative, defined by the relation

$$\mathcal{S}(t,\tau) = \frac{\dddot{t}}{\dot{t}} - \frac{3}{2}\left(\frac{\ddot{t}}{\dot{t}}\right)^2, \quad \dot{t} = \partial_\tau t,$$

noted by conformal invariance,

$$t' = \frac{at+b}{ct+d}, \quad ad - bc \neq 0 \quad \Rightarrow \mathcal{S}(t',\tau) = \mathcal{S}(t,\tau),$$

was invented in complex analysis, but also has a range of applications in differential equations and in mathematical physics, especially in relation to the conformal field theories.

In particular, Schwarzian emerges in the conformal transformations of the energy-momentum tensor

$$T(z) = \left(\frac{d\tilde{z}}{dz}\right)^2 \tilde{T}(\tilde{z}) + \mathcal{S}(\tilde{z}, z).$$

In supersymmetric CFT's, superconformal transformations give rise to the supersymmetric analogs of the Schwarzian.

The bosonic Schwarzian

The bosonic Schwarzian derivative can be straightforwardly constructed using the formalism of nonlinear realizations. As it is invariant with respect to $SO(1,2)$ transformations, one can introduce the conformal algebra in one dimension

$$\mathrm{i}\,[D,P]=P,\quad \mathrm{i}\,[D,K]=-K,\quad \mathrm{i}\,[K,P]=2D$$

and, parameterizing the group element as

$$g=e^{\mathrm{i}tP}e^{\mathrm{i}zK}e^{\mathrm{i}uD},$$

find that the Cartan forms $\omega=g^{-1}dg$ are invariant with respect to $SO(1,2)$ transformations:

$$g^{-1}dg=\mathrm{i}\omega_P P+\mathrm{i}\omega_D D+\mathrm{i}\omega_K K,\;\; \omega_P=e^{-u}dt,\quad \omega_D=du-2zdt,\quad \omega_K=e^{u}\left(dz+z^2dt\right).$$

As these forms are invariant, one can impose the conditions $\omega_D=0$, $\omega_P=d\tau$, where τ is some inert variable. These conditions are evidently algebraic and express u and z in terms of $\dot{t}=\frac{dt}{d\tau}$:

$$u=\log\dot{t},\;\; z=\frac{1}{2}e^{-u}\dot{u}=\frac{\ddot{t}}{2\dot{t}^2}.$$

Then the only remaining form ω_K would reduce to

$$\omega_K=\frac{1}{2}d\tau\left(\frac{\dddot{t}}{\dot{t}}-\frac{3}{2}\left(\frac{\ddot{t}}{\dot{t}}\right)^2\right)=\frac{1}{2}d\tau\mathcal{S}(t,\tau)$$

Supersymmetric Schwarzians

In our previous works [2111.04643, 2112.14481] we have shown that all the known super-Schwarzians can be constructed using the method of nonlinear realizations, starting from suitable superextension of $SO(1,2)$.

The universal method to achieve this is to replace the basic constraint $\omega_P = d\tau$ with its supersymmetry-invariant equivalent, $\omega_P = \triangle\tau \sim d\tau + \mathrm{i} d\theta_i\theta_i$ and add the constraint $\left(\omega_Q\right)_i = d\theta_i$. This choice is dictated by requirement to keep the supersymmetry and Maurer-Cartan equations.

Then $N = 1,2,3,4$ super-Schwarzians can be reproduced by considering $osp(1,2)$, $osp(2|2) \sim su(1,1|1)$, $osp(3|2)$ and $D(1,2,\alpha)$ superalgebras, respectively.

To illustrate the construction of super-Schwarzians using nonlinear realizations formalism, let us consider the simple example of $osp(2|2)$.

$osp(2|2)$ superalgebra

The superalgebra $osp(2|2)$ consists of $d=1$ conformal algebra P, D, K, internal symmetry generator J, supersymmetry and superconformal charges Q and S:

$$
\begin{aligned}
& \mathrm{i}\,[D,P]=P, \quad \mathrm{i}\,[D,K]=-K, \quad \mathrm{i}\,[K,P]=2D, \\
& \left\{Q,\overline{Q}\right\}=2P, \quad \left\{S,\overline{S}\right\}=2K, \quad \left\{Q,\overline{S}\right\}=-2D+2J, \left\{\overline{Q},S\right\}=-2D-2J, \\
& \mathrm{i}\,[J,Q]=\frac{1}{2}Q,\ \mathrm{i}\left[J,\overline{Q}\right]=-\frac{1}{2}\overline{Q}, \quad \mathrm{i}\,[J,S]=\frac{1}{2}S,\ \mathrm{i}\left[J,\overline{S}\right]=-\frac{1}{2}\overline{S}, \\
& \mathrm{i}\,[D,Q]=\frac{1}{2}Q,\ \mathrm{i}\left[D,\overline{Q}\right]=\frac{1}{2}\overline{Q}, \quad \mathrm{i}\,[D,S]=-\frac{1}{2}S,\ \mathrm{i}\left[D,\overline{S}\right]=-\frac{1}{2}\overline{S}, \\
& \mathrm{i}\,[K,Q]=-S,\ \mathrm{i}\left[K,\overline{Q}\right]=-\overline{S}, \quad \mathrm{i}\,[P,S]=Q,\ \mathrm{i}\left[P,\overline{S}\right]=\overline{Q}.
\end{aligned}
$$

We then define the $OSp(2|2)$ group element that corresponds to such an algebra

$$
g=e^{\mathrm{i}tP}\,e^{\xi Q+\bar{\xi}\overline{Q}}\,e^{\psi S+\bar{\psi}\overline{S}}e^{\mathrm{i}zK}e^{\mathrm{i}uD}e^{\phi J},
$$

where t, ξ, $\bar{\xi}$, ψ, $\bar{\psi}$, z, u, ϕ are the superfields that depend on the coordinates τ, θ, $\bar{\theta}$ of $N=2$ superspace with standard realization of supersymmetry.
Then we could find the invariant Cartan forms

$$
\omega=g^{-1}dg=\mathrm{i}\omega_P P+\omega_Q Q+\bar{\omega}_Q\overline{Q}+\mathrm{i}\omega_D D+\omega_J J+\omega_S S+\bar{\omega}_S\overline{S}+\mathrm{i}\omega_K K
$$

and impose constraints on ω_P, ω_Q, $\bar{\omega}_Q$, ω_D.

$osp(2|2)$ constraints

The main constraints read

$$\omega_P = \triangle\tau = d\tau + \mathrm{i}\big(d\theta\bar\theta + d\bar\theta\theta\big),\;\; \omega_Q = d\theta,\;\; \bar\omega_Q = d\bar\theta,\;\; \omega_D = 0.$$

The invariant forms $\triangle\tau$, $d\theta$, $d\bar\theta$ can be obtained as Cartan forms of the group element $\tilde g = e^{\mathrm{i}\tau P} e^{\theta Q + \bar\theta \overline{Q}}$, if necessary. Evaluating forms explicitly, one can find that

$$\omega_P = e^{-u}\triangle t = e^{-u}\left(dt + \mathrm{i}\left(d\bar\xi\xi + d\xi\bar\xi\right)\right) = \triangle\tau \quad\Rightarrow\quad \begin{cases} \dot t + \mathrm{i}\left(\dot{\bar\xi}\xi + \dot\xi\bar\xi\right) = e^u, \\ Dt + \mathrm{i}D\xi\,\bar\xi = 0, \\ \overline{D}t + \mathrm{i}\overline{D}\bar\xi\,\xi = 0, \end{cases}$$

$$\begin{cases} \omega_Q = e^{-\frac12(u-\mathrm{i}\,\phi)}\left(d\xi + \psi\triangle t\right) = d\theta \\ \bar\omega_Q = e^{-\frac12(u+\mathrm{i}\,\phi)}\left(d\bar\xi + \bar\psi\triangle t\right) = d\bar\theta \end{cases} \quad\Rightarrow\quad \begin{cases} \dot\xi + e^u\psi = 0,\; \dot{\bar\xi} + e^u\bar\psi = 0 \\ D\xi = e^{\frac12(u-\mathrm{i}\,\phi)},\; \overline{D}\bar\xi = e^{\frac12(u+\mathrm{i}\,\phi)} \\ \overline{D}\xi = 0,\; D\bar\xi = 0. \end{cases}$$

Most of these constraints are algebraic, together with some chirality conditions.
$\omega_D = 0$ constraint leads to

$$\omega_D = du - 2z\,\triangle t - 2\mathrm{i}(d\xi\bar\psi + d\bar\xi\psi) = 0 \quad\Rightarrow\quad \begin{cases} \dot u - 2e^u z = 0, \\ Du = 2\mathrm{i}\,e^{\frac12(u-\mathrm{i}\,\phi)}\bar\psi, \\ \overline{D}u = 2\mathrm{i}\,e^{\frac12(u+\mathrm{i}\,\phi)}\psi, \end{cases}$$

with only the first of constraints being independent.

$osp(2|2)$ Cartan forms

Using the constraints $\omega_P = \triangle\tau$, $\omega_Q = d\theta$, $\bar{\omega}_Q = d\bar{\theta}$, $\omega_D = 0$ to simplify other Cartan forms, one could find that $d\theta$, $d\bar{\theta}$ projections of ω_J form vanish,

$$\omega_J = d\phi - 2\psi\bar{\psi}\,\triangle t + 2(d\bar{\xi}\psi - d\xi\bar{\psi}) = \mathrm{i}\triangle\tau\mathcal{S}_{N=2},$$

with $\mathcal{S}$ being the known $N = 2$ Schwarzian:

$$\mathcal{S}_{N=2} = \frac{D\dot{\xi}}{D\xi} - \frac{\overline{D}\dot{\bar{\xi}}}{\overline{D}\bar{\xi}} - 2\mathrm{i}\frac{\dot{\xi}\dot{\bar{\xi}}}{D\xi\,\overline{D}\bar{\xi}}.$$

It thus appears in the Cartan forms as a projection with the lowest dimension. It is important to note that all other forms can be written in terms of the Schwarzian and its derivatives:

$$\begin{aligned}
\omega_S &= e^{\frac{u}{2}+\mathrm{i}\frac{\phi}{2}}\left(d\psi - \mathrm{i}\psi\bar{\psi}d\xi + z\left(d\xi + \psi\,\triangle t\right)\right) = -\frac{1}{2}\mathcal{S}_{N=2}\,d\theta - \frac{\mathrm{i}}{2}\overline{D}\mathcal{S}_{N=2}\triangle\tau,\\
\bar{\omega}_S &= e^{\frac{u}{2}-\mathrm{i}\frac{\phi}{2}}\left(d\bar{\psi} + \mathrm{i}\psi\bar{\psi}d\bar{\xi} + z\left(d\bar{\xi} + \bar{\psi}\,\triangle t\right)\right) = \frac{1}{2}\mathcal{S}_{N=2}\,d\bar{\theta} + \frac{\mathrm{i}}{2}D\mathcal{S}_{N=2}\triangle\tau,\\
\omega_K &= e^{u}\Big(dz + z^2\triangle t - \mathrm{i}(\psi\,d\bar{\psi} + \bar{\psi}\,d\psi) + 2\mathrm{i}z\left(d\xi\,\bar{\psi} + d\bar{\xi}\psi\right)\Big) = \\
&= \frac{1}{2}D\mathcal{S}_{N=2}d\theta - \frac{1}{2}\overline{D}\mathcal{S}_{N=2}d\bar{\theta} + \frac{1}{4}\left(\mathrm{i}\left[D,\overline{D}\right]\mathcal{S}_{N=2} - \mathcal{S}^2_{N=2}\right)\triangle\tau.
\end{aligned}$$

Therefore, $\mathcal{S}_{N=2}$ is the only nontrivial superconformal invariant.

$osp(2|2)$ Maurer-Cartan equations

It is important to note that the structure of Cartan forms in not a coincidence: it is possible to find the expressions for all the forms in terms of the Schwarzian $\mathcal{S}_{N=2}$ even without referring to specific group parametrization. The Cartan forms of any group, by construction, satisfy the Maurer-Cartan equations. We find it convenient to introduce two commuting differentials to write these equations as

$$d_2\omega_1 - d_1\omega_2 = \left[\omega_1, \omega_2\right], \;\; \omega_1 = g^{-1} d_1 g, \;\; \omega_2 = g^{-1} d_2 g, \;\; d_1 d_2 = d_2 d_1.$$

At the same time, substituting

$$\omega = \mathrm{i}\omega_P P + \omega_Q Q + \bar\omega_Q \overline{Q} + \mathrm{i}\omega_D D + \omega_J J + \omega_S S + \bar\omega_S \overline{S} + \mathrm{i}\omega_K K$$

and expanding equation into projections, we may derive nontrivial relations forms ω_P, etc satisfy. To take into account the constraints, we should take $\omega_P = \triangle\tau$, $\omega_Q = d\theta$, $\bar\omega_Q = d\bar\theta$, $\omega_D = 0$, and expand all other forms as linear combinations of $\triangle\tau$, $d\theta$, $d\bar\theta$

$$\omega_J = \mathrm{i}\triangle\tau\mathcal{S} + d\theta\,\Phi - d\bar\theta\overline{\Phi}, \;\; \omega_K = \triangle\tau C + d\theta\Sigma - d\bar\theta\overline{\Sigma},$$
$$\omega_S = \triangle\tau\,\Psi + d\theta A + d\bar\theta B, \;\; \bar\omega_S = \triangle\tau\,\overline{\Psi} + d\theta\overline{B} + d\bar\theta\overline{A}.$$

The first of equations, related to the generator P,

$$d_2\omega_{1P} - d_1\omega_{2P} = -\left(\omega_{1P}\omega_{2D} - \omega_{1D}\omega_{2P}\right) + 2\mathrm{i}\left(\omega_{1Q}\bar\omega_{2Q} + \bar\omega_{1Q}\omega_{2Q}\right)$$

is satisfied identically on constraints, as $d_2\triangle_1\tau - d_1\triangle_2\tau = 2\mathrm{i}d_1\theta\, d_2\bar\theta + 2\mathrm{i}d_1\bar\theta d_2\theta$.

$osp(2|2)$ Maurer-Cartan equations

The Q equation is the first nontrivial. It splits into $\triangle\tau \times d\theta$, $\triangle\tau \times d\bar{\theta}$, $d\theta^2$, $d\theta \times d\bar{\theta}$ parts:

$$\begin{aligned} d_2\omega_{1Q} - d_1\omega_{2Q} &= \omega_{1P}\omega_{2S} - \omega_{2P}\omega_{1S} + \frac{1}{2}\big(\omega_{1D}\omega_{2Q} - \omega_{2D}\omega_{1Q}\big) - \frac{\mathrm{i}}{2}\big(\omega_{1J}\omega_{2Q} - \omega_{2J}\omega_{1Q}\big) \Rightarrow \\ d_2 d_1\theta - d_1 d_2\theta &= 0 = \big(\triangle_1\tau\, d_2\theta - \triangle_2\tau d_1\theta\big)\left(A + \frac{1}{2}\mathcal{S}\right) + \big(\triangle_1\tau\, d_2\bar{\theta} - \triangle_2\tau d_1\bar{\theta}\big)B + \\ & + \mathrm{i} d_1\theta\, d_2\theta\Phi + \frac{\mathrm{i}}{2}\big(d_1\bar{\theta}\, d_2\theta - d_2\bar{\theta}\, d_1\theta\big)\overline{\Phi}. \end{aligned}$$

This equation alone is strong enough to show $d\theta$ and $d\bar{\theta}$ projection of ω_J, Φ and $\overline{\Phi}$, are absent, as well as $d\bar{\theta}$ projection of ω_S (B). Taking into account $\overline{Q}$ equation also, one can use J equation

$$d_2\omega_{1J} - d_1\omega_{2J} = -2\big(\omega_{1Q}\bar{\omega}_{2S} - \bar{\omega}_{1Q}\omega_{2S} - \omega_{1S}\bar{\omega}_{2Q} + \bar{\omega}_{1S}\omega_{2Q}\big)$$

to show that $\Psi = -\frac{\mathrm{i}}{2}\overline{D}\mathcal{S}$, $\overline{\Psi} = \frac{\mathrm{i}}{2}D\mathcal{S}$. Analogously, D equation

$$d_2\omega_{1D} - d_1\omega_{2D} = -2\big(\omega_{1P}\omega_{2K} - \omega_{1K}\omega_{2P}\big) - 2\mathrm{i}\big(\omega_{1Q}\bar{\omega}_{2S} + \bar{\omega}_{1Q}\omega_{2S} + \omega_{1S}\bar{\omega}_{2Q} + \bar{\omega}_{1S}\omega_{2Q}\big)$$

implies just $\Sigma = -\frac{1}{2}D\mathcal{S}$, $\overline{\Sigma} = -\frac{1}{2}\overline{D}\mathcal{S}$.

$osp(2|2)$ Maurer-Cartan equations

The S equation

$$d_2\omega_{1S} - d_1\omega_{2S} = -\omega_{1K}\omega_{2Q} + \omega_{2K}\omega_{1Q} - \frac{1}{2}\big(\omega_{1D}\omega_{2S} - \omega_{2D}\omega_{1S}\big) - \frac{\mathrm{i}}{2}\big(\omega_{1J}\omega_{2S} - \omega_{2J}\omega_{1S}\big),$$

after substitution of previous results, allows to determine the single remaining function C as

$$C = \frac{\mathrm{i}}{4}[D,\overline{D}]\mathcal{S} - \frac{1}{4}\mathcal{S}^2.$$

Therefore, structure of the forms is determined by the algebra and constraints only, and all the forms could written down in terms of a single undetermined and unconstrained superfield $\mathcal{S}$

$$\begin{aligned}
\omega_P &= \triangle\tau, \omega_Q = d\theta,\ \bar{\omega}_Q = d\bar{\theta}, \quad \omega_J = \mathrm{i}\mathcal{S}\triangle\tau,\\
\omega_S &= -\frac{1}{2}\mathcal{S}\, d\theta - \frac{\mathrm{i}}{2}\overline{D}\mathcal{S}\triangle\tau, \quad \bar{\omega}_S = \frac{1}{2}\mathcal{S}\, d\bar{\theta} + \frac{\mathrm{i}}{2}D\mathcal{S}\triangle\tau,\\
\omega_K &= \frac{1}{2}D\mathcal{S}d\theta - \frac{1}{2}\overline{D}\mathcal{S}d\bar{\theta} + \frac{1}{4}\left(\mathrm{i}\left[D,\overline{D}\right]\mathcal{S} - \mathcal{S}^2\right)\triangle\tau.
\end{aligned}$$

The only remaining equation is K one

$$d_2\omega_{1K} - d_1\omega_{2K} = \big(\omega_{1K}\omega_{2D} - \omega_{1D}\omega_{2K}\big) + 2\mathrm{i}\big(\omega_{1S}\bar{\omega}_{2S} + \bar{\omega}_{1S}\omega_{2S}\big),$$

which is satisfied identically upon substitution of previous results.

$N = 3$ and $N = 4$ Schwarzians

The Schwarzians with higher supersymmetry, $N = 3$ and $N = 4$, were constructed in analogous way. At first, one should take the respective algebra, $osp(3|2)$ for $N = 3$ and $D(1,2;\alpha)$ for $N = 4$ and determine using the Maurer-Cartan equations what independent superfield survives in the Cartan forms, and determine in terms of the basic fermion after calculating the Cartan forms explicitly. In contrast to the $N = 2$ case, the $N = 3$ and $D(1,2;\alpha \neq -1)$ Schwarzians are fermions and appear as $d\theta$-projections of forms of internal symmetry generators, being the superfields with the lowest dimension for such algebras

$$\mathcal{S}_{N=3} = \frac{1}{2}\frac{\epsilon_{pqr} D_p\xi_n \, D_q D_r \xi_n}{D_k\xi_l \, D_k\xi_l}, \quad k,l,p,\ldots = 1,2,3;$$

$$\left(\mathcal{S}_{N=4}\right)_{ia} = \frac{[D_{ja}, D_{ci}]\,\xi^{kb} D^{jc}\xi_{kb}}{D^{md}\xi_{ne} D_{md}\xi^{ne}} + 6\mathrm{i}\,(1+2\alpha)\,\frac{\dot{\xi}^{dk} D_{ia}\xi_{dk}}{D^{md}\xi_{ne} D_{md}\xi^{ne}}, \quad i = 1,2, \; a = 1,2.$$

Situation for $\alpha = -1$ is different, as the only one instead of two $su(2)$ subalgebras appears in the commutation relations of supercharges of $D(1,2;-1)$. As a result, $d\theta$ projections of the $su(2)$ form disappear and the Schwarzian is the bosonic $\triangle\tau$ projection, just as in $N = 2$:

$$\mathcal{S}_a{}^b = \left[D_a, \overline{D}^b\right] u - \frac{1}{2}\delta_a^b \left[D_c, \overline{D}^c\right] u, \quad e^u = \frac{1}{2} D_a\xi^b \overline{D}^a \bar{\xi}_b.$$

$OSp(N|2)$ algebra

The known Schwarzians have up to $N=4$ supersymmetry, which related to the properties of underlying conformal field theories: for $N>4$ the primary superfields have to contain components with negative conformal dimension. However, within the algebraic approach, it is not forbidden to go further and check whether the super-Schwarzians exist for $N=6$ or $N=8$ supersymmetry. The most simple point to start, which closes for arbitrary N, is $OSp(N|2)$ superalgebra.

$$\begin{gathered}\left[D,P\right]=-\mathrm{i}P,\quad \left[D,K\right]=\mathrm{i}K,\quad \left[P,K\right]=2\mathrm{i}D,\\ \left\{Q_i,Q_j\right\}=2\delta_{ij}P,\quad \left\{S_i,S_j\right\}=2\delta_{ij}K,\quad \left\{Q_i,S_j\right\}=-2\delta_{ij}D+J_{ij},\\ \left[D,Q_i\right]=-\tfrac{\mathrm{i}}{2}Q_i,\quad \left[D,S_i\right]=\tfrac{\mathrm{i}}{2}S_i,\quad \left[K,Q_i\right]=\mathrm{i}S_i,\quad \left[P,S_i\right]=-\mathrm{i}Q_i\end{gathered}$$

Here, J_{kl} form $SO(N)$ and transform Q_i, S_i as vectors:

$$\begin{gathered}\left[J_{ij},J_{kl}\right]=\mathrm{i}\left(\delta_{ik}J_{jl}-\delta_{jk}J_{il}-\delta_{il}J_{jk}+\delta_{jl}J_{ik}\right),\\ \left[J_{ij},Q_k\right]=\mathrm{i}\left(\delta_{ik}Q_j-\delta_{jk}Q_i\right),\quad \left[J_{ij},S_k\right]=\mathrm{i}\left(\delta_{ik}S_j-\delta_{jk}S_i\right).\end{gathered}$$

Here, $i,j=1\ldots N$. We consider the standard parametrization of the group element

$$g=e^{\mathrm{i}tP}e^{\xi_i Q_i}e^{\psi_i S_i}e^{\mathrm{i}zK}e^{\mathrm{i}uD}e^{\lambda_{ij}J_{ij}}.$$

The invariant Cartan forms $g^{-1}dg$ are defined by the relation

$$\omega=g^{-1}dg=\mathrm{i}\omega_P P+\mathrm{i}\omega_K K+\mathrm{i}\omega_D D+\left(\omega_Q\right)_i Q_i+\left(\omega_S\right)_i S_i+\mathrm{i}\left(\omega_J\right)_{ij}J_{ij}.$$

Maurer-Cartan equations

The form $\omega = g^{-1}dg$ satisfies, by construction, Maurer-Cartan equations. As for $N=2$, let us write them in the form

$$d_2\omega_1 - d_1\omega_2 = \left[\omega_1, \omega_2\right],$$

where $\omega_1 = g^{-1}d_1 g$, $\omega_2 = g^{-1}d_2 g$ and differentials d_1, d_2 mutually commute. Therefore, we could say much about the forms before evaluating them explicitly. Let us assume that after imposing conditions $\omega_P = \triangle\tau = d\tau + \mathrm{i}d\theta_i\theta_i$, $\left(\omega_Q\right)_i = d\theta_i$, $\omega_D = 0$ the forms can be expanded in terms of $\triangle\tau$, $d\theta_i$ as

$$\left(\omega_S\right)_i = \triangle\tau\Psi_i + d\theta_j A_{ij}, \;\; \left(\omega_J\right)_{ij} = \triangle\tau X_{ij} + \mathrm{i}d\theta_k\Sigma_{kij}, \;\; \omega_K = \triangle\tau C + \mathrm{i}d\theta_i\,\Xi_i.$$

After introducing these constraints to the Maurer-Cartan equation, one finds that P equation is satisfied trivially, while Q equation reduces to two projections:

$$\left(\triangle_1\tau d_2\theta_j - \triangle_2\tau d_1\theta_j\right)\left(A_{ij} + 2X_{ij}\right) = 0, \;\; d_1\theta_k d_2\theta_l\left(\Sigma_{kil} + \Sigma_{lik}\right) = 0.$$

The last equation implies that Σ_{ijk} is completely antisymmetric. Another simple equation is D equation, which implies just $\Xi_k = \Psi_k$, some of equations are satisfied due to others.

Solution to Maurer-Cartan equations

The full list of nontrivial equations involves only Σ_{ijk}, X_{ij}, Ψ_i and C:

$$-2\delta_{kl}X_{ij} - \delta_{ik}X_{jl} + \delta_{jk}X_{il} - \delta_{il}X_{jk} + \delta_{jl}X_{ik} = D_k\Sigma_{lij} + D_l\Sigma_{kij} - 2\mathrm{i}\Sigma_{kin}\Sigma_{ljn} - 2\mathrm{i}\Sigma_{lin}\Sigma_{kjn},$$

$$\mathrm{i}D_kX_{ij} + \dot{\Sigma}_{kij} = -2X_{in}\Sigma_{knj} + 2X_{jn}\Sigma_{kni} + \frac{1}{2}\left(\delta_{ik}\Psi_j - \delta_{jk}\Psi_i\right),$$

$$D_k\Psi_i + 2\dot{X}_{ik} = -\delta_{ik}C - 4X_{ij}X_{jk} - 2\mathrm{i}\Sigma_{kij}\Psi_j,$$

$$\mathrm{i}D_kC + \dot{\Psi}_k = -4\Psi_iX_{ik}.$$

It should be noted that applying one derivative on the first equation, one can find D_kX_{ij}, reducing the second equation to δ_{ik} and δ_{jk} parts. Similarly, acting by one derivative on Ψ_i, obtain from the second equation, one can reduce the third to δ_{ik} part. Finally, from the first three equations one can obtain that the fourth is satisfied trivially. This happens without restriction on N, which is unusual for high supersymmetries. We see that all of the variables can be written in terms of Σ_{ijk} and its derivatives, thus making it a candidate for the Schwarzian.

$$X_{ij} = \frac{1}{2-N}\left(D_m\Sigma_{mij} - 2\mathrm{i}\Sigma_{imn}\Sigma_{jmn}\right), \quad \Psi_i = -\frac{2\mathrm{i}}{N-1}\left(D_lX_{il} + 2\mathrm{i}X_{mn}\Sigma_{imn}\right),$$

$$C = -\frac{1}{N}\left(D_j\Psi_j - 4X_{mn}X_{mn}\right).$$

Forms explicitly

To show that actually no on-shell constraints arise on Σ_{ijk}, let us study the Cartan forms explicitly. The forms, which give rise to the constraints $\omega_P = \triangle\tau$, $\left(\omega_Q\right)_i = d\theta_i$, $\omega_D = 0$, read

$$\omega_P = e^{-u}\triangle t = e^{-u}\left(dt + \mathrm{i}d\xi_j\,\xi_j\right),\;\; \left(\omega_Q\right)_i = e^{-u/2}\left(d\xi_j + \triangle t\psi_j\right)M_{ji},$$
$$\omega_D = du - 2\mathrm{i}d\xi_k\,\psi_k - 2z\triangle t,\;\; M_{ij} = \left(e^{2\lambda}\right)_{ij},\;\; M_{ik}M_{jk} = \delta_{ij}.$$

The constraints themselves

$$\dot{t} + \mathrm{i}\dot{\xi}_k\,\xi_k = e^{u},\;\; D_i t + \mathrm{i}D_i\xi_k\,\xi_k = 0,\;\; D_m\xi_k = e^{u/2}\left(M^{-1}\right)_{mk},$$
$$\psi_i = -e^{-u}\dot{\xi}_i,\;\; z = \frac{1}{2}e^{-u}\dot{u},\;\; D_i u = 2\mathrm{i}D_i\xi_j\,\psi_j$$

are partially algebraic, partially follow from the main one $D_i t + \mathrm{i}D_i\xi_k\,\xi_k = 0$. For example, applying D_j to it, one finds

$$D_i\left(D_j t + \mathrm{i}D_j\xi_k\,\xi_k\right) + D_j\left(D_i t + \mathrm{i}D_i\xi_k\,\xi_k\right) = 0 \;\Rightarrow\; \left(\dot{t} + \mathrm{i}\dot{\xi}_k\,\xi_k\right)\delta_{ij} = D_i\xi_k\,D_j\xi_k,$$

and $D_i\xi_j$ is orthogonal up to factor of $e^{u/2}$. Expansion of relation $D_i t + \mathrm{i}D_i\xi_k\,\xi_k = 0$ in terms of components in case $N = 8$ leads to conclusion that it is algebraic itself and allows to express all the components of t, ξ_i in terms of components of a single unconstrained superfield.

The Schwarzian

As the analysis of Maurer-Cartan equations shows, the Schwarzian Σ_{ijk} is the $d\theta_k$ projection of the form $\left(\omega_J\right)_{ij}$, which explicitly reads

$$\left(\omega_J\right)_{kl} = \frac{1}{2}\left(M^{-1}\right)_{km} dM_{ml} + \frac{\mathrm{i}}{2}\left(M^{-1}\right)_{km}\left(M^{-1}\right)_{nl} e^{-u}\left(d\xi_m\,\psi_n - d\xi_n\,\psi_m + \triangle t\psi_m\psi_n\right).$$

Extracting $d\theta_p$-projection,

$$\left(\omega_J\right)_{kl} = \ldots + d\theta_p\Big[\frac{1}{2}\left(M^{-1}\right)_{kn} D_p M_{nl} - \frac{\mathrm{i}}{2}\left(D_p\xi_m\dot{\xi}_n - D_p\xi_n\dot{\xi}_m\right) e^{-u}\left(M^{-1}\right)_{km}\left(M^{-1}\right)_{ln}\Big]$$
$$\equiv \ldots + \mathrm{i} d\theta_p \Sigma_{pkl}$$

one use the relation $D_i t + \mathrm{i} D_i \xi_j \xi_j = 0$ and its consequences to prove that the projection is indeed totally antisymmetric and represent it as a functional of ξ_k only

$$\mathrm{i}\Sigma_{pkl} = -\frac{1}{2} e^{-u} D_{[p} D_k \xi_m\, D_{l]} \xi_m = -\frac{N}{2 D_r\xi_s\, D_r\xi_s} D_{[p} D_k \xi_m\, D_{l]}\xi_m.$$

In the cases of $N = 3$ and $N = 4$ (in particular, for $osp(4|2) = D(1,2;-1/2)$) this result repeats already known ones.

The first property of the Schwarzian

The super-Scwharzians with $N \leq 4$ satisfied two properties, related to their conformal field theory origins.

First, just like the bosonic case

$$\mathcal{S}(t,\tau) = 0 \;\text{ if }\; t = \frac{a\tau + b}{c\tau + d},$$

it should be $\mathcal{S}(\zeta;\tau,\theta) = 0$ if ζ corresponds to a finite superconformal transformation. This property can be straightforwardly derived using the nonlinear realizations construction that allowed us to construct the super-Schwarzian. Note that all the Cartan forms, with exception of $\omega_P = \triangle\tau$, $\left(\omega_Q\right)_i = d\theta_i$ are proportional to the Schwarzian and its derivatives and nullify when $\Sigma_{ijk} = 0$. The remaining forms $\triangle\tau$ and $d\theta_i$ can be obtained from the “boundary” group element

$$g_b = e^{\mathrm{i}\tau P} e^{\theta_i Q_i}, \quad g_b^{-1} dg_b = \mathrm{i}\triangle\tau P + d\theta_i Q_i.$$

Thus Cartan forms of g and g_b are equal if $\Sigma_{ijk} = 0$, and $g = g_0 g_b$, where g_0 is some constant element of $OSp(N|2)$. This can be checked directly by evaluating the Cartan forms of both sides of $g = g_0 g_b$ and taking into account that $g^{-1}dg = g_b^{-1}dg_b$. Thus zero Schwarizians originate from group elements of form $g = g_0 e^{\mathrm{i}\tau P} e^{\theta_i Q_i}$, where g_0 is a constant $OSp(N|2)$ element.

The second property of the Schwarzian

The second property is transformation law of the Schwarian with respect to superconformal transformations of τ and θ_i. If the general superdiffeomorphisms $\tau' = T(\tau,\theta)$, $\theta_i' = \chi_i(\tau,\theta)$ are constrained by $D_i T + \mathrm{i} D_i \chi_j \chi_j = 0$, the odd derivative transforms covariantly $D_i = D_i\chi_j D_j'$ and for the Schwarzian we should have the following composition law

$$\Sigma_{ijk}\left[\zeta(\tau',\theta');\tau,\theta\right] = \Sigma_{ijk}\left[\chi(\tau',\theta');\tau,\theta\right] + M_{[ijk]}^{[mnp]}\Sigma_{mnp}\left[\zeta(\tau',\theta');\tau',\theta'\right]$$

with some structure matrix $M_{[ijk]}^{[mnp]}$, making Σ essentially a connection for superconformal transformations. Analogous property holds for all $N \leq 4$ super-Schwarzians and was used by Schoutens to derive them.
This property can be checked because Σ_{ijk} is explicitly known. If the field $\zeta_i(\tau',\theta')$ satisfies the standard condition $D_i\zeta_k D_j\zeta_k \sim \delta_{ij}$, one can show that

$$D_p\zeta_q D_p\zeta_q = \frac{D_k\chi_l D_k\chi_l}{N} D_p'\zeta_q D_p'\zeta_q$$

and, therefore,

$$\Sigma_{ijk}\left[\zeta(\tau',\theta');\tau,\theta\right] = \Sigma_{ijk}\left[\chi(\tau',\theta');\tau,\theta\right] + \frac{N}{D_r\chi_s D_r\chi_s} D_{[i}\chi_m D_j\chi_n D_{k]}\chi_p\, \Sigma_{mnp}\left[\zeta(\tau',\theta');\tau',\theta'\right]$$

Thus Σ_{ijk} satisfies some composition law, as expected for a super-Schwarzian.

Other Schwarzians with extended supersymmetry?

Among other superextensions of $so(1,2)$ are one infinite series of superalgebras $su(1,1|N)$ with $2N$ supersymmetry and also $osp(4^\star|4)$ and $F(4)$ with 8 supersymmetries.

Quick analysis of Maurer-Cartan equations for $su(1,1|N)$ shows that no Schwarzian exists for such an algebra, even for $N=3$ or $N=4$ (6 or 8 supersymmetries). Much like the case of $N=2$, P equation is satisfied identically, and Q equation implies that $d\theta^a$, $d\bar\theta_b$ projections of $su(N)\times u(1)$ forms are zero. However, next equations imply that $\triangle\tau$ projections of $su(N)\times u(1)$ forms have to satisfy an algebraic relation,

$$\mathcal{S}_b{}^a\delta_c^d = \delta_c^a\mathcal{S}_b{}^d + \delta_b^d\mathcal{S}_c{}^a - \frac{2}{N}\delta_b^a\mathcal{S}_c{}^d - \delta_c^a\delta_b^d s + \frac{1}{N}\delta_b^a\delta_c^d s$$

which are solvable only if $N=1$ or $N=2$. Explicit calculation of the forms shows that standard conditions $\omega_P = \triangle\tau$, $\left(\omega_Q\right)^a = d\theta^a$, unlike $OSp(N|2)$ case, put the system on shell with second order equations of motion.
The Schwarzian for $osp(4^\star|4)$ algebra could not constructed for the same reasons, the case of $F(4)$ was not studied yet.

Conclusion

In this talk, we described how all the known supersymmetric Schwarzians can be constructed by applying method of nonlinear realizations to superconformal algebras $osp(1|2)$, $osp(2|2)$, $osp(3|2)$ and $D(1,2;\alpha)$, and then enforcing almost universal conditions on the Cartan forms

$$\omega_P = \triangle\tau, \quad \left(\omega_Q\right)_i = d\theta_i, \quad \omega_D = 0.$$

These constraints ensure that other Cartan forms can be expressed in terms of one superfield of lowest dimension, namely the super-Schwarzian.

Applying the same method to superalgebras with higher supersymmetry, we find that $OSp(N|2)$ Cartan form with $N > 4$ can be expressed in terms of

$$\mathrm{i}\Sigma_{ijk} = -\frac{N}{2D_p\xi_q\, D_p\xi_q} D_{[i}D_j\xi_m\, D_{k]}\xi_m, \quad i = 1\ldots N.$$

This object also satisfies two properties expected to hold for a super-Schwarzian. In contrast, it was found that for superalgebras $osp(4^\star|4)$ and $su(1,1|N)$, $N > 2$, analogous superfields do not exist.