# One-loop divergences in the six-dimensional $\mathcal{N} = (1, 0)$ hypermultiplet self-coupling model

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## Aims:

- to study the gauge-invariant $\mathcal{N} = (1,0)$ supersymmetric sigma-model in six dimensions
- to develop the manifest gauge invariant formulation of quantum corrections in the model
- to calculate the one-loop divergent contribution to the effective action
- to obtain the leading low-energy contributions to the effective action

The talk is based on:
**A.S. Budekhina, BM**, *Phys.Rev.D 104 (2021)*.

## General motivation

The modern interest to $6D$ field theories is stipulated by the following reasons:

- The problem of describing the quantum structure of six-dimensional supersymmetric gauge theories dimensionally reduced from superstrings and the connection of effective action for the D5-branes at low energies with maximally supersymmetric Yang-Mills theory in six dimensions. [**N.Seiberg (1996), E. Witten (1996); N. Seiberg, (1997)**].

- Lagrangian description of the interacting multiple $M5$-branes is related to $6D$, $\mathcal{N} = (2, 0)$ supersymmetric gauge theory. The theory includes self-dual non-Abelian antisymmetric tensor and it is not constructed still (see e.g. reviews [**J. Bagger, N. Lambert, S. Mikhu, C. Papageorgakis (2013), G. Moore (2012)**]).

- Revised critical $O(N)$ behavior in six dimensions [**L. Fei, S. Giombi, I. Klebanov (2014), S. Giombi, I. Klebanov, S. Pufu, G. Tarnopolsky (2020)**] and UV completion of the $CP(N)$ sigma-model [**J.Gracey (2020), H. Khachatryan (2019)**].

- The six-dimensional gauge-invariant $\mathcal{N}=(1,0)$ supersymmetric sigma-model possess by hyper-Kähler manifold as a target space and was steadied in terms of physical components in [**G. Sierra, P.K. Townsend, (1983)**].

- The off-shell formulation of the model is possible using the harmonic superspace [**A. Galperin, E. Ivanov, S. Kalitsyn, V. Ogievetsky, E. Sokatchev, (1984)**]. Classification hyper-Kähler metrics in such approach is provided in accordance with the analytic superfield self-interactions potential in general $d=4, \mathcal{N}=2$ hypermultiplet theory with self-interaction [**A. Galperin, E. Ivanov, V. Ogievetsky, E. Sokatchev, (1985)**].

- The study $d=1$ hyper-Kähler sigma-models with extended supersymmetry with hypermultiplet (**see, e.g.,[S. Fedoruk, E. Ivanov, A. Smilga, (2014),(2018)] and also for a review [A. Smilga,***Differential Geometry Through Supersymmetric Glasses***,World Scientific (2020)]**) and tensor multiplet (see, e.g, **[ S. Krivonos, A. Shcherbakov (2008)] and [G. Moore, (2012)]** for a review).

Our aim is to develop gauge invariant technique to study quantum corrections to the effective action in the theory.

**P.S. Howe, G. Sierra, P.K. Townsend, (1983).**

## $6D$ Minkowski space

- Coordinates $x^M$, $M = 0, 1, 2, 3, 4, 5$
- Metric $\eta_{MN} = diag(1, -1, -1, -1, -1, -1)$
- Proper Lorentz group $SO(1, 5)$

## Two types of $6D$ Spinors

- Left $(1, 0)$ spinors $\psi_a$, $a = 1, 2, 3, 4$
- Right $(0, 1)$ spinors $\phi^a$, $a = 1, 2, 3, 4$

## Dirac matrices

- $8 \times 8$ Dirac matrices $\Gamma_M$,

$$\Gamma_M \Gamma_N + \Gamma_N \Gamma_M = 2\eta_{MN}$$

- Representation of the Dirac matrices

$$\Gamma_M = \begin{pmatrix} 0 & \tilde{\gamma}_M \\ -\gamma_M & 0 \end{pmatrix},$$

- Antisymmetric $4 \times 4$ Weyl-type matrices $\gamma_M$ and $\tilde{\gamma}_M$,

$$\gamma_M \tilde{\gamma}_N + \gamma_N \tilde{\gamma}_M = -2\eta_{MN}$$

$$(\tilde{\gamma}_M)^{ab} = \frac{1}{2}\epsilon^{abcd}(\gamma_M)_{cd}$$

- Spinor representation of the vectors, $V_{ab} = \frac{1}{2}(\gamma^M)_{ab}V_M$ .

## $6D$ superalgebra

- Two types of independent supercharges $Q^I_a, Q^a_J$, $I = 1, ..., 2m; J = 1, ..., 2n$
- $\mathcal{N} = (m, n)$ supersymmetry
- Anticommutational relations for supercharges

$$\{Q^I_a, Q^K_b\} = 2\Omega^{IK} P_{ab}$$

$$\{Q^a_J, Q^b_L\} = 2\Omega_{JL} P^{ab}$$

Matrix $\Omega_{IK}$ belongs to $USp(2n)$ group (R-symmetry group), $\Omega_{IK}\Omega^{KJ} = \delta^J_I$

- $\mathcal{N} = (1, 0)$ superspace, $I = i$, coordinates $z = (x^M, \theta^a_i), i = 1, 2$
- Basic spinor derivatives

$$D^i_a = \frac{\partial}{\partial\theta^a_i} - i\theta^{ib}\partial_{ab}, \quad \{D^i_a, D^j_b\} = -2i\Omega^{ij}\partial_{ab}$$

## Harmonic superspace

### $4D$

**A.Galperin, E. Ivanov, S. Kalitsyn, V. Ogievetsky, E. Sokatchev, (1984).**

**A.Galperin, E. Ivanov, V. Ogievetsky, E. Sokatchev,** *Harmonic Superspace*, **(2001).**

General purpose:
to formulate $\mathcal{N} = 2$ models in terms of unconstrained $\mathcal{N} = 2$ superfields.
General idea:
to use the parameters $u^{\pm i}(i = 1, 2)$ (harmonics) related to $SU(2)$ automorphism group of the $\mathcal{N} = 2$ superalgebra and parameterizing the 2-sphere, $u^{+i}u_i^- = 1$.

### $6D$

**P.S. Howe, K.S. Stelle, P.C. West, (1985).**
**B.M. Zupnik, (1986); (1999).**

**G. Bossard, E. Ivanov, A. Smilga, (2015).**

Note! Pure spinor approach to describe $6D$ SYM theories, [**M. Cederwall, (2018)**].

## $\mathcal{N} = (1,0)$ harmonic superspace

- $USp(2) \sim SU(2), I = i$ The same harmonics $u^{\pm i}$ as in $4D, \mathcal{N} = 2$ supersymmetry
- Harmonic $6D, (1,0)$ superspace with coordinates $Z = (x^M, \theta^a_i, u^{\pm i})$
- Analytic basis $Z_{(an)} = (x^M_{(an)}, \theta^{\pm a}, u^\pm_i)$,
  $x^M_{(an)} = x^M + \frac{i}{2}\theta^{-a}(\gamma^M)_{ab}\theta^{+b}, \quad \theta^{\pm a} = u^\pm_i \theta^{ai}$
  The coordinates $\zeta = (x^M_{(an)}, \theta^{+a}, u^\pm_i)$ form a subspace closed under $(1,0)$ supersymmetry
- The harmonic derivatives

$$D^{++} = u^{+i}\frac{\partial}{\partial u^{-i}} + i\theta^{+a}\theta^{+b}\partial_{ab} + \theta^{+a}\frac{\partial}{\partial \theta^{-a}},$$

$$D^0 = u^{+i}\frac{\partial}{\partial u^{+i}} - u^{-i}\frac{\partial}{\partial u^{-i}} + \theta^{+a}\frac{\partial}{\partial \theta^{+a}} - \theta^{-a}\frac{\partial}{\partial \theta^{-a}}$$

- Spinor derivatives in the analytic basis

$$D^+_a = \frac{\partial}{\partial \theta^{-a}}, \quad D^-_a = -\frac{\partial}{\partial \theta^{+a}} - 2i\partial_{ab}\theta^{-b}, \quad \{D^+_a, D^-_b\} = 2i\partial_{ab}$$

- Analytic superfields $\phi$ do not depend on $\theta^{-\alpha}$, $D^+_\alpha \phi = 0$

Hypermultiplet and Vector multiplet in $6D$

- On-shell hypermultiplet:

$$q^{+} = f^{+} + \theta^{a+}\psi_{a} + ldots,$$

- On-shell vector multiplet:

$$V^{++} = \theta^{+a}\theta^{+b}A_{ab} + 2(\theta^{+})_{a}\lambda^{-a} + \ldots.$$

# The model

The action for the model is

$$S_0[q^+, V^{++}] = \frac{1}{4f^2} \int d^{14}z \frac{du_1 du_2}{(u_1^+ u_2^+)^2} V^{++}(z, u_1) V^{++}(z, u_2) - \int d\zeta^{(-4)} \Big( \tilde{q}^+ D^{++} q^+ + i \tilde{q}^+ V^{++} q^+ + L^{(+4)} \Big), \quad (1)$$

where the coupling constant $f$ has the dimension of the inverse mass. The gauge transformations

$$\delta V^{++} = -D^{++}\lambda\,, \quad \delta q^+ = i\lambda\, q^+\,, \quad \delta \tilde{q}^+ = -i\lambda\, \tilde{q}^+\,, \quad (2)$$

with a real analytic gauge parameter $\lambda = \lambda(\zeta, u)$.
Gauge invariance restrict the structure of the potential

$$L^{(+4)} = L^{(+4)}(\tilde{q}^+ q^+).$$

# Background field method

Aim: construction of gauge invariant effective action, (see, e.g., [**B.DeWitt (1965)**]).
Realization:

- The superfields $V^{++}, q^{+}$ are splitting into the sum of the background $V^{++}, Q^{+}$ and the quantum $v^{++}, q^{+}$ superfields

$$V^{++} \to V^{++} + f v^{++}, \qquad q^{+} \to Q^{+} + f q^{+}$$

- The action is expending in a power series in quantum fields.
- The gauge-fixing function are imposed only on quantum superfiled

$$\mathcal{F}_{\tau}^{(+4)} = D^{++} v_{\tau}^{++} = e^{-ib} (\nabla^{++} v^{++}) e^{ib} = e^{-ib} \mathcal{F}^{(+4)} e^{ib} \ ,$$

- Faddev-Popov procedure is used. The effective action is constructed analogous to one in $4D, \mathcal{N}=2$ SYM theory [**I.L.Buchbinder, E.I. Buchbinder, S.M. Kuzenko, B.A. Ovrut, (1998)**].

## Background field method

The gauge fixing action for the 'quantum' field $v^{++}$

$$S_{\rm gf} = -\frac{1}{4}\int d^{14}z \frac{du_1 du_2}{(u_1^+ u_2^+)^2} v^{++}(z,u_1) v^{++}(z,u_2) + \frac{1}{8}\int d^{14}z\, du\, v^{++}(D^{--})^2 v^{++} \,.$$

Introducing the covariant harmonic derivative

$$\mathcal{D}^{++} = D^{++} + iV^{++} + i\Psi^{++} \tag{4}$$

and use the notations

$$\Psi^{++} = -i\frac{\partial L^{(+4)}(\tilde{Q}^+ Q^+)}{\partial(\tilde{Q}^+ Q^+)}\,, \quad \Psi = \frac{\partial^2 L^{(+4)}(\tilde{Q}^+ Q^+)}{\partial(\tilde{Q}^+ Q^+)^2}\,. \tag{5}$$

We obtain the quadratic part of the action into the matrix form

$$S_2[Q^+;V^{++}] = -\frac{1}{2}\int d\zeta^{(-4)} \left( \widetilde{q^+ \quad \tilde{q}^+} \right) \left(\mathcal{D}^{++}\cdot \mathbf{1} + \mathcal{Q}^{++}\right) \begin{pmatrix} q^+ \\ \tilde{q}^+ \end{pmatrix}, \tag{6}$$

where $\mathcal{Q}^{++}$ is defined as follows

$$\mathcal{Q}^{++} = \Psi \begin{pmatrix} \tilde{Q}^+ Q^+ & (Q^+)^2 \\ -(\tilde{Q}^+)^2 & -\tilde{Q}^+ Q^+ \end{pmatrix} \tag{7}$$

# Background field method

We remove such terms by the change of the quantum hypermultiplet variables in the path integral (see, e.g., **[I.L. Buchbinder, E.A. Ivanov, B.S. Merzlikin, K.V. Stepanyantz, (2017)]**)

$$\left( \begin{array}{c} q^{+} \\ \tilde{q}^{+} \end{array} \right) \rightarrow \left( \begin{array}{c} q^{+} \\ \tilde{q}^{+} \end{array} \right) - f \int d\zeta_2^{(-4)} \mathbf{G}^{(1,1)}(1|2) \left( \begin{array}{c} iQ^{+} \\ -i\tilde{Q}^{+} \end{array} \right)_2 v_2^{++}. \tag{8}$$

The Green function $\mathbf{G}(1|2)$ satisfies the equation

$$\Big(\mathcal{D}^{++} \cdot \mathbf{1} + \mathcal{Q}^{++}\Big)\mathbf{G}^{(1,1)}(1|2) = \delta^{(3,1)}(1|2) \cdot \mathbf{1}\,. \tag{9}$$

A formal solution of the equation (9) has a form

$$\mathbf{G}^{(1,1)} = \mathcal{G}^{(1,1)} \cdot \mathbf{1} - \mathcal{G}^{(1,1)}\mathcal{Q}^{++}\mathcal{G}^{(1,1)} + \ldots, \tag{10}$$

where the Green function $\mathcal{G}^{(1,1)}$ satisfies the equation

$$\mathcal{G}^{++}\mathcal{G}^{(1,1)}(1|2) = \delta^{(3,1)}(1|2)\,. \tag{11}$$

One-loop quantum correction to the classical action (1)

$$\Gamma^{(1)} = \frac{i}{2}\mathrm{Tr}_{(3,1)} \ln\left(\mathcal{D}^{++}\cdot\mathbf{1} + \mathcal{Q}^{++}\right) + \frac{i}{2}\mathrm{Tr}_{(2,2)} \ln\left(\partial^2 - 4f^2\widetilde{Q}^{+}\mathcal{G}^{(1,1)}Q^{+} + \ldots\right),$$

reduced to

$$\Gamma^{(1)} = i\mathrm{Tr}_{(3,1)} \ln \mathcal{G}^{++} + \frac{i}{2}\mathrm{Tr}_{(2,2)} \ln\left(\partial^2 - 4f^2\widetilde{Q}^{+}\mathcal{G}L^{(1,1)}Q^{+} + \ldots\right). \qquad (12)$$

# Background field method

For convenience, we introduce a notation

$$\mathcal{V}^{++} = V^{++} + \Psi^{++}. \tag{13}$$

Then we consider the non-analytic gauge connection $\mathcal{V}^{--}$ by the rule

$$D^{--}\mathcal{V}^{++} = D^{++}\mathcal{V}^{--}\,, \tag{14}$$

which can be solved exactly in the abelian case

$$\mathcal{V}^{--} = \int du_1 \frac{\mathcal{V}^{++}(u_1)}{(u^+u_1^+)^2}\,, \tag{15}$$

and introduce the gauge field strength $\mathcal{W}^{+a}$

$$\mathcal{W}^{+a} = -\tfrac{i}{6}\varepsilon^{abcd}D_b^+D_c^+D_d^+\mathcal{V}^{--}\,, \quad \mathcal{W}^{-a} := \nabla^{--}\mathcal{W}^{+a}\,. \tag{16}$$

and the analytic superfield $\mathcal{F}^{++} = D_a^+\mathcal{W}^{+a}$ with the properties

$$D^{++}\mathcal{F}^{++} = 0, \quad D^{++}\mathcal{W}^{+a} = 0, \quad D^{++}\mathcal{W}^{-a} = 0\,, \tag{17}$$

where $\mathcal{W}^{-a} = D^{--}\mathcal{W}^{+a}$. Also we separate the contributions

$$\mathcal{W}^{+a} = \mathcal{W}_V^{+a} + \mathcal{W}_Q^{+a}\,, \quad \mathcal{F}^{++} = \mathcal{F}_V^{++} + \mathcal{F}_Q^{++}\,. \tag{18}$$

The divergent contributions from quantum hypermultiplet

$$\Gamma_{q,\,\mathrm{div}} = \frac{1}{6(4\pi)^3\varepsilon}\int d^{14}z\, du\, \mathcal{V}^{--}\partial^2\mathcal{V}^{++} = \frac{1}{6(4\pi)^3\varepsilon}\int d\zeta^{(-4)}(\mathcal{F}^{++})^2 \qquad (19)$$

taking into account the connection of the variation $\delta\mathcal{V}^{++}$ with $\delta\mathcal{V}^{--}$

$$\delta\mathcal{V}^{--} = \tfrac{1}{2}(D^{--})^2\delta\mathcal{V}^{++} - \tfrac{1}{2}D^{++}(D^{--}\delta\mathcal{V}^{--})\,, \qquad (20)$$

and the properties (17) for the superfield $\mathcal{F}^{++}$ we can rewrite (19) in the form

$$\Gamma_{q,\,\mathrm{div}} = \frac{1}{6(4\pi)^3\varepsilon}\int d\zeta^{(-4)}(\mathcal{F}^{++}_V + \mathcal{F}^{++}_Q)^2\,. \qquad (21)$$

The divergent contributions from quantum gauge multiplet reads

$$\Gamma_{v,\,\mathrm{div}} = \frac{2if^2}{(4\pi)^3\varepsilon}\int d\zeta^{(-4)}\,\widetilde{Q}^{+}\mathcal{F}^{++}Q^{+}\,. \qquad (22)$$

Finally, we combine two contributions (19) and (22)

$$\Gamma^{(1)}_{\rm div} = \frac{1}{6(4\pi)^3\varepsilon}\int d\zeta^{(-4)}(\mathcal{F}^{++}_V + \mathcal{F}^{++}_Q)\big(\mathcal{F}^{++}_V + \mathcal{F}^{++}_Q + 12if^2\widetilde{Q}^+Q^+\big), \quad (23)$$

Let us discuss the final result (23). The classical equations of motion for the background superfields have the form

$$\mathcal{F}^{++}_V - 2if^2\widetilde{Q}^+Q^+ = 0\,, \quad \mathcal{D}^{++}Q^+ = 0\,, \quad (24)$$

where the covariant harmonic derivative $\mathcal{D}$ was introduced in (4).

As an example, we consider the case [**A. Galperin, E. Ivanov, V. Ogievetsky, E. Sokatchev, (1985)**]

$$L^{(+4)} = \tfrac{1}{2}(\tilde{Q}^{+}Q^{+})^{2},$$

$$\Psi^{++} = -i\tilde{Q}^{+}Q^{+}\,, \quad \Psi^{--} = \int du_2 \frac{\Psi_2^{++}}{(u_1^{+}u_2^{+})^2} = -i\tilde{Q}^{-}Q^{-}\,. \tag{25}$$

The divergent contributions reads

$$\begin{aligned}\Gamma^{(1)}_{\rm div}[Q^{+}] \;&=\; \frac{1}{6(4\pi)^3\varepsilon}\int d^{14}z\,du\left(\tilde{Q}^{-}Q^{-}\partial^2(\tilde{Q}^{+}Q^{+}) + 16f^2\tilde{Q}^{-}Q^{-}\tilde{Q}^{+}Q^{+}\right)\\ &\quad -\frac{14f^4}{3(4\pi)^3\varepsilon}\int d\zeta^{(-4)}(\tilde{Q}^{+}Q^{+})^2\,,\end{aligned} \tag{26}$$

The leading low-energy finite contributions in case of constant background is

$$\Gamma^{(1)}_{\rm lead}[Q^{+}] = \frac{1}{32\pi^3 m^2}\int d\zeta^{(-4)}(\mathcal{W}_Q^{+})^4 = 0\,. \tag{27}$$

However, in four dimensions the one-loop effective action of the hypermultiplet with gauge-invariant self-interacting term contains the divergent contribution of the following form

$$\Gamma^{(1)}_{4D,\,\mathrm{div}} = \frac{1}{32\pi^2\varepsilon} \int d^{12}z\, du\, \mathcal{V}^{--}\mathcal{V}^{++}\,. \tag{28}$$

and leading finite contribution

$$\Gamma^{(1)}_{4D,\,\mathrm{lead}} \sim \int d^{12}z\, du\, \mathcal{W}^2 \ln \mathcal{W}^2\,. \tag{29}$$

# Conclusion and outlook

- We developed the manifest covariant superfield approach to study the gauge-invariant $\mathcal{N} = (1,0)$ hyper-Kähler sigma-model in six-dimensions
- We calculate the divergent contribution to one-loop effective action and simply demonstrate that leading low-energy contribution do not appear in the model
- The procedure can straightforwardly be applied to the hypermultiplet models in divers dimensions
- The procedure can be generalized to the arbitrary number of hypermultiplets [**I.L. Buchbinder, A.S. Budekhina, BM,** *Eur.Phys.J.C 82 (2022)*]
- The question of the dimension reduction of the model from six dimensions to lower ones is open