Supersymmetries and Quantum Symmetries

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The higher covariant derivative regularization for various supersymmetric theories

Supersymmetric theories at the quantum level

Investigation of quantum corrections in supersymmetric theories is very important both for theory and for phenomenology. Certainly, for calculating quantum corrections a theory should be regularized.

Dimensional regularization breaks supersymmetry and is not convenient for calculations in supersymmetric theories. That is why supersymmetric theories are mostly regularized by dimensional reduction. However, dimensional reduction is not self-consistent.

W.Siegel, Phys.Lett. **B 84** (1979) 193; **B 94** (1980) 37.

Removing of the inconsistencies leads to the loss of explicit supersymmetry:

L.V.Avdeev, G.A.Chochia, A.A.Vladimirov, Phys.Lett. **B 105** (1981) 272.

As a consequence, supersymmetry can be broken by quantum corrections in higher loops.

L.V.Avdeev, Phys.Lett. **B 117** (1982) 317;
L.V.Avdeev, A.A.Vladimirov, Nucl.Phys. **B 219** (1983) 262.

The higher covariant derivative regularization

Here we will discuss the application of the higher covariant derivative method for the regularization of various supersymmetric theories. We will argue that this regularization allows to reveal some interesting features of quantum corrections which cannot be seen in the case of using dimensional reduction.

The higher covariant derivative regularization was proposed by A.A.Slavnov

A.A.Slavnov, Nucl.Phys. **B31**, (1971), 301;
Theor.Math.Phys. **13** (1972) 1064.

By construction, it includes insertion of the Pauli–Villars determinants for removing residual one-loop divergencies

A.A.Slavnov, Theor.Math.Phys. **33**, (1977), 977.

Unlike dimensional reduction, this regularization is self-consistent. It can be formulated in a manifestly supersymmetric way in terms of $\mathcal{N}=1$ superfields

V.K.Krivoshchekov, Theor.Math.Phys. **36** (1978) 745;
P.West, Nucl.Phys. B268, (1986), 113.

Moreover, the exact Novikov, Shifman, Vainshtein, and Zakharov (NSVZ) β-function

V.Novikov, M.A.Shifman, A.Vainshtein, V.I.Zakharov, Nucl.Phys. **B 229** (1983) 381; Phys.Lett. **B 166** (1985) 329; M.A.Shifman, A.I.Vainshtein, Nucl.Phys. **B 277** (1986) 456; D.R.T.Jones, Phys.Lett. **B 123** (1983) 45.

can naturally be obtained in the case of using the higher covariant derivative regularization. It relates the β-function and the anomalous dimension of the matter superfields in $\mathcal{N}=1$ supersymmetric gauge theories,

$$\beta(\alpha,\lambda) = -\frac{\alpha^2\Big(3C_2 - T(R) + C(R)_i{}^j(\gamma_\phi)_j{}^i(\alpha,\lambda)/r\Big)}{2\pi(1-C_2\alpha/2\pi)}.$$

Here α and λ are the gauge and Yukawa coupling constants, respectively, and we use the notation

$$\mathrm{tr}\,(T^A T^B) \equiv T(R)\,\delta^{AB}; \qquad (T^A)_i{}^k (T^A)_k{}^j \equiv C(R)_i{}^j;$$
$$f^{ACD} f^{BCD} \equiv C_2 \delta^{AB}; \qquad r \equiv \delta_{AA} = \dim G.$$

Explicit calculation and the problem of constructing an NSVZ scheme

Three- and four-loop calculations in $\mathcal{N}=1$ supersymmetric theories made with dimensional reduction supplemented by modified minimal subtraction (i.e. in the so-called $\overline{\mbox{DR}}$-scheme)

L.V.Avdeev, O.V.Tarasov, Phys.Lett. **112 B** (1982) 356; I.Jack, D.R.T.Jones, C.G.North, Phys.Lett **B386** (1996) 138; Nucl.Phys. **B 486** (1997) 479; R.V.Harlander, D.R.T.Jones, P.Kant, L.Mihaila, M.Steinhauser, JHEP **0612** (2006) 024.

revealed that the NSVZ relation in the $\overline{\mbox{DR}}$-scheme holds only in the one- and two-loop approximations, where the β-function is scheme independent.

However, in the three- and four-loop approximations it is possible to restore the NSVZ relation with the help of a specially tuned finite renormalization of the gauge coupling constant. Note that a possibility of making this finite renormalization is highly nontrivial.

This implies that the NSVZ relation holds only in some special renormalization schemes, which are usually called "NSVZ schemes", and the $\overline{\mbox{DR}}$-scheme is not NSVZ.

Now, let us discuss how one can derive the NSVZ equation in all orders and construct some all-loop NSVZ schemes with the help of the higher covariant derivative regularization.

Renormalizable $\mathcal{N}=1$ supersymmetric gauge theories with matter superfields at the classical level are described by the action

$$S=\frac{1}{2e_0^2}\,\mbox{Re}\,\mbox{tr}\int d^4x\,d^2\theta\,W^a W_a+\frac{1}{4}\int d^4x\,d^4\theta\,\phi^{*i}(e^{2V})_i{}^j\phi_j$$
$$+\Big\{\int d^4x\,d^2\theta\,\Big(\frac{1}{4}m_0^{ij}\phi_i\phi_j+\frac{1}{6}\lambda_0^{ijk}\phi_i\phi_j\phi_k\Big)+\mbox{c.c.}\Big\}.$$

We assume that the gauge group is simple, and the chiral matter superfields ϕ_i lie in its representation R. The gauge and Yukawa coupling constants are denoted by e_0 and λ_0^{ijk}, respectively. The strength of the gauge superfield V is defined by the equation

$$W_a\equiv\frac{1}{8}\bar D^2\Big(e^{-2V}D_a e^{2V}\Big).$$

The theory under consideration is gauge invariant if the (bare) masses and Yukawa couplings satisfy the conditions

$$m_0^{im}(T^A)_m{}^j+m_0^{mj}(T^A)_m{}^i=0;$$
$$\lambda_0^{ijm}(T^A)_m{}^k+\lambda_0^{imk}(T^A)_m{}^j+\lambda_0^{mjk}(T^A)_m{}^i=0.$$

The background superfield method and the nonlinear renormalization

For quantizing the theory it is convenient to use the background field method. Moreover, it is necessary to take into account nonlinear renormalization of the quantum gauge superfield

O. Piguet and K. Sibold, Nucl.Phys. **B197** (1982) 257; 272;
I.V.Tyutin, Yad.Fiz. **37** (1983) 761.

This can be done with the help of the replacement $e^{2V} \to e^{2\mathcal{F}(V)} e^{2\boldsymbol{V}}$, where $\boldsymbol{V}$ and V are the background and quantum gauge superfields, respectively, and the function $\mathcal{F}(V)$ includes an infinite set of parameters needed for describing the nonlinear renormalization. In the lowest order

J.W.Juer and D.Storey, Phys.Lett. **119B** (1982) 125; Nucl. Phys. **B216** (1983) 185.

$$\mathcal{F}(V)^A = V^A + e_0^2\, y_0\, G^{ABCD}\, V^B V^C V^D + \ldots,$$

where y_0 is one of the constant entering this set, and G^{ABCD} is a certain function of the structure constants.

The background gauge invariance

$$\phi_i \to (e^A)_i{}^j \phi_j; \qquad V \to e^{-A^+} V e^{A^+}; \qquad e^{2\boldsymbol{V}} \to e^{-A^+} e^{2\boldsymbol{V}} e^{-A}.$$

parameterized by a chiral superfield A remains a manifest symmetry of the effective action.

For constructing the regularized theory we first add to its action terms with higher derivatives,

$$S_{\text{reg}} = \frac{1}{2e_0^2}\,\text{Re}\,\text{tr}\int d^4x\,d^2\theta\,W^a\,\Big(e^{-2\boldsymbol{V}}e^{-2\mathcal{F}(V)}\Big)_{Adj}\,R\Big(-\frac{\bar\nabla^2\nabla^2}{16\Lambda^2}\Big)_{Adj}$$
$$\times\Big(e^{2\mathcal{F}(V)}e^{2\boldsymbol{V}}\Big)_{Adj}W_a + \frac{1}{4}\int d^4x\,d^4\theta\,\phi^{*i}\Big[F\Big(-\frac{\bar\nabla^2\nabla^2}{16\Lambda^2}\Big)e^{2\mathcal{F}(V)}e^{2\boldsymbol{V}}\Big]_i{}^j\phi_j$$
$$+\Big[\int d^4x\,d^2\theta\,\Big(\frac{1}{4}m_0^{ij}\phi_i\phi_j + \frac{1}{6}\lambda_0^{ijk}\phi_i\phi_j\phi_k\Big) + \text{c.c.}\Big],$$

where the covariant derivatives are defined as

$$\nabla_a = D_a;\qquad \bar\nabla_{\dot a} = e^{2\mathcal{F}(V)}e^{2\boldsymbol{V}}\,\bar D_{\dot a}e^{-2\boldsymbol{V}}\,e^{-2\mathcal{F}(V)}.$$

Gauge is fixed by adding the term

$$S_{\text{gf}} = -\frac{1}{16\xi_0 e_0^2}\,\text{tr}\int d^4x\,d^4\theta\,\boldsymbol{\nabla}^2 V K\Big(-\frac{\bar{\boldsymbol{\nabla}}^2\boldsymbol{\nabla}^2}{16\Lambda^2}\Big)_{Adj}\bar{\boldsymbol{\nabla}}^2 V.$$

Also it is necessary to introduce the Faddeev-Popov and Nielsen–Kalosh ghosts. The regulator functions $R(x)$, $F(x)$, and $K(x)$ should rapidly increase at infinity and satisfy the condition $R(0) = F(0) = K(0) = 1$.

The Pauli–Villars determinants in the non-Abelian case

For regularizing residual one-loop divergences we insert into the generating functional two Pauli–Villars determinants,

$$Z = \int D\mu\, \mathsf{Det}(PV, M_\varphi)^{-1} \mathsf{Det}(PV, M)^c \times \exp\Big\{ i\Big(S_{\mathsf{reg}} + S_{\mathsf{gf}} + S_{\mathsf{FP}} + S_{\mathsf{NK}} + S_{\mathsf{sources}} \Big)\Big\},$$

where $D\mu$ is the functional integration measure, and

$$\mathsf{Det}(PV, M_\varphi)^{-1} \equiv \int D\varphi_1\, D\varphi_2\, D\varphi_3\, \exp(iS_\varphi);$$

$$\mathsf{Det}(PV, M)^{-1} \equiv \int D\Phi\, \exp(iS_\Phi).$$

Here we use chiral commuting Pauli–Villars superfields.
The superfields $\varphi_{1,2,3}$ belong to the adjoint representation and cancel one-loop divergences coming from gauge and ghost loops. The superfields Φ_i lie in a representation R_{PV} and cancel divergences coming from a loop of the matter superfields if $c = T(R)/T(R_{\mathsf{PV}})$. The masses of these superfields are

$$M_\varphi = a_\varphi \Lambda; \qquad M = a\Lambda,$$

where the coefficients a_φ and a do not depend on couplings.

It is important to distinguish renormalization group functions (RGFs) defined in terms of the bare couplings α_0 and λ_0,

$$\beta(\alpha_0,\lambda_0) \equiv \frac{d\alpha_0}{d\ln\Lambda}\Big|_{\alpha,\lambda=\text{const}}; \qquad \gamma_x(\alpha_0,\lambda_0) \equiv -\frac{d\ln Z_x}{d\ln\Lambda}\Big|_{\alpha,\lambda=\text{const}},$$

and RGFs standardly defined in terms of the renormalized couplings α and λ,

$$\widetilde\beta(\alpha,\lambda) \equiv \frac{d\alpha}{d\ln\mu}\Big|_{\alpha_0,\lambda_0=\text{const}}; \qquad \widetilde\gamma_x(\alpha,\lambda) \equiv \frac{d\ln Z_x}{d\ln\mu}\Big|_{\alpha_0,\lambda_0=\text{const}}.$$

A.L.Kataev and K.S., Nucl.Phys. **B875** (2013) 459.

RGFs defined in terms of the bare couplings do not depend on a renormalization prescription for a fixed regularization, but depend on a regularization.

RGFs defined in terms of the renormalized couplings depend both on a regularization and on a renormalization prescription.

The HD+MSL scheme. The main statements.

Both definitions of RGFs give the same functions in the HD+MSL-scheme, when a theory is regularized by Higher Derivatives, and divergences are removed by Minimal Subtractions of Logarithms. This means that the renormalization constants include only powers of $\ln\Lambda/\mu$, where μ is a renormalization point.

$$\widetilde{\beta}(\alpha,\lambda)\Big|_{\mbox{HD+MSL}} = \beta(\alpha_0\to\alpha,\lambda_0\to\lambda);$$

$$\widetilde{\gamma}_x(\alpha,\lambda)\Big|_{\mbox{HD+MSL}} = \gamma_x(\alpha_0\to\alpha,\lambda_0\to\lambda).$$

Here we will briefly describe the proof of the following statements:

1. NSVZ equation is valid for RGFs defined in terms of the bare couplings in the case of using the higher covariant derivative regularization for an arbitrary renormalization prescription.

2. For RGFs defined in terms of the renormalized couplings some NSVZ schemes are given by the HD+MSL prescription. (MSL can supplement various versions of the higher covariant derivative regularization.)

The all-loop derivation of the NSVZ equation: the main steps

1. First, one proves the ultraviolet finiteness of triple vertices with two external lines of the Faddeev–Popov ghosts and one external line of the quantum gauge superfield.

2. Next, it is necessary to rewrite the NSVZ relation in the equivalent form

$$\frac{\beta(\alpha_0,\lambda_0)}{\alpha_0^2} = -\frac{1}{2\pi}\Big(3C_2 - T(R) - 2C_2\gamma_c(\alpha_0,\lambda_0)$$
$$-2C_2\gamma_V(\alpha_0,\lambda_0) + C(R)_i{}^j(\gamma_\phi)_j{}^i(\alpha_0,\lambda_0)/r\Big).$$

K.S., Nucl.Phys. **B909** (2016) 316.

3. After this we prove that the β-function is determined by integrals of double total derivatives with respect to loop momenta and present a method for constructing this integrals.

K.S., JHEP **10** (2019) 011.

4. Then the NSVZ equation is obtained by summing singular contributions.

5. Finally, an NSVZ scheme is constructed.

K.S., Eur.Phys.J. **C80** (2020) 10, 911.

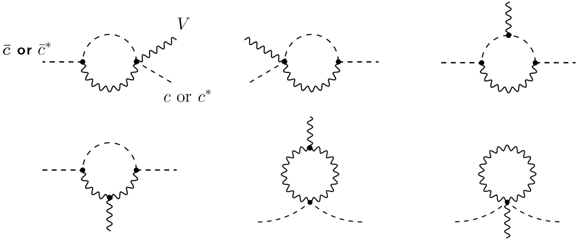
Non-renormalization of the three-point gauge-ghost vertices

The all-order finiteness of triple vertices in which two external lines correspond to the Faddeev–Popov ghosts and one external line corresponds to the quantum gauge superfield has been proved in the paper

K.S., Nucl.Phys. **B909** (2016) 316.

using the Slavnov–Taylor identities and rules for calculating supergraphs. The result is valid for the superfield formulation of the theory in the general ξ-gauge.

The one-loop contribution to these vertices comes from the superdiagrams presented below. The ultraviolet finiteness of their sum has been verified by an explicit calculation



Non-renormalization of the triple gauge-ghost vertices and the new form of the NSVZ β-function

There are 4 vertices of the considered structure, $\bar{c}Vc$, $\bar{c}^+Vc$, $\bar{c}Vc^+$, and $\bar{c}^+Vc^+$. All of them have the same renormalization constant $Z_\alpha^{-1/2}Z_cZ_V$. Therefore, due to their finiteness

$$\frac{d}{d\ln\Lambda}(Z_\alpha^{-1/2}Z_cZ_V) = 0,$$

where

$$\frac{1}{\alpha_0} = \frac{Z_\alpha}{\alpha}; \qquad \boldsymbol{V} = \boldsymbol{V}_R; \qquad V = Z_V Z_\alpha^{-1/2} V_R; \qquad \bar{c}c = Z_c Z_\alpha^{-1}\bar{c}_R c_R.$$

The non-Abelian NSVZ equation can equivalently be rewritten as

$$\frac{\beta(\alpha_0,\lambda_0)}{\alpha_0^2} = -\frac{3C_2 - T(R) + C(R)_i{}^j(\gamma_\phi)_j{}^i(\alpha_0,\lambda_0)/r}{2\pi} + \frac{C_2}{2\pi}\cdot\frac{\beta(\alpha_0,\lambda_0)}{\alpha_0}.$$

The β-function in the right hand side can be expressed in terms of the charge renormalization constant Z_α:

$$\beta(\alpha_0,\lambda_0) = \frac{d\alpha_0(\alpha,\lambda,\Lambda/\mu)}{d\ln\Lambda}\Big|_{\alpha,\lambda=\text{const}} = -\alpha_0\frac{d\ln Z_\alpha}{d\ln\Lambda}\Big|_{\alpha,\lambda=\text{const}}.$$

Non-renormalization of the triple gauge-ghost vertices and the new form of the NSVZ β-function

Using the finiteness of the triple gauge-ghost vertices we obtain

$$\beta(\alpha_0,\lambda_0) = -2\alpha_0 \frac{d\ln(Z_c Z_V)}{d\ln\Lambda}\Big|_{\alpha,\lambda=\text{const}} = 2\alpha_0\Big(\gamma_c(\alpha_0,\lambda_0) + \gamma_V(\alpha_0,\lambda_0)\Big),$$

where γ_c and γ_V are the anomalous dimensions of the Faddeev–Popov ghosts and of the quantum gauge superfield (defined in terms of the bare couplings), respectively.
Substituting this expression into the the right hand side we obtain the equivalent form of the NSVZ equation

$$\frac{\beta(\alpha_0,\lambda_0)}{\alpha_0^2} = -\frac{1}{2\pi}\Big(3C_2 - T(R) - 2C_2\gamma_c(\alpha_0,\lambda_0) - 2C_2\gamma_V(\alpha_0,\lambda_0) + C(R)_i{}^j(\gamma_\phi)_j{}^i(\alpha_0,\lambda_0)/r\Big).$$

It relates the β-function in a certain loop to the anomalous dimensions of quantum superfields in the previous loop, because the right hand side does not contain a denominator depending on couplings.

The β-function of $\mathcal{N}=1$ supersymmetric gauge theories as an integral of double total derivatives

A key observation needed for derivation of the NSVZ relation is that in the case of using the higher covariant derivative regularization the integrals giving the β-function defined in terms of the bare couplings are integrals of double total derivatives in $\mathcal{N}=1$ supersymmetric gauge theories. This was first noted in

A.A.Soloshenko, K.S., ArXiv: hep-th/0304083v1 (the factorization into total derivatives);
A.V.Smilga, A.I.Vainshtein, Nucl.Phys. **B 704** (2005) 445 (the factorization into double total derivatives).

The all-loop proof of this statement has been done in

K.S., Nucl. Phys. **852** (2011) 71.

for the Abelian case and in

K.S., JHEP **10** (2019) 011.

for general non-Abelian gauge theories.

As an example, at the next slide we present the three-loop expression for the β-function of $\mathcal{N}=1$ SQED with N_f flavors.

The three-loop β-function of $\mathcal{N}=1$ SQED as an integral of double total derivatives

$$\frac{\beta(\alpha_0)}{\alpha_0^2} = N_f \frac{d}{d\ln\Lambda}\Big\{2\pi\int\frac{d^4Q}{(2\pi)^4}\frac{\partial}{\partial Q^\mu}\frac{\partial}{\partial Q_\mu}\frac{\ln(Q^2+M^2)}{Q^2} + 4\pi\int\frac{d^4Q}{(2\pi)^4}\frac{d^4K}{(2\pi)^4}\frac{e^2}{K^2R_K^2}$$

$$\times\frac{\partial}{\partial Q^\mu}\frac{\partial}{\partial Q_\mu}\Big(\frac{1}{Q^2(K+Q)^2}-\frac{1}{(Q^2+M^2)((K+Q)^2+M^2)}\Big)\Big[R_K\Big(1+\frac{e^2N_f}{4\pi^2}\ln\frac{\Lambda}{\mu}\Big)$$

$$-2e^2N_f\Big(\int\frac{d^4L}{(2\pi)^4}\frac{1}{L^2(K+L)^2}-\int\frac{d^4L}{(2\pi)^4}\frac{1}{(L^2+M^2)((K+L)^2+M^2)}\Big)\Big]$$

$$+4\pi\int\frac{d^4Q}{(2\pi)^4}\frac{d^4K}{(2\pi)^4}\frac{d^4L}{(2\pi)^4}\frac{e^4}{K^2R_KL^2R_L}\frac{\partial}{\partial Q^\mu}\frac{\partial}{\partial Q_\mu}\Big\{\Big(-\frac{2K^2}{Q^2(Q+K)^2(Q+K+L)^2}$$

$$\times\frac{1}{(Q+L)^2}+\frac{2}{Q^2(Q+K)^2(Q+L)^2}\Big)-\Big(-\frac{2(K^2+M^2)}{((Q+K)^2+M^2)((Q+L)^2+M^2)}$$

$$\times\frac{1}{(Q^2+M^2)((Q+K+L)^2+M^2)}+\frac{2}{(Q^2+M^2)((Q+K)^2+M^2)((Q+L)^2+M^2)}$$

$$-\frac{4M^2}{(Q^2+M^2)^2((Q+K)^2+M^2)((Q+L)^2+M^2)}\Big)+O(e^6)\Big\}$$

Integrals of double total derivatives and a graphical interpretation of the NSVZ relation for $\mathcal{N}=1$ SQED

The integrals of double total derivatives do not vanish due to singularities of the integrands. Really, if $f(Q^2)$ is a non-singular function which rapidly decrease at infinity, then

$$\int \frac{d^4Q}{(2\pi)^4} \frac{\partial}{\partial Q^\mu} \frac{\partial}{\partial Q_\mu}\Big(\frac{f(Q^2)}{Q^2}\Big) = \int\limits_{S^3_\varepsilon} \frac{dS^\mu}{(2\pi)^4}\Big(- \frac{2Q_\mu}{Q^4} f(Q^2) + \frac{2Q_\mu}{Q^2} f'(Q^2)\Big)$$
$$= \frac{1}{4\pi^2} f(0) \ne 0.$$

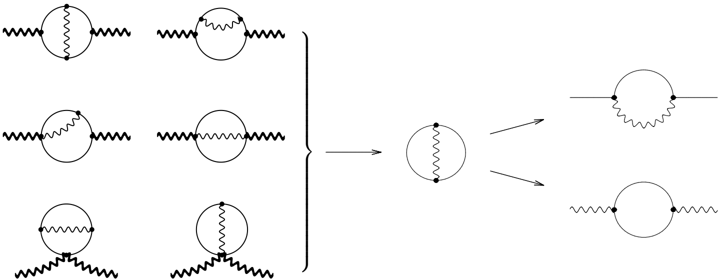
Due to similar equations the double total derivatives effectively cut lines of quantum superfields. As a result, we obtain diagrams contributing to various anomalous dimensions, in which a number of loops is less by 1. For example, in the Abelian case this gives the NSVZ β-function

$$\beta(\alpha) = \frac{\alpha^2 N_f}{\pi}\Big(1-\gamma(\alpha)\Big).$$

M.A.Shifman, A.I.Vainshtein, V.I.Zakharov, JETP Lett. **42** (1985) 224; Phys.Lett. **B 166** (1986) 334.

This allows to give a simple qualitative interpretation of the new form of the NSVZ equation:

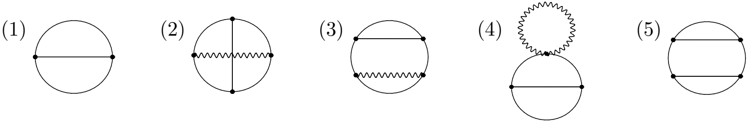
For each vacuum supergraph the NSVZ equation relates a contribution to the β-function obtained by attaching two external lines of the background gauge superfield to the corresponding contribution to the anomalous dimension of quantum superfields obtained by all various cuts of internal lines:



In the non-Abelian case internal lines can correspond to the quantum gauge superfield, the Faddeev–Popov ghosts, and the matter superfields, but all features are the same as in the Abelian case.

An example of a certain contribution to the β-function

The two- and three-loop contributions to the β-function which depend on the Yukawa couplings are generated by the vacuum supergraphs



Here we write down the contributions of the supergraphs (1) and (5) which determine the three-loop part of the β-function quartic in the Yukawa couplings

V.Yu.Shakhmanov, K.S., Nucl.Phys., **B920**, (2017), 345;
A.E.Kazantsev, V.Yu.Shakhmanov, K.S., JHEP 1804 (2018) 130.

$$\frac{\Delta\beta(\alpha_0,\lambda_0)}{\alpha_0^2} = -\frac{2\pi}{r}C(R)_i{}^j\frac{d}{d\ln\Lambda}\int\frac{d^4K}{(2\pi)^4}\frac{d^4Q}{(2\pi)^4}\,\lambda_0^{imn}\lambda^*_{0jmn}\frac{\partial}{\partial Q_\mu}\frac{\partial}{\partial Q^\mu}\Big(\frac{1}{K^2}$$
$$\times\frac{1}{F_K\,Q^2F_Q\,(Q+K)^2F_{Q+K}}\Big) + \frac{4\pi}{r}C(R)_i{}^j\frac{d}{d\ln\Lambda}\int\frac{d^4K}{(2\pi)^4}\frac{d^4L}{(2\pi)^4}\frac{d^4Q}{(2\pi)^4}\Big[\lambda_0^{iab}$$
$$\times\lambda^*_{0kab}\lambda_0^{kcd}\lambda^*_{0jcd}\Big(\frac{\partial}{\partial K_\mu}\frac{\partial}{\partial K^\mu} - \frac{\partial}{\partial Q_\mu}\frac{\partial}{\partial Q^\mu}\Big) + 2\lambda_0^{iab}\lambda^*_{0jac}\lambda_0^{cde}\lambda^*_{0bde}\frac{\partial}{\partial Q_\mu}\frac{\partial}{\partial Q^\mu}\Big]$$
$$\times\frac{1}{K^2F_K^2\,Q^2F_Q\,(Q+K)^2F_{Q+K}\,L^2F_L\,(L+K)^2F_{L+K}} = -\frac{1}{2\pi r}C(R)_i{}^j(\Delta\gamma_\phi)_j{}^i.$$

Deriving the new form of the exact NSVZ β-function by summing singular contributions

Due to the factorization into integrals of double total derivatives the function β/α^2 can be found by summing singular contributions. In all loops this has been done in

K.S., Eur.Phys.J. **C80** (2020) 10, 911.

The result can be written as

$$\frac{\beta(\alpha_0,\lambda_0)}{\alpha_0^2}-\frac{\beta_{\textbf{1-loop}}(\alpha_0)}{\alpha_0^2}$$

$$=\underbrace{\frac{1}{\pi}C_2\gamma_V(\alpha_0,\lambda_0)}_{\text{gauge propagators}}+\underbrace{\frac{1}{\pi}C_2\gamma_c(\alpha_0,\lambda_0)}_{\text{Faddeev–Popov ghost propagators}}-\underbrace{\frac{1}{2\pi r}C(R)_i{}^j(\gamma_\phi)_j{}^i(\alpha_0,\lambda_0)}_{\text{matter propagators}}.$$

Conditions required for validity of the NSVZ relation in all loops

Thus, the NSVZ relation

$$\frac{\beta(\alpha_0,\lambda_0)}{\alpha_0^2} = -\frac{1}{2\pi}\Big(3C_2 - T(R) - 2C_2\gamma_c(\alpha_0,\lambda_0) \\ -2C_2\gamma_V(\alpha_0,\lambda_0) + C(R)_i{}^j(\gamma_\phi)_j{}^i(\alpha_0,\lambda_0)/r\Big),$$

and, therefore, the NSVZ relation

$$\beta(\alpha_0,\lambda_0) = -\frac{\alpha_0^2\Big(3C_2 - T(R) + C(R)_i{}^j(\gamma_\phi)_j{}^i(\alpha_0,\lambda_0)/r\Big)}{2\pi(1 - C_2\alpha_0/2\pi)}$$

are valid in all orders of the perturbation theory for RGFs defined in terms of the bare couplings if a theory is regularized by higher covariant derivatives.

Consequently, for RGFs defined in terms of the renormalized couplings, similar equations hold in the HD+MSL scheme in all orders of the perturbation theory.

The NSVZ relations for theories with multiple gauge couplings

The NSVZ equations can also be written for theories with multiple gauge couplings. In this case a number of the NSVZ equations is equal to a number of (simple or $U(1)$) factors in the gauge group $G = G_1 \times G_2 \times \ldots \times G_n$. They can be written in the form

D.Korneev, D.Plotnikov, K.S., N.Tereshina, JHEP **10** (2021) 046.

$$\frac{\beta_K(\alpha,\lambda)}{\alpha_K^2} = -\frac{1}{2\pi(1 - C_2(G_K)\alpha_K/2\pi)}\Big[3C_2(G_K) - \sum_{\mathsf{a}} \boldsymbol{T}_{\mathsf{a}K}\Big(1 - \gamma_{\mathsf{a}}(\alpha,\lambda)\Big)\Big],$$

where the subscript a numerates chiral matter superfields in irreducible representations of simple G_I,

$$\boldsymbol{T}_K(R) = \sum_{\mathsf{a}} \boldsymbol{T}_{\mathsf{a}K},$$

and we use the notation

$$\boldsymbol{T}_{\mathsf{a}K} = \begin{cases} \delta_{i_1}{}^{i_1} \ldots \delta_{i_{K-1}}{}^{i_{K-1}} T_K(R_{\mathsf{a}K})\, \delta_{i_{K+1}}{}^{i_{K+1}} \ldots \delta_{i_n}{}^{i_n} & \text{if } G_K \text{ is simple;} \\ \delta_{i_1}{}^{i_1} \ldots \delta_{i_{K-1}}{}^{i_{K-1}}\, q_{\mathsf{a}K}^2\, \delta_{i_{K+1}}{}^{i_{K+1}} \ldots \delta_{i_n}{}^{i_n} & \text{if } G_K = U(1). \end{cases}$$

The NSVZ relations for $\mathcal{N}=1$ SQCD+SQED

As a particular case one can consider $\mathcal{N}=1$ SQCD interacting with the Abelian gauge superfield. This theory is based on the gauge group $G\times U(1)$ and is described by the action

$$S=\frac{1}{2g^2}\,\mbox{Re}\,\mbox{tr}\int d^4x\,d^2\theta\,W^aW_a+\frac{1}{4e^2}\,\mbox{Re}\int d^4x\,d^2\theta\,\boldsymbol{W}^a\boldsymbol{W}_a$$
$$+\sum_{\mathsf{a}=1}^{N_f}\frac{1}{4}\int d^4x\,d^4\theta\,\Big(\phi_{\mathsf{a}}^+e^{2V+2q_{\mathsf{a}}\boldsymbol{V}}\phi_{\mathsf{a}}+\widetilde\phi_{\mathsf{a}}^+e^{-2V-2q_{\mathsf{a}}\boldsymbol{V}}\widetilde\phi_{\mathsf{a}}\Big).$$

The NSVZ equations for this model take the form

$$\frac{\beta_s(\alpha_s,\alpha)}{\alpha_s^2}=-\frac{1}{2\pi(1-C_2\alpha_s/2\pi)}\Big[3C_2-2T(R)\sum_{\mathsf{a}=1}^{N_f}\Big(1-\gamma_{\mathsf{a}}(\alpha_s,\alpha)\Big)\Big];$$
$$\frac{\beta(\alpha_s,\alpha)}{\alpha^2}=\frac{1}{\pi}\,\mbox{dim}\,R\sum_{\mathsf{a}=1}^{N_f}q_{\mathsf{a}}^2\Big(1-\gamma_{\mathsf{a}}(\alpha_s,\alpha)\Big),$$

where $\alpha_s\equiv g^2/4\pi$ is an analog of the strong coupling constant, and $\alpha=e^2/4\pi$ is an analog of the electromagnetic coupling constant.

The NSVZ relation for the Adler D-function in $\mathcal{N}=1$ SQCD+SQED

The Adler D-function encodes quantum corrections to the electromagnetic coupling constant appearing due to the strong interaction,

$$D(\alpha_s) = \frac{3\pi}{2}\lim_{\alpha\to 0}\frac{\beta(\alpha_s,\alpha)}{\alpha^2}.$$

Taking into account that in the limit $\alpha\to 0$ all chiral matter superfields have the same anomalous dimension, $\lim\limits_{\alpha\to 0}\gamma_{\mathsf{a}}(\alpha_s,\alpha) = \gamma(\alpha_s)$, from the NSVZ equation for $\beta(\alpha,\alpha_s)$ we obtain the all-loop relation

$$D(\alpha_s) = \frac{3}{2}\,\mbox{dim}\, R\sum_{\mathsf{a}=1}^{N_f} q_{\mathsf{a}}^2\Big(1-\gamma(\alpha_s)\Big)$$

M.Shifman, K.S. Phys. Rev. Lett. **114** (2015) no.5, 051601; Phys. Rev. D **91** (2015), 105008.

see also

A.L.Kataev, A.E.Kazantsev, K.S., Nucl. Phys. B **926** (2018), 295.

Various definitions of the Adler D-function and the validity of the NSVZ equation for various renormalization prescriptions were analyzed in

S.S.Aleshin, A.L.Kataev, K.S., JHEP **03** (2019), 196.

Another form of the NSVZ equations for theories with multiple gauge couplings

The non-renormalization theorem for the triple gauge-ghost vertices is also valid for theories with multiple gauge couplings, so that for all $k = 1, \ldots, n$

$$Z_{\alpha_K}^{-1/2} Z_{c_K} Z_{V_K} = 1.$$

Therefore, the NSVZ equations can be rewritten as relations between the β-functions in a certain loop and the anomalous dimensions of the quantum superfields in the previous loop,

$$\frac{\beta_K(\alpha,\lambda)}{\alpha_K^2} = -\frac{1}{2\pi}\Big[C_2(G_K)\Big(3 - 2\gamma_{V_K}(\alpha,\lambda) - 2\gamma_{c_K}(\alpha,\lambda)\Big) - \sum_{\mathsf{a}} \boldsymbol{T}_{\mathsf{a}K}\Big(1 - \gamma_{\mathsf{a}}(\alpha,\lambda)\Big)\Big].$$

It seems that (as for the case of theories with a single gauge couplings) the NSVZ equations are valid for RGFs defined in terms of the bare coupling in the case of using the higher covariant derivative regularization, and for RGFs defined in terms of the renormalized couplings in the HD+MSL scheme.

Let us present the all-order exact NSVZ β-functions for some phenomenologically interesting theories. For instance, for MSSM they are given by the equations

$$\frac{\beta_3(\alpha,\lambda)}{\alpha_3^2} = -\frac{1}{2\pi(1-3\alpha_3/2\pi)}\Big[3+\mbox{tr}\Big(\gamma_{Q_I}(\alpha,\lambda)+\frac{1}{2}\gamma_{U_I}(\alpha,\lambda)+\frac{1}{2}\gamma_{D_I}(\alpha,\lambda)\Big)\Big];$$

$$\frac{\beta_2(\alpha,\lambda)}{\alpha_2^2} = -\frac{1}{2\pi(1-\alpha_2/\pi)}\Big[-1+\mbox{tr}\Big(\frac{3}{2}\gamma_{Q_I}(\alpha,\lambda)+\frac{1}{2}\gamma_{L_I}(\alpha,\lambda)\Big)+\frac{1}{2}\gamma_{H_u}(\alpha,\lambda)$$
$$+\frac{1}{2}\gamma_{H_d}(\alpha,\lambda)\Big];$$

$$\frac{\beta_1(\alpha,\lambda)}{\alpha_1^2} = -\frac{3}{5}\cdot\frac{1}{2\pi}\Big[-11+\mbox{tr}\Big(\frac{1}{6}\gamma_{Q_I}(\alpha,\lambda)+\frac{4}{3}\gamma_{U_I}(\alpha,\lambda)+\frac{1}{3}\gamma_{D_I}(\alpha,\lambda)$$
$$+\frac{1}{2}\gamma_{L_I}(\alpha,\lambda)+\gamma_{E_I}(\alpha,\lambda)\Big)+\frac{1}{2}\gamma_{H_u}(\alpha,\lambda)+\frac{1}{2}\gamma_{H_d}(\alpha,\lambda)\Big],$$

where the traces are taken with respect to the generation indices.

(In a different form) they were first presented in

M. A. Shifman, Int. J. Mod. Phys. A **11** (1996), 5761.

and correctly reproduce the (scheme-independent) two-loop contributions.

The exact NSVZ β-functions for the flipped $SU(5)$ model

As another example we consider the flipped $SU(5)$ Grand Unification Theory.

S. M. Barr, Phys. Lett. B **112** (1982), 219; I. Antoniadis, J. R. Ellis, J. S. Hagelin and D. V. Nanopoulos, Phys. Lett. B **194** (1987), 231.

The quark and lepton superfields belong to the representation $3 \times \big(\overline{10}(1) + 5(-3) + 1(5)\big)$ of the gauge group $SU(5) \times U(1)$. Also the theory includes Higgs superfields H and $\widetilde{H}$ in $10(-1)$ and $\bar{1}0(1)$; h and $\widetilde{h}$ in $5(2)$ and $\bar{5}(-2)$, and four singlets ϕ. The exact NSVZ β-functions for this model are

$$\frac{\beta_5(\alpha,\lambda)}{\alpha_5^2} = -\frac{1}{2\pi(1-5\alpha_5/2\pi)}\Big[5 + \mbox{tr}\Big(\frac{3}{2}\gamma_{\overline{10}_I}(\alpha,\lambda) + \frac{1}{2}\gamma_{5_I}(\alpha,\lambda)\Big) + \frac{3}{2}\gamma_H(\alpha,\lambda) + \frac{3}{2}\gamma_{\widetilde{H}}(\alpha,\lambda) + \frac{1}{2}\gamma_h(\alpha,\lambda) + \frac{1}{2}\gamma_{\widetilde{h}}(\alpha,\lambda)\Big];$$

$$\frac{\beta_1(\alpha,\lambda)}{\alpha_1^2} = \frac{1}{8}\cdot\frac{1}{2\pi}\Big[60 - \mbox{tr}\Big(2\gamma_{\overline{10}_I}(\alpha,\lambda) + 9\gamma_{5_I}(\alpha,\lambda) + 5\gamma_{E_I}(\alpha,\lambda)\Big) - 2\gamma_H(\alpha,\lambda) - 2\gamma_{\widetilde{H}}(\alpha,\lambda) - 4\gamma_h(\alpha,\lambda) - 4\gamma_{\widetilde{h}}(\alpha,\lambda)\Big].$$

D.Korneev, D.Plotnikov, K.S., N.Tereshina, JHEP **10** (2021) 046.

The two-loop anomalous dimension of the matter superfields with the higher derivative regularization

For theories with a single gauge coupling the two-loop anomalous dimension defined in terms of the bare coupling constant for $\mathcal{N}=1$ supersymmetric theories regularized by higher derivatives has been calculated in

A.E.Kazantsev, K.S., JHEP **2006** (2020) 108.

$$(\gamma_\phi)_i{}^j(\alpha_0,\lambda_0) = -\frac{\alpha_0}{\pi}C(R)_i{}^j + \frac{1}{4\pi^2}\lambda^*_{0imn}\lambda_0^{jmn} + \frac{\alpha_0^2}{2\pi^2}\left[C(R)^2\right]_i{}^j - \frac{1}{16\pi^4}$$
$$\times\lambda^*_{0iac}\lambda_0^{jab}\lambda^*_{0bde}\lambda_0^{cde} - \frac{3\alpha_0^2}{2\pi^2}C_2C(R)_i{}^j\Big(\ln a_\varphi + 1 + \frac{A}{2}\Big) + \frac{\alpha_0^2}{2\pi^2}T(R)C(R)_i{}^j$$
$$\times\Big(\ln a + 1 + \frac{A}{2}\Big) - \frac{\alpha_0}{8\pi^3}\lambda^*_{0lmn}\lambda_0^{jmn}C(R)_i{}^l(1-B+A) + \frac{\alpha_0}{4\pi^3}\lambda^*_{0imn}\lambda_0^{jml}$$
$$\times C(R)_l{}^n(1-A+B) + O\Big(\alpha_0^3,\alpha_0^2\lambda_0^2,\alpha_0\lambda_0^4,\lambda_0^6\Big),$$

where

$$A = \int\limits_0^\infty dx\,\ln x\,\frac{d}{dx}\frac{1}{R(x)};\qquad B = \int\limits_0^\infty dx\,\ln x\,\frac{d}{dx}\frac{1}{F^2(x)}\qquad a = \frac{M}{\Lambda};\qquad a_\varphi = \frac{M_\varphi}{\Lambda}.$$

Obtaining the three-loop β-function from the NSVZ equation

If the anomalous dimension of the matter superfields defined in terms of the bare couplings has been calculated in L-loops with the higher derivative regularization, then it is possible to construct the $(L+1)$-loop β-function from the NSVZ equation without loop calculations. For example, in the three-loop approximation

$$\frac{\beta(\alpha_0,\lambda_0)}{\alpha_0^2} = -\frac{1}{2\pi}\Big(3C_2 - T(R)\Big) + \frac{\alpha_0}{4\pi^2}\Big\{ -3C_2^2 + \frac{1}{r}C_2\,\mbox{tr}\,C(R) + \frac{2}{r}\,\mbox{tr}\left[C(R)^2\right]\Big\}$$
$$-\frac{1}{8\pi^3 r}C(R)_j{}^i\lambda^*_{0imn}\lambda_0^{jmn} + \frac{\alpha_0^2}{8\pi^3}\Big\{ -3C_2^3 + \frac{1}{r}C_2^2\,\mbox{tr}\,C(R) - \frac{2}{r}\,\mbox{tr}\left[C(R)^3\right] + \frac{2}{r}$$
$$\times C_2\,\mbox{tr}\left[C(R)^2\right]\Big(3\ln a_\varphi + 4 + \frac{3A}{2}\Big) - \frac{2}{r^2}\,\mbox{tr}\,C(R)\,\mbox{tr}\left[C(R)^2\right]\Big(\ln a + 1 + \frac{A}{2}\Big)\Big\}$$
$$-\frac{\alpha_0 C_2}{16\pi^4 r}C(R)_j{}^i\lambda^*_{0imn}\lambda_0^{jmn} + \frac{\alpha_0}{16\pi^4 r}\left[C(R)^2\right]_j{}^i\lambda^*_{0imn}\lambda_0^{jmn}\Big(1+A-B\Big) - \frac{\alpha_0}{8\pi^4 r}$$
$$\times C(R)_j{}^i C(R)_l{}^n\lambda^*_{0imn}\lambda_0^{jml}\Big(1-A+B\Big) + \frac{1}{32\pi^5 r}C(R)_j{}^i\lambda^*_{0iac}\lambda_0^{jab}\lambda^*_{0bde}\lambda_0^{cde}$$
$$+O\Big(\alpha_0^3,\,\alpha_0^2\lambda_0^2,\,\alpha_0\lambda_0^4,\,\lambda_0^6\Big).$$

Certainly, RGFs defined in terms of the renormalized couplings can also be calculated for an arbitrary renormalization prescription.

Obtaining RGFs defined in terms of the renormalized couplings

To calculate RGFs defined in terms of the renormalized couplings, first, we integrate the equations

$$\beta(\alpha_0,\lambda_0) \equiv \frac{d\alpha_0}{d\ln\Lambda}\Big|_{\alpha,\lambda=\text{const}}; \qquad (\gamma_\phi)_i{}^j(\alpha_0,\lambda_0) \equiv -\frac{d(\ln Z_\phi)_i{}^j}{d\ln\Lambda}\Big|_{\alpha,\lambda=\text{const}}$$

and obtain the expressions for the renormalized gauge coupling constant and $(\ln Z_\phi)_i{}^j$. They depend on a set of finite constants which determine a subtraction scheme in the considered approximation. Next, we substitute the expressions obtained in this way into the equations

$$\widetilde\beta(\alpha,\lambda) \equiv \frac{d\alpha}{d\ln\mu}\Big|_{\alpha_0,\lambda_0=\text{const}}; \qquad (\widetilde\gamma_\phi)_i{}^j(\alpha,\lambda) \equiv \frac{d(\ln Z_\phi)_i{}^j}{d\ln\mu}\Big|_{\alpha_0,\lambda_0=\text{const}}.$$

These RGFs will nontrivially depend on the finite constants due to the scheme dependence.

The results for the two-loop anomalous dimension and the three-loop β-function defined in terms of the renormalized couplings are rather large and are not presented here. They can be found in

A.E.Kazantsev, K.S., JHEP **2006** (2020) 108.

Finite constants fixing a renormalization prescription

The finite constants in the lowest approximation are defined by the equations

$$(\ln Z_\phi)_i{}^j(\alpha,\lambda) = \frac{\alpha}{\pi}C(R)_i{}^j\Big(\ln\frac{\Lambda}{\mu}+g_{11}\Big) - \frac{1}{4\pi^2}\lambda^*_{imn}\lambda^{jmn}\Big(\ln\frac{\Lambda}{\mu}+g_{12}\Big)$$
$$+O(\alpha^2,\alpha\lambda^2,\lambda^4);$$
$$\frac{1}{\alpha}-\frac{1}{\alpha_0} = -\frac{3}{2\pi}C_2\Big(\ln\frac{\Lambda}{\mu}+b_{11}\Big)+\frac{1}{2\pi}T(R)\Big(\ln\frac{\Lambda}{\mu}+b_{12}\Big)-\frac{3\alpha}{4\pi^2}C_2^2\Big(\ln\frac{\Lambda}{\mu}$$
$$+b_{21}\Big)+\frac{\alpha}{4\pi^2 r}C_2\,\mathrm{tr}\,C(R)\Big(\ln\frac{\Lambda}{\mu}+b_{22}\Big)+\frac{\alpha}{2\pi^2 r}\mathrm{tr}\left[C(R)^2\right]\Big(\ln\frac{\Lambda}{\mu}+b_{23}\Big)$$
$$-\frac{1}{8\pi^3 r}C(R)_j{}^i\lambda^*_{imn}\lambda^{jmn}\Big(\ln\frac{\Lambda}{\mu}+b_{24}\Big)+O(\alpha^2,\alpha\lambda^2,\lambda^4).$$

In the HD+MSL scheme all finite constants g_i and b_i are equal to 0, and the NSVZ relation is valid in the $O(\alpha^2,\alpha\lambda^2,\lambda^4)$ approximation. For other schemes this in general is not true.

Here (at the next slide) we present the result for considered RGFs only for a particular case, namely, for one-loop finite $\mathcal{N}=1$ supersymmetric theories, see

P.West, Phys.Lett. **B 137** (1984) 371;
A.Parkes, P.West, Phys.Lett. **B 138** (1984) 99.

An important particular case is theories finite in the one-loop approximation which satisfy the conditions

$$T(R) = 3C_2; \qquad \lambda^*_{imn}\lambda^{jmn} = 4\pi\alpha C(R)_i{}^j.$$

In this case the two-loop anomalous dimension and the three-loop β-function defined in terms of the renormalized couplings have the form

$$(\widetilde{\gamma}_\phi)_i{}^j(\alpha,\lambda) = -\frac{3\alpha^2}{2\pi^2}C_2 C(R)_i{}^j\Big(\ln\frac{a_\varphi}{a} - b_{11} + b_{12}\Big) - \frac{\alpha}{4\pi^2}\Big(\frac{1}{\pi}\lambda^*_{imn}\lambda^{jml}C(R)_l{}^n$$
$$+2\alpha\left[C(R)^2\right]_i{}^j\Big)\Big(A - B - 2g_{12} + 2g_{11}\Big) + O\Big(\alpha^3,\alpha^2\lambda^2,\alpha\lambda^4,\lambda^6\Big);$$
$$\frac{\widetilde{\beta}(\alpha,\lambda)}{\alpha^2} = \frac{3\alpha^2}{4\pi^3 r}C_2\,\mbox{tr}\left[C(R)^2\right]\Big(\ln\frac{a_\varphi}{a} - b_{11} + b_{12}\Big) + \frac{\alpha}{8\pi^3 r}\Big(\frac{1}{\pi}C(R)_j{}^i C(R)_l{}^n$$
$$\times\lambda^*_{imn}\lambda^{jml} + 2\alpha\,\mbox{tr}\left[C(R)^3\right]\Big)\Big(A - B - 2g_{12} + 2g_{11}\Big) + O\Big(\alpha^3,\alpha^2\lambda^2,\alpha\lambda^4,\lambda^6\Big).$$

We see that in this case the NSVZ equation is satisfied in the lowest nontrivial approximation for an arbitrary renormalization presription,

$$\frac{\beta(\alpha,\lambda)}{\alpha^2} = -\frac{1}{2\pi r}C(R)_i{}^j(\gamma_\phi)_j{}^i(\alpha,\lambda) + O(\alpha^3,\alpha^2\lambda^2,\alpha\lambda^4,\lambda^6).$$

The NSVZ equation for theories finite in the lowest loops

For $\mathcal{N} = 1$ supersymmetric theories finite in the one-loop approximation it is possible to tune a subtraction scheme so that the theory will be all-loop finite

D.I.Kazakov, Phys. Lett. B **179** (1986) 352; A.V.Ermushev, D.I.Kazakov, O.V.Tarasov, Nucl.Phys. B **281** (1987) 72; C.Lucchesi, O.Piguet, K.Sibold, Helv.Phys.Acta **61** (1988) 321; Phys.Lett. B **201** (1988) 241.

If a subtraction scheme is tuned in such a way that the β-function vanishes in the first L loops and the anomalous dimension of the matter superfields vanishes in the first $(L-1)$ loops, then

K.S., Eur.Phys.J. C **81** (2021) 571.

for an arbitrary renormalization prescription the $(L+1)$-loop gauge β-function satisfies the equation

$$\frac{\beta_{L+1}(\alpha,\lambda)}{\alpha^2} = -\frac{1}{2\pi r} C(R)_i{}^j (\gamma_{\phi,L})_j{}^i(\alpha,\lambda).$$

Therefore, if a theory is finite in a certain approximation, its β-function vanishes in the next order. This exactly agrees with the earlier known result of

A.J.Parkes, P.West, Nucl.Phys. B **256** (1985) 340;
M.T.Grisaru, B.Milewski and D.Zanon, Phys.Lett. **155B** (1985) 357.

Three-loop MSSM β-functions for an arbitrary supersymmetric renormalization prescription

Starting from the two-loop expressions for the anomalous dimensions of the matter superfields it is possible to find the three-loop MSSM β-functions for an arbitrary supersymmetric renormalization prescription supplementing the higher covariant derivative regularization

O.Haneychuk, V.Shirokova, K.S., arXiv:2207.11944[hep-ph].

The result is very large and depends on both regularization parameters and finite constants fixing a subtraction scheme. For certain values of these finite constants it reproduces the $\overline{\text{DR}}$ result obtained earlier. Also it is possible to construct a class of the NSVZ schemes, which are related to the HD+MSL scheme by the finite renormalizations satisfying the constraint

$$\frac{1}{\alpha'_K} - \frac{1}{\alpha_K} + \frac{C_2(G_K)}{2\pi}\ln\frac{\alpha'_K}{\alpha_K} - \frac{1}{2\pi}\sum_{\mathsf{a}} \boldsymbol{T}_{\mathsf{a}K}\ln z_{\mathsf{a}} = B_K\,,$$

where B_K are some constants.
As an example, we present the three-loop expression for the function $\widetilde{\beta}_3$.

MSSM, three-loop β_3

$$\frac{\widetilde{\beta}_3(\alpha,Y)}{\alpha_3^2} = -\frac{1}{2\pi}\Big\{3 - \frac{11\alpha_1}{20\pi} - \frac{9\alpha_2}{4\pi} - \frac{7\alpha_3}{2\pi} + \frac{1}{8\pi^2}\,\mathrm{tr}\Big(2\,Y_U^+ Y_U + 2\,Y_D^+ Y_D\Big) + \frac{1}{2\pi^2}\Big[\frac{137\alpha_1^2}{1200}$$
$$+\frac{27\alpha_2^2}{16} + \frac{\alpha_3^2}{6} + \frac{3\alpha_1\alpha_2}{40} - \frac{11\alpha_1\alpha_3}{60} - \frac{3\alpha_2\alpha_3}{4} + \frac{363\alpha_1^2}{100}\Big(\ln a_1 + 1 + \frac{A}{2} + b_{2,31} - b_{1,1}\Big) + \frac{9\alpha_2^2}{4}$$
$$\times\Big(-6\ln a_{\varphi,2} + 7\ln a_2 + 1 + \frac{A}{2} + b_{2,32} - b_{1,2}\Big) - 24\alpha_3^2\Big(3\ln a_{\varphi,3} - 2\ln a_3 + 1 + \frac{A}{2} + \frac{7}{16}b_{2,33}$$
$$-\frac{7}{16}b_{1,3}\Big)\Big] + \frac{1}{8\pi^3}\mathrm{tr}\Big(Y_U Y_U^+\Big)\Big[\frac{3\alpha_1}{20} + \frac{3\alpha_2}{4} + 3\alpha_3 + \frac{13\alpha_1}{30}\Big(B - A + 2b_{2,3U} - 2j_{U1}\Big) + \frac{3\alpha_2}{2}$$
$$\times\Big(B - A + 2b_{2,3U} - 2j_{U2}\Big) + \frac{8\alpha_3}{3}\Big(B - A + 2b_{2,3U} - 2j_{U3}\Big)\Big] + \frac{1}{8\pi^3}\mathrm{tr}\Big(Y_D Y_D^+\Big)\Big[\frac{3\alpha_1}{20} + \frac{3\alpha_2}{4}$$
$$+3\alpha_3 + \frac{7\alpha_1}{30}\Big(B - A + 2b_{2,3D} - 2j_{D1}\Big) + \frac{3\alpha_2}{2}\Big(B - A + 2b_{2,3D} - 2j_{D2}\Big) + \frac{8\alpha_3}{3}\Big(B - A$$
$$+2b_{2,3D} - 2j_{D3}\Big)\Big] - \frac{1}{(8\pi^2)^2}\Big[\frac{3}{2}\mathrm{tr}\Big((Y_U Y_U^+)^2\Big)\Big(1 + 4b_{2,3U} - 4j_{UU}\Big) + \frac{3}{2}\mathrm{tr}\Big((Y_D Y_D^+)^2\Big)\Big(1$$
$$+4b_{2,3D} - 4j_{DD}\Big) + 3\Big(\mathrm{tr}(Y_U Y_U^+)\Big)^2\Big(1 + 2b_{2,3U} - 2j_{UtU}\Big) + 3\Big(\mathrm{tr}(Y_D Y_D^+)\Big)^2\Big(1 + 2b_{2,3D}$$
$$-2j_{DtD}\Big) + \mathrm{tr}\Big(Y_E Y_E^+\Big)\,\mathrm{tr}\Big(Y_D Y_D^+\Big)\Big(1 + 2b_{2,3D} - 2j_{DtE}\Big) + \mathrm{tr}\Big(Y_D Y_D^+ Y_U Y_U^+\Big)\Big(1 + 2b_{2,3U}$$
$$+2b_{2,3D} - 2j_{UD} - 2j_{DU}\Big)\Big]\Big\} + O(\alpha^3, \alpha^2 Y^2, \alpha Y^4, Y^6).$$

The four-loop β-function for $\mathcal{N}=1$ SQED with N_f flavors

Similarly, it is possible to obtain the four-loop β-function of $\mathcal{N}=1$ SQED with N_f flavors starting from the three-loop anomalous dimension

I.Shirokov and K.S., JHEP 04 (2022) 108.

The result (for RGFs defined in terms of the renormalized coupling constant) is

$$\widetilde\gamma(\alpha) = -\frac{\alpha}{\pi} + \frac{\alpha^2}{2\pi^2} + \frac{\alpha^2 N_f}{\pi^2}\Big(\ln a + 1 + \frac{A_1}{2} + g_{1,0} - b_{1,0}\Big) - \frac{\alpha^3}{2\pi^3} + \frac{\alpha^3 N_f}{\pi^3}$$
$$\times\Big(-\ln a - \frac{3}{4} - C - b_{2,0} + b_{1,0} - g_{2,0} + g_{1,0}\Big) + \frac{\alpha^3 (N_f)^2}{\pi^3}\Big\{ -\Big(\ln a + 1 - b_{1,0}\Big)^2$$
$$+\frac{A_2}{4} - D_1 \ln a - D_2 + b_{1,0} A_1 - g_{2,1}\Big\} + O(\alpha^4);$$

$$\frac{\widetilde\beta(\alpha)}{\alpha^2} = \frac{N_f}{\pi} + \frac{\alpha N_f}{\pi^2} - \frac{\alpha^2 N_f}{2\pi^3} - \frac{\alpha^2 (N_f)^2}{\pi^3}\Big(\ln a + 1 + \frac{A_1}{2} + b_{2,0} - b_{1,0}\Big)$$
$$+\frac{\alpha^3 N_f}{2\pi^4} + \frac{\alpha^3 (N_f)^2}{\pi^4}\Big(\ln a + \frac{3}{4} + C + b_{3,0} - b_{1,0}\Big) + \frac{\alpha^3 (N_f)^3}{\pi^4}\Big\{\Big(\ln a + 1 - b_{1,0}\Big)^2$$
$$-\frac{A_2}{4} + D_1 \ln a + D_2 - b_{1,0} A_1 + b_{3,1}\Big\} + O(\alpha^4).$$

Notations

Here the notations are

$$A_1 \equiv \int\limits_0^\infty dx\ \ln x\ \frac{d}{dx}\Big(\frac{1}{R(x)}\Big); \qquad A_2 \equiv \int\limits_0^\infty dx\ \ln^2 x\ \frac{d}{dx}\Big(\frac{1}{R(x)}\Big);$$

$$C \equiv \int\limits_0^1 dx \int\limits_0^\infty dy\ x\ \ln y\ \frac{d}{dy}\Big(\frac{1}{R(y)R(x^2 y)}\Big); \qquad D_1 \equiv \int\limits_0^\infty dx\ \ln x\ \frac{d}{dx}\Big(\frac{1}{R^2(x)}\Big);$$

$$D_2 \equiv \int\limits_0^\infty dx\ \ln x\ \frac{d}{dx}\left\{\frac{1}{R^2(x)}\Big[-\frac{1}{2}\Big(1-R(x)\Big)\ln x + \sqrt{1+\frac{4a^2}{x}}\,\text{arctanh}\sqrt{\frac{x}{x+4a^2}}\,\Big]\right\}.$$

and the finite constants are defined by the equations

$$\ln Z = \frac{\alpha}{\pi}\Big(\ln\frac{\Lambda}{\mu} + g_{1,0}\Big) - \frac{\alpha^2}{2\pi^2}\Big(\ln\frac{\Lambda}{\mu} + g_{2,0} + N_f g_{2,1}\Big) - \frac{\alpha^2 N_f}{\pi^2}\Big(\ln a + 1 + \frac{A_1}{2} - b_{1,0}\Big)\ln\frac{\Lambda}{\mu} + \frac{\alpha^2 N_f}{2\pi^2}\ln^2\frac{\Lambda}{\mu} + O(\alpha^3).$$

$$\frac{1}{\alpha_0} = \frac{1}{\alpha} - \frac{N_f}{\pi}\Big(\ln\frac{\Lambda}{\mu} + b_{1,0}\Big) - \frac{\alpha N_f}{\pi^2}\Big(\ln\frac{\Lambda}{\mu} + b_{2,0}\Big) + \frac{\alpha^2 N_f}{2\pi^3}\Big(\ln\frac{\Lambda}{\mu} + b_{3,0} + N_f b_{3,1}\Big) + \frac{\alpha^2 (N_f)^2}{\pi^3}\Big(\ln a + 1 + \frac{A_1}{2} - b_{1,0}\Big)\ln\frac{\Lambda}{\mu} - \frac{\alpha^2 (N_f)^2}{2\pi^3}\ln^2\frac{\Lambda}{\mu} + O(\alpha^3).$$

We see that the terms in the anomalous dimension without N_f and terms in the β-function proportional to $(N_f)^1$ are scheme-independent in agreement with the general all-loop statement proved in

A.L.Kataev and K.S., Phys. Lett. B **730** (2014), 184; Theor. Math. Phys. **181** (2014), 1531 [Teor. Mat. Fiz. **181** (2014) 475].

From the explicit above expressions for RGFs we see that by a special choice of the finite constants b_i and g_i it is possible to remove all terms proportional to $(N_f)^k$ with $k\geq 1$ in the anomalous dimension and all terms proportional to $(N_f)^k$ with $k\geq 2$ in the β-function. Then we obtain the simplest, so-called minimal scheme, in which

$$\widetilde\gamma(\alpha) = -\frac{\alpha}{\pi} + \frac{\alpha^2}{2\pi^2} - \frac{\alpha^3}{2\pi^3} + O(\alpha^4);$$

$$\widetilde\beta(\alpha) = \frac{\alpha^2 N_f}{\pi} + \frac{\alpha^3 N_f}{\pi^2} - \frac{\alpha^4 N_f}{2\pi^3} + \frac{\alpha^5 N_f}{2\pi^4} + O(\alpha^6).$$

The minimal renormalization scheme for the considered theory can be chosen in all orders of the perturbation theory. This scheme is NSVZ in all orders.

$\mathcal{N}=2$ supersymmetric gauge theories in $\mathcal{N}=1$ superspace

$\mathcal{N}=2$ supersymmetric theories can be considered as a particular case of $\mathcal{N}=1$ supersymmetric theories. Therefore, they can be formulated in terms of $\mathcal{N}=1$ superfields,

$$S=\frac{1}{2e_0^2}\mathrm{tr}\Big(\mathrm{Re}\int d^4x\,d^2\theta\,W^a W_a+\int d^4x\,d^4\theta\,\Phi^+ e^{2V}\Phi\,e^{-2V}\Big)+\frac{1}{4}\int d^4x\,d^4\theta$$
$$\times\Big(\phi^+ e^{2V}\phi+\widetilde\phi^+ e^{-2V^T}\widetilde\phi\Big)+\Big[\int d^4x\,d^2\theta\,\Big(\frac{i}{\sqrt{2}}\widetilde\phi^t\Phi\phi+\frac{1}{2}m_0\widetilde\phi^T\phi\Big)+\mathrm{c.c.}\Big]$$

Here the chiral superfield Φ in the adjoint representation is an $\mathcal{N}=2$ superpartner of the gauge superfield V. The chiral superfields ϕ and $\widetilde\phi$ in the representations R_0 and $\bar R_0$ form an $\mathcal{N}=2$ hypermultiplet.

Therefore, we obtain an $\mathcal{N}=1$ supersymmetric theory with chiral matter superfields in the reducible representation

$$R=Adj+R_0+\bar R_0,$$

containing nontrivial Yukawa interaction.

In this formulation only $\mathcal{N}=1$ supersymmetry is manifest, while the second supersymmetry is hidden.

Special features of quantization and renormalization in $\mathcal{N}=1$ superspace

A higher derivative term S_Λ invariant under both supersymmetries has been constructed in

I.L.Buchbinder and K.S., Nucl.Phys. **B883** (2014) 20.

However, with the help of the $\mathcal{N}=1$ superfield technique it is impossible to quantize a theory in the $\mathcal{N}=2$ supersymmetric way. Therefore, in this case quantum corrections can break the hidden supersymmetry.

In this case from the NSVZ equation and the non-renormalization of superpotential we obtain

$$\beta(\alpha_0) = -\frac{\alpha_0^2}{\pi}\Big(C_2 - T(R_0)\Big)\Big(1+\frac{1}{2}\,\gamma_\Phi(\alpha_0)\Big) = -\frac{\alpha_0^2}{\pi}\Big(C_2 - T(R_0)\Big)\Big(1-\gamma_\phi(\alpha_0)\Big).$$

This implies that, in general, higher loop $(L>1)$ contributions to the β-function do not vanish and are determined by the function $\gamma_\Phi(\alpha_0)$.
To construct a renormalization prescription for which the β-function (defined in terms of the bare coupling) vanishes beyond the one-loop approximation, one should use a manifestly $\mathcal{N}=2$ supersymmetric regularization and quantization.

Explicitly $\mathcal{N} = 2$ supersymmetric formulation in the harmonic superspace

$\mathcal{N} = 2$ supersymmetry is a manifest symmetry in the case of using $\mathcal{N} = 2$ harmonic superspace

A.Galperin, E.Ivanov, S.Kalitzin, V.Ogievetsky and E.Sokatchev, Class.Quant.Grav. **1** (1984) 469.

with the coordinates $(x^\mu, \theta^i_a, \bar\theta_{i\dot a}, u_i^\pm)$, where $u_i^- = (u^{+i})^*$ and $u^{+i}u_i^- = 1$. With the help of the harmonic superspace one can quantize the theory in a manifestly $\mathcal{N} = 2$ supersymmetric way. That is why the harmonic superspace technique together with the background superfield method allow having manifest $\mathcal{N} = 2$ supersymmetry and gauge invariance at all steps of calculating quantum corrections.

A.S.Galperin, E.A.Ivanov, V.I.Ogievetsky and E.S.Sokatchev, Harmonic superspace. Cambridge University Press (2001) 306p.

The higher covariant derivative regularization can also be formulated in the harmonic superspace

I.L.Buchbinder, N.G.Pletnev and K.S., Phys.Lett. **B751** (2015) 434.

$\mathcal{N}=2$ non-renormalization theorem and the NSVZ β-function

The higher covariant derivative regularization allows to prove simply the $\mathcal{N}=2$ non-renormalization theorem starting from the NSVZ β-function.

The degree of divergence (for non-regularized theory) in the harmonic superspace is written as

I.L.Buchbinder, S.M.Kuzenko and B.A.Ovrut, Phys.Lett. **B433** (1998) 335.

$$\omega_0 = -N_q - N_c - \frac{1}{2}N_D,$$

where N_q is a number of external hypermultiplet lines, N_c is a number of external ghost lines, and N_D is a number of spinor derivatives acting on external lines. Therefore, all superdiagrams containing hypermultiplet external lines are finite, so that $\gamma_\phi(\alpha_0)=0$. Consequently,

$$\frac{\beta(\alpha_0)}{\alpha_0^2} = -\frac{1}{\pi}\Big(C_2 - T(R)\Big)\Big(1-\gamma_\phi(\alpha_0)\Big) = -\frac{1}{\pi}\Big(C_2-T(R)\Big).$$

This implies that the β-function is non-trivial only in the one-loop approximation.

It is convenient to formulate $6D$, $\mathcal{N}=(1,0)$ supersymmetrc theories in $6D$, $\mathcal{N}=(1,0)$ harmonic superspace because in this case $\mathcal{N}=(1,0)$ supersymmetry is a manifest symmetry even at the quantum level.

The harmonic superspace is parameterized by the coordinates x^M, $M=1,\ldots 6$, θ^{ai}, and $u^\pm$ (such that $u^{+i}u_i^-=1$, $u_i^-\equiv(u^{+i})^*$).

The $6D$, $\mathcal{N}=(1,0)$ supersymmetric gauge theories in the harmonic superspace are described by the action

B. M. Zupnik, Sov. J. Nucl. Phys. **44** (1986) 512 [Yad. Fiz. **44** (1986) 794].

$$S=\frac{1}{f_0^2}\sum_{n=2}^{\infty}\frac{(-i)^n}{n}\mbox{tr}\int d^{14}z\,du_1\ldots du_n\,\frac{V^{++}(z,u_1)\ldots V^{++}(z,u_n)}{(u_1^+u_2^+)\ldots(u_n^+u_1^+)}$$
$$-\int d\zeta^{(-4)}du\,\widetilde{q}^+\nabla^{++}q^+,$$

where $\nabla^{++}q^+=D^{++}q^++iV^{++}Q^+$; V^{++} is an analytic gauge superfield, and q^+ is an analytic hypermultiplet superfield, and we use the notation

$$\int d^{14}z=\int d^6x\,d^8\theta;\qquad\int d\zeta^{(-4)}\equiv\int d^6x\,d^4\theta^+.$$

The higher covariant regularization for $6D$, $\mathcal{N}=(1,0)$ theories

The coupling constant f_0 in six dimensions has the dimension m^{-1}. This implies that the considered theory is not renormalizable, because the degree of divergence increases as a number of loops L increases,

I.L.Buchbinder, E.A.Ivanov, B.S.Merzlikin, K.S., Nucl. Phys. B **921** (2017), 127.

$$\omega_0 = 2L - N_q - N_c - \frac{1}{2}N_D,$$

where N_q, N_c, and N_D are the numbers of external hypermultiplets, ghosts, and spinor derivatives acting on external lines, respectively.
Next, we introduce the gauge superfield strength $F^{++} \equiv (D^+)^4 V^{--}$, where

$$(D^+)^4 = -\frac{1}{24}\varepsilon^{abcd} D_a^+ D_b^+ D_c^+ D_d^+;$$
$$V^{--}(z,u) \equiv \sum_{n=1}^{\infty}(-i)^{n+1}\int du_1 \ldots du_n \, \frac{V^{++}(z,u_1)\ldots V^{++}(z,u_n)}{(u^+u_1^+)(u_1^+u_2^+)\ldots(u_n^+u^+)}.$$

Using this superfield we construct the higher derivative term

$$S_\Lambda = \frac{1}{\Lambda^{2n}}\int d\zeta^{(-4)} du \, F^{++}\widehat{\Box}^{\,n} F^{++},$$

where the operator $\widehat{\Box} \equiv \frac{1}{2}(D^+)^4(\nabla^{--})^2$ is reduced to the Laplace operator if it acts on the subspace of analytic superfields.

The background-quantum splitting is linear, $V^{++} = \boldsymbol{V}^{++} + v^{++}$, and the gauge fixing term can be chosen in the form invariant under the background gauge transformations,

$$S_{\text{gf}} = -\frac{1}{2f_0^2\xi_0}\text{tr}\int d^{14}z\, du_1 du_2 \frac{(u_1^- u_2^-)}{(u_1^+ u_2^+)^3} D_2^{++} v_{2,\tau}^{++}\Big[1+\Big(\frac{\widehat{\Box}_{\tau,1}}{\Lambda^2}\Big)^{n+1}\Big] D_1^{++} v_{1,\tau}^{++}.$$

The corresponding Faddeev–Popov ghost action is given by the expression

$$S_{\text{FP}} = \text{tr}\int d\zeta^{(-4)}\, du\, b\boldsymbol{\nabla}^{++}\Big(\boldsymbol{\nabla}^{++}c + i[v^{++},c]\Big),$$

where the covariant derivatives are defined as $\boldsymbol{\nabla}^{\pm\pm} \equiv D^{\pm\pm} + i\boldsymbol{V}^{\pm\pm}$.
Also it is necessary to introduce the Nielsen–Kallosh determinant

$$\Delta_{NK} \equiv \text{Det}^{1/2}\Big[\,\widehat{\Box}\,\Big(1+\Big(\frac{\widehat{\Box}}{\Lambda^2}\Big)^{n+1}\Big)\Big]\int D\varphi \exp\big(iS_{\text{NK}}\big).$$

where

$$S_{\text{NK}} = -\frac{1}{2}\text{tr}\int d\zeta^{(-4)}\, du\,(\boldsymbol{\nabla}^{++}\varphi)^2.$$

The degree of divergence for the theory with the action $S+S_\Lambda$ is

$$\omega = 2 - 2n(L-1) - (n+2)(N_c+N_q) - \frac{1}{2}N_D.$$

Therefore, for $n \geq 1$ divergences are present only in the one-loop approximation. They are removed by inserting the Pauli–Villars determinant

$$\mbox{Det}^{-1}(PV,M) \equiv \int D\widetilde{Q}^{++} DQ^{++} \exp(iS_Q),$$

where S_Q can be defined as

$$S_Q = \frac{1}{\Lambda^2}\Big\{ \int d\zeta^{(-4)}\, du\, \widetilde{Q}^{++}\big(\,\widehat{\Box} + M^2\big)^2 Q^{++} - \int d^{14}z\, du_1 du_2\, \frac{i}{(u_1^+ u_2^+)^2} \widetilde{Q}^{++}_{1,\tau}\big(F^{++}\nabla^{--}Q^{++}\big)_{2,\tau}\Big\},$$

Then the generating functional is given by the expression

$$Z = \int Dv^{++}\, D\widetilde{q}^{+}\, Dq^{+}\, Db\, Dc\, D\varphi\, \mbox{Det}^{1/2}\Big[\,\widehat{\Box}\,\Big(1+\Big(\frac{\widehat{\Box}}{\Lambda^2}\Big)^{n+1}\Big)\Big] \times \mbox{Det}(PV,M)^x \exp\Big[i(S+S_\Lambda+S_{\mbox{gf}}+S_{\mbox{FP}}+S_{\mbox{NK}}+S_{\mbox{sources}})\Big],$$

where x should be chosen so that all one-loop divergences cancel each other.

Conclusion

- The higher covariant derivative regularization allows revealing some interesting features of supersymmetric theories and deriving some all-loop results.
- The integrals giving the β-function(s) of supersymmetric theories are integrals of double total derivatives with this regularization.
- RGFs defined in terms of the bare couplings satisfy the NSVZ relation in theories regularized by higher derivatives in all loops.
- Some all-order NSVZ schemes are given by the HD+MSL prescription.
- Validity of the NSVZ equation with the higher covariant derivative regularization allows to essentially simplify some multiloop calculations.
- The higher covariant derivative regularization can be constructed in the harmonic superspace, in particular for $4D$, $\mathcal{N}=2$ and $6D$, $\mathcal{N}=(1,0)$ theories.
- NSVZ equation and the higher covariant derivative regularization in the harmonic superspace allows to give a simple proof of the non-renormalization theorems for $4D$ theories with extended supersymmetry.

Thank you for the attention!