Supersymmetries and Quantum Symmetries

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Poincaré group and operators of position and spin

Alexander J. Silenko

(in collaboration with **Pengming Zhang** and **Liping Zou**)

Joint Institute for Nuclear Research, Dubna, Russia

Research Institute for Nuclear Problems, BSU, Minsk, Belarus

Institute of Modern Physics CAS, Lanzhou, China

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OUTLINE

- Position and spin in relativistic quantum mechanics
- Poincaré group and fundamental quantities in classical and quantum physics
- Comparison of classical and quantum-mechanical descriptions of a Dirac particle in external fields
- Connection between spin and geometry
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Position and spin in relativistic quantum mechanics

Many scientists are sure that in the Dirac representation wave eigenfunctions have the probabilistic interpretation ($\rho_D(r)=|\Psi_D(r)|^2$) and the Dirac radius vector *r* corresponds to the classical variable, radius vector *R*. This point of view is presented in a lot of publications, while it has been proven many years ago (Foldy, Wouthuysen, 1950) **that only wave eigenfunctions in the FW representation have the probabilistic interpretation ($\rho_{FW}(x)=|\Psi_{FW}(x)|^2$) and the radius vector *x* in this representation corresponds to the classical variable *R*.**

We follow the paper

L. Zou, P. Zhang, A.J. Silenko, Position and spin in relativistic quantum mechanics, Phys. Rev. A **101, 032117 (2020).**

PHYSICAL REVIEW LETTERS **121,** 043202 (2018)

Relativistic Quantum Dynamics of Twisted Electron Beams in Arbitrary Electric and Magnetic Fields

Alexander J. Silenko,[1,2,3,*] Pengming Zhang,[1,4,†] and Liping Zou[1,‡]

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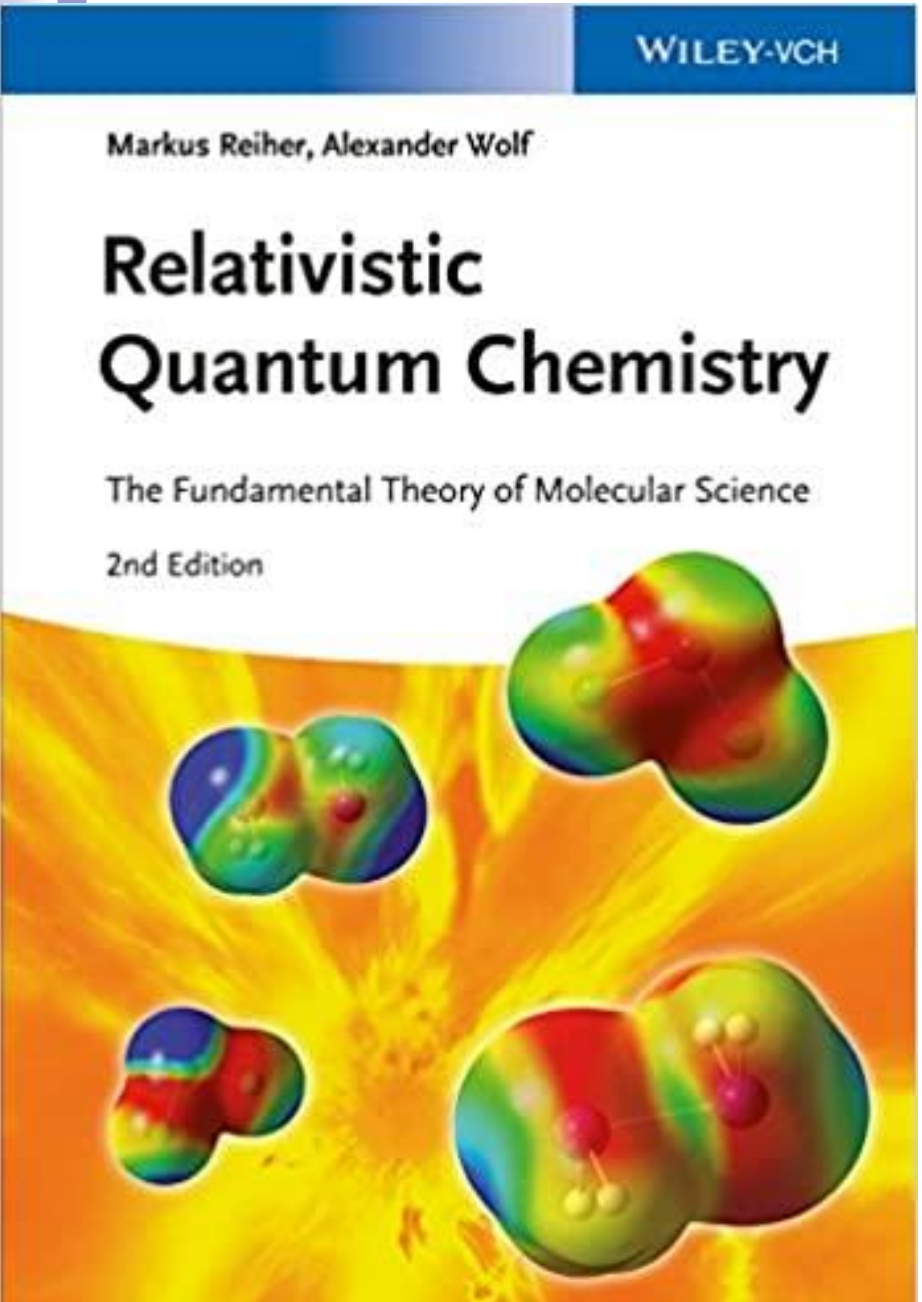
Comment on "Relativistic Quantum Dynamics of Twisted Electron Beams in Arbitrary Electric and Magnetic Fields"

Iwo Bialynicki-Birula[1] and Zofia Bialynicka-Birula[2]

[1]Center for Theoretical Physics
Polish Academy of Sciences
[2]Institute of Physics, Polish Academy of Sciences

Contrary opinions!



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Poincaré group and fundamental quantities in classical and quantum physics

Poincaré group and fundamental quantities in classical physics

There are ten independent fundamental quantities P_μ =(H,P), $J_{\mu\nu}$ (μ,ν =0,1,2,3) describing the momentum and total angular momentum and characteristic for the dynamical system. The antisymmetric tensor $J_{\mu\nu}$ is defined by the two vectors, J and K. As a result, there are the ten infinitesimal generators of the Poincare group (inhomogeneous Lorentz group), namely, the generators of the infinitesimal space translations P = (P_i), the generator of the infinitesimal time translation H, the generators of infinitesimal rotations J = (J_i), and the generators of infinitesimal Lorentz transformations (boosts) K = (K_i) (i = 1,2,3). These ten generators satisfy the following Poisson brackets:

$$\{P_i, P_j\} = 0, \quad \{P_i, H\} = 0, \quad \{J_i, H\} = 0,$$

$$\{J_i, J_j\} = e_{ijk} J_k, \quad \{J_i, P_j\} = e_{ijk} P_k, \quad \{J_i, K_j\} = e_{ijk} K_k,$$

$$\{K_i, H\} = P_i, \quad \{K_i, K_j\} = -e_{ijk} J_k, \quad \{K_i, P_j\} = \delta_{ij} H.$$

The only additional equation which should be satisfied defines the orbital and spin parts of the total angular momentum:

$$\boldsymbol{J} = \boldsymbol{L} + \boldsymbol{S}, \qquad \boldsymbol{L} \equiv \boldsymbol{Q} \times \boldsymbol{P}.$$

There is some latitude in the definition of the position, orbital angular momentum (OAM), and spin.

M. H. L. Pryce, The mass-centre in the restricted theory of relativity and its connexion with the quantum theory of elementary particles, Proc. R. Soc. London A 195, 62 (1948).

A consideration of the particle position variables Q_i **brings the following Poisson brackets:**

$$\{Q_i, P_j\} = \delta_{ij}, \quad \{Q_i, J_j\} = e_{ijk} Q_k,$$

$$\{Q_i, K_j\} = \frac{1}{2}(Q_j\{Q_i, H\} + \{Q_i, H\}Q_j) - t\delta_{ij}.$$

It follows from the above equations that

$$\{L_i, P_j\} = e_{ijk} P_k, \qquad \{S_i, P_j\} = 0.$$

The Poisson brackets for the conventional particle position defining *the center of charge* are equal to zero:

$$\{Q_i, Q_j\} = 0.$$

As a result, for a free particle

$$\{Q_i, L_j\} = e_{ijk} Q_k, \quad \{Q_i, S_j\} = 0, \quad \{P_i, S_j\} = 0,$$

$$\{L_i, L_j\} = e_{ijk} L_k, \quad \{S_i, S_j\} = e_{ijk} S_k.$$

$$\{L_i, S_j\} = 0.$$

No spin-orbit interaction for a free particle

The main variables of a free spinning particle in classical mechanics are specified by

$$H = \sqrt{m^2 + \boldsymbol{P}^2}, \qquad \boldsymbol{K} = \boldsymbol{Q}H - \frac{\boldsymbol{S} \times \boldsymbol{P}}{m + H} - t\boldsymbol{P}.$$

L. L. Foldy, Synthesis of Covariant Particle Equations, Phys. Rev. 102, 568 (1956).
L. L. Foldy, Relativistic Particle Systems with Interaction, Phys. Rev. 122, 275 (1961).

Poincaré group and fundamental operators in relativistic quantum mechanics

The operators being counterparts of fundamental classical variables should satisfy the relations

$$[p_i,p_j]=0,\quad [p_i,\mathcal{H}]=0,\quad [j_i,\mathcal{H}]=0,$$

$$[j_i,j_j]=ie_{ijk}j_k,\quad [j_i,p_j]=ie_{ijk}p_k,\quad [j_i,K_j]=ie_{ijk}K_k,$$

$$[K_i,\mathcal{H}]=ip_i,\quad [K_i,K_j]=-ie_{ijk}j_k,\quad [K_i,p_j]=i\delta_{ij}\mathcal{H},$$

$$[q_i,p_j]=i\delta_{ij},\quad [q_i,j_j]=ie_{ijk}q_k,\quad [q_i,s_j]=0,\quad [j_i,p_j]=ie_{ijk}p_k,$$

$$[s_i,p_j]=0,\quad [l_i,l_j]=ie_{ijk}l_k,\quad [s_i,s_j]=ie_{ijk}s_k,\quad [l_i,s_j]=0,$$

$$[q_i,q_j]=0,$$

key definitions

$$[q_i,K_j]=\frac{1}{2}\left(q_j\,[q_i,\mathcal{H}]+[q_i,\mathcal{H}]\,q_j\right)-it\delta_{ij}.$$

There is a difference for the Dirac and FW representations!

Let us first consider the set of operators *p*, H_D, *j*, *K*, *q*, s_D, where $\mathbf{s}_D = (1/2)\hbar\boldsymbol{\Sigma}$ and all these operators are defined in the Dirac representation (in particular, the position operator is the Dirac radius vector r). Evidently, some of the above commutators are not satisfied by these operators.

$$\mathcal{H}_D = \beta m + \alpha \cdot p.$$

A consideration of the set of operators *p*, H_{FW}, *j*, *K*, *q*, *s* defined in the FW representation leads to an opposite conclusion. In this representation, the definition of s is the same ($\mathbf{s} = (1/2)\hbar\boldsymbol{\Sigma}$) and the position operator q is equal to the FW radius vector x. We can check that all the commutators are now satisfied.

$$\mathcal{H}_{FW} = \beta\sqrt{m^2 + p^2}, \qquad p \equiv -i\hbar\frac{\partial}{\partial r}.$$

The counterparts of the classical Hamiltonian, the position vector, the orbital angular momentum (OAM), and the spin are the operators H_{FW}, $\boldsymbol{x}$, $(\boldsymbol{x}\times\boldsymbol{p})$, and $\boldsymbol{s}=(1/2)\hbar\boldsymbol{\Sigma}$ defined in the FW representation. The operators p and J are not changed by the transformation from the Dirac representation to the FW one.

Evidently, the FW Hamiltonian commutes with the OAM and spin operators.

In the Dirac representation, the fundamental operators are defined by the transformation of the corresponding FW operators. This transformation is inverse with respect to the FW one and is performed by the operator U^{-1}_{FW} (***with*** $\Psi_{FW}=U_{FW}\Psi_D$).

The Dirac operators of the position and the spin are equal to

$$q = \boldsymbol{X} = \boldsymbol{r} - \frac{\boldsymbol{\Sigma} \times \boldsymbol{p}}{2\epsilon(\epsilon + m)} + \frac{i\boldsymbol{\gamma}}{2\epsilon} - \frac{i(\boldsymbol{\gamma} \cdot \boldsymbol{p})\boldsymbol{p}}{2\epsilon^2(\epsilon + m)},$$

$$\boldsymbol{S} = \frac{m}{2\epsilon}\boldsymbol{\Sigma} - i\frac{\boldsymbol{\gamma} \times \boldsymbol{p}}{2\epsilon} + \frac{\boldsymbol{p}(\boldsymbol{\Sigma} \cdot \boldsymbol{p})}{2\epsilon(\epsilon + m)}, \quad \epsilon = \sqrt{m^2 + \boldsymbol{p}^2}.$$

The conventional spin operator corresponding to the classical rest-frame spin commutes with the OAM operator, the Hamiltonian, and the position and momentum operators *in any representation*.

Comparison of classical and quantum-mechanical descriptions of a Dirac particle in external fields

Equation of spin motion in the classical limit

$$\frac{d\boldsymbol{\zeta}}{dt} = (\boldsymbol{\Omega}_{MDM} + \boldsymbol{\Omega}_{EDM}) \times \boldsymbol{\zeta},$$

$$\boldsymbol{\Omega}_{MDM} = \frac{e}{m}\left[\left(\frac{1}{\gamma+1} + a\right)\mathbf{v}\times\mathbf{E} - \left(\frac{1}{\gamma} + a\right)\mathbf{B} + \frac{a\gamma}{\gamma+1}(\mathbf{v}\cdot\mathbf{B})\mathbf{v}\right],$$

$$\boldsymbol{\Omega}_{EDM} = -\frac{e\eta}{2m}\left[\boldsymbol{E} - \frac{\gamma}{\gamma+1}(\mathbf{v}\cdot\boldsymbol{E})\mathbf{v} + \mathbf{v}\times\boldsymbol{B}\right],$$

$$\eta = 2mcd/(e\hbar s) = 4mcd/(e\hbar)$$

We obtains the Thomas-Bargmann-Mishel-Telegdi equation *added by terms describing the electric dipole moment*.

In the classical limit, the related equations of spin motion for spin-½ and spin-1 particles coincide with the classical equation of spin motion for particles with the EDM.

- **In the classical limit, one obtains the Thomas-Bargmann-Mishel-Telegdi equation *added by terms describing the electric dipole moment*. The rigorous derivation of the classical equation is presented in**

T. Fukuyama, A.J. Silenko, Derivation of generalized Thomas–Bargmann–Michel–Telegdi equation for a particle with electric dipole moment. Int. J. Mod. Phys. A 28, 1350147 (2013);

A.J. Silenko, Spin precession of a particle with an electric dipole moment: contributions from classical electrodynamics and from the Thomas effect. Phys. Scr. 90, 065303 (2015).

The perfect agreement with classical equations takes also place at the quantum-mechanical description of a Dirac particle in noninertial frames and arbitrary gravitational fields in the Foldy-Wouthuysen representation.

The comparison of classical and quantum-mechanical descriptions of a Dirac particle in external fields leads to results fully supporting the conclusions made. An analysis of spin-0 and spin-1 particles in external fields also presents arguments in favor of these conclusions. In contrast to the results for a free particle, the particle spin motion in electric and magnetic field is sensitive to the Thomas effect and unambiguously shows that the fundamental spin operator is defined in the particle rest frame. The analysis presented excludes the possibility of a definition of this operator in the instantaneously accompanying frame.

Connection between spin and geometry

The additional equation which should be satisfied defines the orbital and spin parts of the total angular momentum:

$$\boldsymbol{J} = \boldsymbol{L} + \boldsymbol{S}, \quad \boldsymbol{L} \equiv \boldsymbol{Q} \times \boldsymbol{P}.$$

The quantities $\boldsymbol{l}$ and $\boldsymbol{s}$ forming the total angular momentum $\boldsymbol{j}$ have different physical meanings. The OAM $\boldsymbol{l}$ is the spatial part of the antisymmetric tensor $L^{\mu\nu} = (-\boldsymbol{\kappa}, -\boldsymbol{l})$ with $\boldsymbol{\kappa} = (\boldsymbol{q}\mathcal{H} + \mathcal{H}\boldsymbol{q})/2 - t\boldsymbol{p}$ and is noninvariant relative to Lorentz transformations. The rest-frame spin $\boldsymbol{s}$ is invariant relative to such transformations. It is natural to constitute the total angular momentum from spatial parts of the two antisymmetric tensors, $L^{\mu\nu}$ and $S^{\mu\nu}$:

$$J^{\mu\nu} = L^{\mu\nu} + S^{\mu\nu} = x^{\mu}p^{\nu} - x^{\nu}p^{\mu} + S^{\mu\nu}.$$

Since the spatial part of the spin tensor is presented by the vector ζ, the definition of this vector is analogous to the definition of the total angular momentum j. The corresponding operators of the position and OAM should be redefined in order to avoid a change of the operator j:

$$j = l + s = \mathcal{L} + \zeta, \qquad \mathcal{L} = \mathcal{X} \times p.$$

$$\mathcal{X}_{\mathrm{FW}} = x + \frac{s \times p}{m(\epsilon + m)}, \qquad \mathcal{L}_{\mathrm{FW}} = \mathcal{X}_{\mathrm{FW}} \times p,$$

where x is the FW center-of-charge position operator

The geometry based on the center-of-mass position operator is noncommutative.

In some papers, the projected spin operator has been considered. It is possible to project some operators onto positive- and negative-energy subspaces, eliminating the cross terms corresponding to the electron-positron transitions. In particular, the projected radius vector operator is given by

$$\mathfrak{R} = \Pi^{+}\boldsymbol{r}\Pi^{+} + \Pi^{-}\boldsymbol{r}\Pi^{-},$$

where the projectors are given by

$$\Pi^{\pm} = \frac{1}{2}U_{\mathrm{FW}}^{\dagger}(1 \pm \beta)U_{\mathrm{FW}} = \frac{1}{2}\left(1 \pm \beta\frac{m}{\epsilon}\right) \pm \frac{\boldsymbol{\alpha}\cdot\boldsymbol{p}}{2\epsilon}.$$

The projected operators of the radius vector (position) and spin are equal to

$$\mathfrak{R}_{\rm FW} = x - \frac{\Sigma \times p}{2\epsilon(\epsilon + m)} \qquad \mathfrak{S}_{\rm FW} = \frac{m\Sigma}{2\epsilon} + \frac{p(p \cdot \Sigma)}{2\epsilon(\epsilon + m)},$$

The projected OAM operator is given by

$$\mathfrak{L}_{\rm FW} = \mathfrak{R}_{\rm FW} \times p.$$

The geometry based on the projected operators is also noncommutative.

Discussion

Summary

- The use of the Poincaré group unambiguously shows that quantum-mechanical counterparts of fundamental classical variables are the corresponding operators in the Foldy-Wouthuysen representation but not in the Dirac one.
- The comparison of classical and quantum-mechanical descriptions of a Dirac particle in external fields confirms this conclusion.
- Only wave functions in the Foldy-Wouthuysen representation (but not in the Dirac one) have the probabilistic interpretation.
- The geometries based on the center-of-mass position operator and on the projected operators are noncommutative.

Thank you for your attention