TOPOLOGICAL PHASES IN SOLID STATE PHYSICS

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Introduction

The topological phase theory is based on the homotopy approach to the study of properties of solid states. A key property of such states used in this theory is the existence of the energy gap stable under small deformations. It motivates the application of topological methods to the investigation of topological phases.

The topological phases are defined in the following way. Denote by G the symmetry group and consider the set Ham_G of classes of homotopy equivalent G-symmetric Hamiltonians satisfying the energy gap condition. It is possible to introduce on this set a natural stacking operation such that Ham_G, provided with this operation, becomes an Abelian monoid (i.e. an Abelian semigroup with the neutral element). The group of invertible elements of this monoid is precisely the topological phase. The initial ideas, lying in the base of the theory of topological phases, were formulated by Alexei Kitaev in his talks.

It turns out that the family (F_d) of d-dimensional topological phases forms an Ω-spectrum. In other words, it is a collection of topological spaces F_d having the property that the loop space ΩF_{d+1} is homotopy equivalent to the space F_d. This fact opens a way to wide use of algebraic topology methods for the study of topological phases. More concretely, one can associate with any Ω-spectrum the generalized cohomology theory, determined by the functor h^d, which assigns to the topological space X the set $[X, F_d]$ of classes of homotopy equivalent maps $X \to F_d$.

Topological phases

We consider the quantum mechanical systems described by the Hamiltonians H invariant under the action of the symmetry group G. The Hamiltonians H are given by selfadjoint operators defined on a Hilbert space $\mathcal{H}$ and the group G acts on $\mathcal{H}$ by the unitary or anti-unitary operators. Apart from G-invariance condition we shall impose on Hamiltonians H some other restrictions, the most important of them is the gap condition requiring that the point 0 should not belong to the spectrum of H. We shall call the G-symmetric gapped Hamiltonians admissible. It is useful to describe the properties of admissible Hamiltonians in terms of their ground states, i.e. the eigenstates with minimal energy. Such states will be also called admissible.

We introduce on the set of admissible ground states the following stacking operation. Suppose that we are given the two admissible ground states Φ_0 and Φ_1 with associate admissible Hamiltonians H_0 and H_1, acting in Hilbert spaces $\mathcal{H}_0$ and $\mathcal{H}_1$ respectively. The stacking of these two states is the ground state of the form

$$\Phi = \Phi_0 \otimes \Phi_1$$

associated with the Hamiltonian H, acting in the tensor product

$$\mathcal{H} = \mathcal{H}_0 \otimes \mathcal{H}_1.$$

The symmetry group G acts in $\mathcal{H}$ as the tensor product of representations of G in the Hilbert spaces $\mathcal{H}_0$ and $\mathcal{H}_1$ and the operator H is given by the equality

$$H = H_0 \otimes I + I \otimes H_1.$$

The constructed ground state Φ and Hamiltonian H are G-symmetric and gapped if the initial ground states Φ_0, Φ_1 and Hamiltonians H_0, H_1 were of this type.

Consider next continuous deformations of admissible Hamiltonians, i.e. continuous paths of the form H_t, $0 \le t \le 1$, in the class of admissible Hamiltonians.

Denote by Ham_G the set of classes of homotopy equivalent admissible Hamiltonians and the corresponding ground states. The stacking operation, introduced above, can be pushed down to a binary operation on Ham_G. Denote by $[\Phi]$ the class in Ham_G containing the ground state Φ and by $[\Phi_1] + [\Phi_2]$ the stacking of the ground states $[\Phi_1]$ and $[\Phi_2]$. This operation has the following properties:

1. **associativity**: for any admissible ground states $[\Phi_1]$, $[\Phi_2]$, $[\Phi_3]$ the following relation holds

$$([\Phi_1] + [\Phi_2]) + [\Phi_3] = [\Phi_1] + ([\Phi_2] + [\Phi_3]);$$

2. **commutativity**: for any admissible ground states $[\Phi_1]$, $[\Phi_2]$ the following equality takes place

$$[\Phi_1] + [\Phi_2] = [\Phi_2] + [\Phi_1];$$

3. **existence of the neutral element**: there exists a trivial ground state $[0]$ such that for any admissible ground state $[\Phi]$ we have the equality

$$[0] + [\Phi] = [\Phi] + [0] = [\Phi].$$

Below we shall consider a concrete example of the lattice system having the formulated properties. Here we point out only the analogy of the given construction with the definition of the semigroup $\mathrm{Vect}_s(X)$ of vector bundles over the topological space X defined up to the stable equivalence. By this analogy the stacking operation corresponds to the direct sum of bundles while the trivial state corresponds to the trivial bundle.

We shall call by the SRE(short range entangled)-state the admissible ground state which is homotopic to the trivial one in the class of admissible states.

Taking into account the G-symmetry, we shall call by SPT(symmetry protected topological)-phase (or G-protected topological phase) the class in Ham_G such that any of its representatives is an SRE-state if we ignore the G-symmetry. In other words, if any representative of this phase can be connected with the trivial state if one forgets the G-symmetricity condition.

Here is another, more formal definition. As it was pointed out before the space Ham_G, provided with the stacking operation, is an Abelian monoid. The group of the invertible elements of the monoid Ham_G is called the SPT-phase. It is an Abelian group with respect to stacking operation.

To describe the many-particle states we shall need the bosonic Fock space.

The bosonic Fock space over the Hilbert space $\mathcal{H}$ is defined as the completion

$$\mathcal{B}(\mathcal{H}) = \overline{\mathfrak{S}(\mathcal{H})} = \overline{\bigoplus_p \mathfrak{S}^p(\mathcal{H})}$$

where $\mathfrak{S}^p(\mathcal{H})$ is the subspace of p-particle states of the form

$$\mathfrak{S}^p(\mathcal{H}) = \mathrm{span}\{v_1 \otimes \ldots v_p, v_j \in \mathcal{H}\}.$$

The inner product $(\cdot\,,\cdot)$ in the space $\mathcal{H}$ is extended in a natural way to the inner product on the Fock space $\mathcal{B}(\mathcal{H})$. Namely, it is defined on monomials of the same degree as

$$(v_1 \otimes \ldots v_p, v_1' \otimes \ldots v_p') = \sum_{\sigma} (v_1, v_{i_1}') \cdot \ldots \cdot (v_p, v_{i_p}')$$

where the summation is taken over all permutations $\sigma = \{i_1, \ldots, i_p\}$ of the set $\{1, \ldots, p\}$ (the inner product of monomials of different degrees is set to zero). The inner product on monomials is extended by linearity to the whole algebra $\mathfrak{S}(\mathcal{H})$. The bosonic Fock space $\mathcal{B}(\mathcal{H})$ is the completion of the algebra $\mathfrak{S}(\mathcal{H})$ with respect to the introduced norm.

If $\{w_n\}$, $n = 1, \ldots$, is an orthonormal basis of the space $\mathcal{H}$ then we can take for the orthonormal basis of the Fock space $\mathcal{B}(\mathcal{H})$ the collection of monomials of the form

$$P_K(v) = \frac{1}{\sqrt{K!}} (v, w_1)^{k_1} \cdot \ldots (v, w_n)^{k_n}$$

where $v \in \mathcal{H}$, $K = (k_1, \ldots, k_n)$, $k_i \in \mathbb{N}$, and $K! = k_1! \ldots, k_n!$.

Denote by $a_i^\dagger$ the operator of creation of a particle in the state w_i given by the operator of multiplication by the inner product with w_i. The adjoint operator of annihilation of a particle in the state w_i coincides with the operator $-\partial_{w_i}$ where ∂_{w_i} is the operator of differentiation in the direction of w_i. These operators satisfy the standard commutation relation

$$[a_i^\dagger, a_j] = \delta_{ij}$$

(the commutators of other operators are equal to zero). An arbitrary linear operator $O : \mathcal{H} \to \mathcal{H}$ can be extended to a linear operator $\hat{O} : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ given on monomials by the formula

$$\hat{O}(v_1 \otimes \ldots \otimes v_p) = (Ov_1) \otimes \ldots \otimes (Ov_p)$$

with subsequent extension by linearity and completion to the whole space $\mathcal{B}(\mathcal{H})$. In terms of creation and annihilation operators this operator is written in the form

$$\hat{O} = \sum_{i,j} O_{ij} a_i^\dagger a_j. \tag{1}$$

Suppose that we have a lattice $\mathcal{L}$ in $\mathbb{R}^d$, i.e. a discrete Abelian group, isomorphic to $\mathbb{Z}^d$, which acts on $\mathbb{R}^d$ by translations by vectors $\gamma \in \mathcal{L}$. Denote by G the symmetry group of the Hamiltonian. The class of admissible Hamiltonians H consists in this case of d-dimensional, local, G-symmetric, gapped selfadjoint operators acting in the Hilbert space $\mathcal{H}$ and the Fock space $\mathcal{B}(\mathcal{H})$. The admissible operators are given by the formula (1) in which the number of terms in the sum does not exceed a common constant k (locality condition). The Hamiltonian H is called bosonic if there exists a (finite-dimensional) Hilbert space $\mathcal{H} = \mathcal{H}_\gamma$ associated with every $\gamma \in \mathcal{H}$.

The trivial state, called also the trivial product, is the state of the form

$$\bigotimes_{\gamma\in\mathcal{L}} \Phi_\gamma \in \bigotimes_{\gamma\in\mathcal{L}} \mathcal{H}_\gamma$$

without entanglement between the states corresponding to different γ's. For any pair of such states there exists a path, connecting them in the space of trivial states. In this setting the SRE-state is the ground state of a local gapped Hamiltonian which can be connected by a path to the trivial state.

In the same terms the d-dimensional G-invariant topological phase is called by the G-protected topological phase or SPT-phase if any of its representatives is an SRE-state if we ignore the G-symmetry, i.e. it can be connected by a path with the trivial state.

Recall that the symmetry group G acts on the Hilbert space $\mathcal{H}$ by the unitary or anti-unitary transformations. It is convenient to introduce the homomorphism $\phi : G \to \mathbb{Z}_2 = \{\pm 1\}$ indicating that for $\phi(g) = +1$ the element $g \in G$ acts on $\mathcal{H}$ as a unitary operator while for $\phi(g) = -1$ it acts as an anti-unitary operator. Apart from that the group G may contain the symmetry with respect to the time inversion, given by the homomorphism $T : G \to \{\pm 1\}$, and the symmetry with respect to the charge conjugation, given by the homomorphism $C : G \to \{\pm 1\}$.

The G-protected SPT-phases have the functoriality property which means that being given a homomorphism $\varphi : G' \to G$ of symmetry groups the composition of φ with a representation of the group G in the Hilbert space $\mathcal{H}$ generates the representation of the group G' in the same Hilbert space defining in this way the G'-protected SPT-phase.

We generalize now the initial problem by including a (real or complex) C^*-algebra $\mathcal{A}$ into the play. We shall consider the pairs $(G, \mathcal{A})$ in which the action of G on the algebra $\mathcal{A}$ is given by the homomorphism $\alpha : G \to \mathrm{Aut}(\mathcal{A})$ into the group of linear $*$-automorphisms of the algebra $\mathcal{A}$.

A covariant representation of the pair $(G, \mathcal{A})$ is a non-degenerate $*$-representation of the algebra $\mathcal{A}$ by bounded linear operators in the Hilbert space $\mathcal{H}$ given by the homomorphism θ.

Suppose now that the algebra $\mathcal{A}$ is graded, i.e. $\mathcal{A} = \mathcal{A}_0 \oplus \mathcal{A}_1$ where $\mathcal{A}_0$, $\mathcal{A}_1$ are selfadjoint closed subspaces satisfying the relations

$$\mathcal{A}_i \mathcal{A}_j \subset \mathcal{A}_{(i+j)(\mathrm{mod}\,2)}.$$

Denote by $\mathrm{Aut}(\mathcal{A})$ the group of even $*$-automorphisms of the algebra $\mathcal{A}$, i.e. $*$-automorphisms of the algebra $\mathcal{A}$ preserving the decomposition $\mathcal{A} = \mathcal{A}_0 \oplus \mathcal{A}_1$.

A graded covariant representation of the system $(G, \mathcal{A}, c)$, where c is the homomorphism $G \to \{\pm 1\}$, is the graded $*$-representation of the algebra $\mathcal{A}$ in the graded Hilbert space $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1$ satisfying the condition that $\theta(g)$ is an even operator for $c(g) = +1$ and an odd operator for $c(g) = -1$.

Up to this moment we were considering only the bosonic SPT-phases. However the introduced notions are easily extended to the fermionic case. A fermionic Hamiltonian H is defined in terms of a finite collection of creation and annihilation operators acting in the fermionic Fock space $\mathcal{F}(\mathcal{H})$.

Topological phases

By definition the Ω-spectrum is the family of pointed topological spaces (T_n), $n \in \mathbb{Z}$, having the following property: for any $n \in \mathbb{Z}$ the pointed topological spaces

$$T_n \sim \Omega T_{n+1}$$

are homotopy equivalent where ΩT_{n+1} is the loop space of the topological space T_{n+1}.

With every Ω-spectrum it is associated the generalized cohomology theory determined by the contravariant functor h^n. This functor assigns to any pair of pointed topological spaces (X, Y) with $Y \subset X$ the Abelian group

$$h^n(X, Y) = [(X, Y), (T_n, *)]$$

where on the right stands the set of homotopy classes of continuous maps $(X, Y) \to (T_n, *)$ sending Y to the marked point $*$.

To take into account the action of the symmetry group G suppose that it acts on the pair (X, Y) by a continuous homeomorphism φ. In this case we can introduce the G-invariant generalized cohomology theory given by the functor defined by the equality

$$h^n_G(X, Y) = h^n(EG \times_G X, EG \times_G Y)$$

where $EG \to BG$ is the classifying bundle in which EG is the contractible space, being a principal G-bundle over the classifying space BG, and $EG \times_G X$ denotes the quotient $(EG \times X)/G$. In particular, for $X = *$ we get

$$h^n_G(*) = h^n(BG).$$

Denote by F_d the space of d-dimensional SRE-states. Then for such spaces we have the property

$$F_d \sim \Omega F_{d+1} \quad \text{for } d \geq 0.$$

If this property is fulfilled for $d \geq 0$ then we can define by induction the spaces F_d for all $d \in \mathbb{Z}$ and the family of the spaces $(F_d)_{d\in\mathbb{Z}}$ will form an Ω-spectrum.

Here what is known about the homotopy groups of the spaces F_d. The group $\pi_0(F_d)$ classifies the d-dimensional SPT-phases without symmetry. In lower dimensions this group is equal to

$$\pi_0(F_0) = 0,\ \pi_0(F_1) = 0,\ \pi_0(F_2) = \mathbb{Z},\ \pi_0(F_3) = 0$$

(the group $\mathbb{Z}$ in dimension 2 is generated by the so called E_8-phase).

Note that the condition $F_d \sim \Omega F_{d+1}$ implies that

$$\pi_k(F_d) \cong \pi_{k+1}(F_{d+1}).$$

The space F_0 is identified with the infinite-dimensional projective space

$$F_0 = \mathbb{CP}^\infty$$

and the other spaces F_d of lower dimensions are described in terms of the Eilenberg–Mac Lane spaces $K(\mathbb{Z}, n)$ as

$$F_1 = K(\mathbb{Z}, 3), \;\; F_2 = K(\mathbb{Z}, 4) \times \mathbb{Z}, \;\; F_3 = K(\mathbb{Z}, 5) \times \mathrm{U}(1).$$

If we denote by $\mathbf{SPT}^{d}(G)$ the Abelian group of d-dimensional G-protected SPT-phases and by $H^{n}(G,\mathbb{Z})$ the n-dimensional cohomology group of the group G then the lower groups $\mathbf{SPT}^{d}(G)$ will be described as follows:

$$SPT^{0}(G) = H^{2}(G,\mathbb{Z}),\ SPT^{1}(G) = H^{3}(G,\mathbb{Z}),$$
$$SPT^{2}(G) = H^{4}(G,\mathbb{Z}) \oplus H^{0}(G,\mathbb{Z}),\ SPT^{3}(G) = H^{5}(G,\mathbb{Z}) \oplus H^{1}(G,\mathbb{Z}).$$

Consider as an example of application of the introduced notions the fermionic systems with the so called hourglass symmetries. These are the symmetry groups including the charge conjugation symmetry U(1), time reversion symmetry T with $T^2 = -1$ and glide symmetry given by the composition of the translation to half-period with reflection.

As an example of the systems with glide symmetry we can take the three-dimensional system in which the planes with constant coordinate $x \in \mathbb{Z}$ are occupied by the two-dimensional systems (quantum spin Hall insulators), and the planes with constant coordinate $x \in \mathbb{Z} + 1/2$ are occupied with their mirror reflections. The obtained system is invariant under the glide given by the map: $(x, y, z) \mapsto (x + \frac{1}{2}, -y, z)$. We call this procedure the alternating fibers construction.

The constructed system may be described in terms of the diagram

$$\mathbb{Z}_2 \longrightarrow \mathbb{Z}_4 \longrightarrow \mathbb{Z}_2$$

connecting topological insulators in two and three dimensions. In dimension 2 the generator of the first group $\mathbb{Z}_2$ is the quantum spin Hall insulator (QSH-phase). We can assign to it the three-dimensional system, described above, corresponding to the group $\mathbb{Z}_4$ and having the hourglass symmetry. Transition from the group $\mathbb{Z}_4$ to the second group $\mathbb{Z}_2$ is done by "forgetting" the glide symmetry.

This construction may be extended to arbitrary symmetry groups G. Denote as above by $\mathrm{SPT}^d(G)$ the Abelian group of d-dimensional G-protected SPT-phases, and by $\mathrm{SPT}^d(\mathbb{Z} \times G)$ the same group with glide symmetry added. Then there is the following exact sequence of homomorphisms

$$
\begin{aligned}
0 \longrightarrow \mathrm{SPT}^{d-1}(G)/2\mathrm{SPT}^{d-1}(G) \xrightarrow{\alpha} \mathrm{SPT}^d(\mathbb{Z} \times G) \xrightarrow{\beta} \\
\xrightarrow{\beta} \{[c] \in \mathrm{SPT}^d(G) : 2[c] = 0\} \longrightarrow 0. \quad (2)
\end{aligned}
$$

Here the homomorphism β is generated by the forgetting map, and homomorphism α is generated by the alternating fibers construction.

As one more example of application of topological phases we give a description of Wigner–Dyson classes (class A). In this case $d = 3$ and the group $G = \mathrm{U}(1)$ corresponds to the charge preservation. In the two-dimensional case the fermionic phases of this type are classified by the 1st Chern class of the Bloch bundle over the Brillouin zone.

The phases with odd Chern classes represent the non-trivial element in the first term of the exact sequence written above. This phase may be pulled up with the help of the alternating fibers construction to the three-dimensional phase including the gliding. The obtained three-dimensional phases are characterized by a topological invariant $\kappa \in \mathbb{Z}_2$. The phases with non-trivial invariant κ are called Möbius.

Using the exact sequence (2), we can obtain the complete classification of two-dimensional and three-dimensional U(1)-protected fermionic SPT-phases. Namely:

$$\boldsymbol{SPT^2(\mathrm{U}(1)) \cong \mathbb{Z} \oplus \mathbb{Z}, \ SPT^3(\mathrm{U}(1)) \cong 0}$$

where the first group $\mathbb{Z}$ is generated by the phase with zero Chern class while the second group $\mathbb{Z}$ corresponds to $\boldsymbol{E_8}$-phase, mentioned above. After adding the gliding we shall obtain the relation

$$\boldsymbol{SPT^3(\mathbb{Z} \times \mathrm{U}(1)) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2.}$$

Consider one more Wigner–Dyson class (denoted by AII). Here again $d = 3$ and the group G is generated by the U(1)-symmetry and T-symmetry. In this case the fermionic G-protected phases admit the following classification

$$SPT^2(G) \cong \mathbb{Z}_2,\ SPT^3(G) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$$

where the group $\mathbb{Z}_2$ in the two-dimensional case is generated by the QSH-phase and three groups $\mathbb{Z}_2$ in the three-dimensional case correspond to zonal insulators.

It may be also shown that

$$SPT^3(\mathbb{Z} \times G) \cong \mathbb{Z}_4 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2.$$

Connection with K-theory

Consider the Hamiltonians H acting in the Hilbert space $\mathcal{H}$ and satisfying the gap condition. Denote by Γ the spectral flattening $\operatorname{sgn} H$ of the Hamiltonian H. In other words, Γ is the grading operator, belonging to the same phase as H, with the spectrum consisting of two points $\{+1, -1\}$. (Continuous deformation of Hamiltonian H to its spectral flattening $\Gamma = \operatorname{sgn} H$ may be given by an explicit formula). The space of grading operators Γ, acting in the Hilbert space $\mathcal{H}$, is denoted by $\mathrm{Grad}(\mathcal{H})$.

Two grading operators Γ_1, Γ_2 are called homotopic if they can be connected by the continuous path inside $\mathrm{Grad}(\mathcal{H})$. The triple $(\mathcal{H}, \Gamma_1, \Gamma_2)$ with $\Gamma_1, \Gamma_2 \in \mathrm{Grad}(\mathcal{H})$ is called the ordered difference between the grading operators Γ_1, Γ_2 or the corresponding Hamiltonians H_1, H_2. If in this triple Γ_1 is homotopic to Γ_2 we call such triple $(\mathcal{H}, \Gamma_1, \Gamma_2)$ trivial.

We can extend the definition given above by including into consideration the symmetry group G. Namely, denote by $\mathcal{A}$ the C^*-algebra on which the group G acts by the representation $\alpha : G \to \operatorname{Aut} \mathcal{A}$. Let W be a finitely generated $\mathcal{A}$-module and $\operatorname{Grad}_{\mathcal{A}}(W)$ denotes the space of $\mathcal{A}$-compatible grading operators acting in W. The above definitions, related to $\operatorname{Grad}(\mathcal{H})$, immediately extend to the case $\operatorname{Grad}_{\mathcal{A}}(W)$. The direct sum operation provides $\operatorname{Grad}_{\mathcal{A}}(W)$ with the structure of the Abelian monoid.

Tiang has proposed the following definition of K-functor. Denote by $K_0(\mathcal{A})$ the quotient of the monoid $\mathrm{Grad}_{\mathcal{A}}(W)$ with respect to the equivalence relation determined by trivial triples. In more detail, the triple (W, Γ_1, Γ_2) is equivalent to the triple $(W', \Gamma_1', \Gamma_2')$ if there exist the trivial triples (V, Δ_1, Δ_2) $(V', \Delta_1', \Delta_2')$ such that

$$(W \oplus V, \Gamma_1 \oplus \Delta_1, \Gamma_2 \oplus \Delta_2) = (W' \oplus V', \Gamma_1' \oplus \Delta_1', \Gamma_2' \oplus \Delta_2')$$

in $\mathrm{Grad}_{\mathcal{A}}(W)$. The group $K_0(\mathcal{A})$ is called the group of differences of $\mathcal{A}$-compatible gapped Hamiltonians. This group is Abelian and $-[W, \Gamma_1, \Gamma_2] = [W, \Gamma_2, \Gamma_1]$. It satisfies also the condition

$$[W, \Gamma_1, \Gamma_2] + [W, \Gamma_2, \Gamma_3] = [W, \Gamma_1, \Gamma_3]$$

in $K_0(\mathcal{A})$.

Let G be the symmetry group of the Hamiltonian. We shall provide it with the following homomorphisms:

1. $\phi : G \to \{\pm 1\}$ is responsible for the unitarity of the action of the element $g \in G$: this action is unitary if $\phi(g) = +1$, and it is anti-unitary if $\phi(g) = -1$;
2. $c : G \to \{\pm 1\}$ is responsible for the charge preservation: the action of $g \in G$ commutes with the Hamiltonian if $c(g) = +1$, and it anti-commutes with it if $c(g) = -1$;
3. $\tau : G \to \{\pm 1\}$ is responsible for the preservation of time direction: the action of $g \in G$ preserves the time direction if $\tau(g) = +1$, and it inverts it if $\tau(g) = -1$.

Consider a concrete example of the group G called the CT-group. It is generated by the unit and three generators T, C, S where

1. $\phi(T) = -1$, $c(T) = +1$;
2. $\phi(C) = -1$, $c(C) = -1$;
3. $\phi(S) = +1$, $c(S) = +1$.

The generators T, C and $S = CT = TC$ correspond to the symmetries of the time inversion, charge conjugation and chiral symmetry respectively. We are interested in the graded representations of the CT-group and its subgroups. We shall denote the operators, corresponding to the generators of the group G, by $\hat{T}$, $\hat{C}$ and $\hat{S}$ respectively.

Then we have the following possibilities: $\hat{T}^2 = \pm 1$, $\hat{C}^2 = \pm 1$ and $\hat{S} = \hat{C}\hat{T} = \hat{T}\hat{C}$. The family of pairwise anti-commuting odd operators $\{\hat{C}, i\hat{C}, i\hat{C}\hat{T}\}$ generates the graded representation of the real Clifford algebra $\mathrm{Cl}_{r,s}$ where r (resp. s) is the number of the negatively (resp. positively) determined selfadjoint generators so that the representation of the full CT-group G coincides with the graded $*$-representation of the Clifford algebra $\mathrm{Cl}_{r,s}$.

In the case of the subgroup $A = \{1, C\}$ we can take for the odd generators the representation operators $\{\hat{C}, i\hat{C}\}$ with $\hat{C}^2 = \pm 1$ generating the graded representation of the Clifford algebras $\mathrm{Cl}_{0,2}$ or $\mathrm{Cl}_{2,0}$. In the case of the subgroup $A = \{1, S\}$ we have necessarily $\hat{S}^2 = +1$ so that the obtained representation coincides with the graded representation of the complex Clifford algebra $\mathbb{C}\mathrm{l}_1$. And in the case of the subgroup $A = \{1, T\}$ we have two choices for $\hat{T}^2 = \pm 1$. The family of operators $\{i, \hat{T}, i\hat{T}\hat{\Gamma}\}$, where $\hat{\Gamma}$ is the grading operator, generates the non-graded representation of the Clifford algebra $\mathrm{Cl}_{1,2}$ for $\hat{T}^2 = +1$ and of $\mathrm{Cl}_{3,0}$ for $\hat{T}^2 = -1$.

In the case of zonal insulators the group G has the form $G = \mathcal{L} \rtimes P$ where P is a compact point symmetry group. The Pontryagin dual group $\hat{\mathcal{L}}$ coincides with the Brillouin torus $\mathbb{T}^d$, and the K-functor $\mathrm{K}_0(\mathcal{A})$ of the algebra $\mathcal{A} = \mathbb{C} \rtimes G$ coincides with the usual K-functor $K_0(\mathcal{A})$. The graded finitely-generated $\mathcal{A}$-module is in this case the graded $C(\mathbb{T}^d)$-module of sections of the Bloch bundle over $\mathbb{T}^d$.

We point out the relation of this construction with Clifford algebras. To see it suppose that $G = \mathcal{L} \rtimes A$ where A is one of the subgroups of CT-group so that the algebra $\mathcal{A} = \mathbb{C} \rtimes G$. Then the group $\mathrm{K}_0(\mathcal{A})$ coincides in the case of complex Clifford algebras with $K^{-n}(\mathbb{T}^d)$, and in the case of real Clifford algebras $\mathrm{Cl}_{r,s}$ with $KR^{(r-s)(\mathrm{mod}\, 8)}(\mathbb{T}^d)$.