# Presymplectic structures and intrinsic Lagrangians for massive fields


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10 august 2022

Based on the article of the same name by Maxim Grigoriev and Vyacheslav Gritzaenko

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# Introduction

In the context of modern field theory one often encounters a problem of constructing a variational principle for a given system of partial differential equations (inverse problem of variation calculus).

| PDEs can be defined in an invariant way by specifying the equation manifold $\mathcal{M}$ equipped with the involutive Cartan distribution $D$ [1] | Lagrangian formulation depends on explicit set of variables |
|---|---|

The invariant approach to Lagrangian formulation is based on a distinguished realization of a given PDE as a surface in the jet-bundle of the equation manifold itself and geometric structure determined on it. A natural candidate for a role of this geometric structure is the presymplectic structure. [2]

[1] A. Vinogradov, 1977; M. Vasiliev, 1988
[2] J. Kijowski and W.M. Tulczyjew, 1979

# Jet-bundle

Total derivative on $J^\infty(F \times X)$:

$$D_a = \frac{\partial}{\partial x^a} + \phi^i_a \frac{\partial}{\partial \phi^i} + \phi^i_{ab} \frac{\partial}{\partial \phi^i_a} + \dots \tag{1}$$

The decomposition of the tangent space into the direct sum of the vertical and horizontal subspaces induces an additional degree on the algebra $\bigwedge(J^\infty)$ of local differential forms on $J^\infty$ and hence induces the decomposition of the de Rham differential

$$d = d_h + d_v\,, \quad d_h^2 = d_v^2 = 0\,, \quad d_h d_v + d_v d_h = 0 \tag{2}$$

$$d_h = dx^a D_a \tag{3}$$

# Presymplectic potential

The first variational formula can be written in a following way [3]:

$$d_v\mathcal{L} = d_v\phi^i E_i - d_h\hat{\chi}\,, \tag{4}$$

where $\mathcal{L}$ is $(n, 0)$ form, $\hat{\chi}$ is $(n-1, 1)$ form and $E_i$ is Euler-Lagrange operator.

$$E_i = \frac{\partial \mathcal{L}}{\partial \phi^i} - D_a \frac{\partial \mathcal{L}}{\partial \phi^i_a} + D_a D_b \frac{\partial \mathcal{L}}{\partial \phi^i_{ab}} - \ldots \tag{5}$$

It can be shown that

$$\hat{\chi} = ((\frac{\partial \mathcal{L}}{\partial \phi_a} - D_b \frac{\partial \mathcal{L}}{\partial \phi_{ab}})d_v\phi + \frac{\partial \mathcal{L}}{\partial \phi_{ab}} d_v\phi_b)(dx)^{n-1}_a \tag{6}$$

[3] I. M. Anderson, 1989

# Presymplectic structure

$\omega$ is $(n-1,2)$ form:

$$\hat{\omega} = d_v \hat{\chi} = d(\hat{\chi} + \mathcal{L}) - d_v \phi^i E_i \tag{7}$$

Pulling this back on equation manifold $\mathcal{M}$ one can find

$$\omega = d(\chi + l) \tag{8}$$

$l$ can be obtained from $\chi$:

$$d_h \chi = -d_v l \tag{9}$$

# Intrinsic action

If $\sigma : X \to \mathcal{M}$ is a section then one defined the following action functional [4]:

$$S^C[\sigma] = \int \sigma^*(\chi + l) \tag{10}$$

In coordinates:

$$S^C[\psi] = \int (d\psi^A(x)\chi_A(\psi(x), x, dx) - \mathcal{H}(\psi(x), x, dx)), \tag{11}$$

$$\chi = d_v\psi^A\chi_A\,,\ \mathcal{H} = d_h\psi^A\chi_A - l \tag{12}$$

Despite the fact that $\mathcal{M}$ is generically infinite-dimensional the intrinsic action depends only on a finite number of coordinates because $\chi$ is local. It is natural to gauge fix those fields on which the action does not depend.

[4] M. Grigoriev, 2016

# Example: metric gravity

Action for Einstein-Hilbert gravity:

$$S = \int d^n x \sqrt{-g} (R(g_{\mu\nu}) - 2\Lambda) \tag{13}$$

$$\hat{\chi} = \sqrt{-g} (\Gamma^{\rho\mu\nu} - \frac{1}{2} g^{\rho\mu} \Gamma_{\lambda}{}^{\lambda\nu} - \frac{1}{2} g^{\rho\nu} \Gamma_{\lambda}{}^{\lambda\mu} + \frac{1}{2} g^{\mu\nu} \Gamma_{\lambda}{}^{\lambda\rho} - \frac{1}{2} g^{\mu\nu} \Gamma^{\rho\lambda}{}_{\lambda}) d_v g_{\mu\nu} (dx)^n_\rho . \tag{14}$$

$x^\mu$, $g_{\mu\nu}$, $\Gamma^\lambda{}_{\mu\nu}$, ... can be chosen as coordinates on $\mathcal{M}$. The intrinsic action takes the following form:

$$S^C[g_{\mu\nu}, \Gamma^\lambda{}_{\mu\nu}] = \int d^n x \sqrt{-g} (R(g_{\mu\nu}, \Gamma^\lambda{}_{\mu\nu}) - 2\Lambda) \tag{15}$$

# Example: massive spin 1 field

$$\Box A^{\nu} - m^2 A^{\nu} = 0\,, \quad \partial_{\mu} A^{\mu} = 0\,. \tag{16}$$

This system has 5 equations for 4 variables. Still, action exists:

$$S[A^{\nu}] = \int d^n x(-\frac{1}{4} F^{\mu\nu} F_{\mu\nu} + \frac{1}{2} m^2 A^{\nu} A_{\nu}) \tag{17}$$

$$\partial_{\mu} F^{\mu\nu} - m^2 A^{\nu} = 0 \tag{18}$$

Making all same steps we obtain:

$$S^C[A^{\nu}, F^{\mu\nu}] = \int d^n x(-\frac{1}{2}(\partial^{\mu} A^{\nu} - \partial^{\nu} A^{\mu}) F_{\mu\nu} + \frac{1}{4} F^{\mu\nu} F_{\mu\nu} - \frac{1}{2} m^2 A^{\nu} A_{\nu})\,. \tag{19}$$

# Massive spin-2 field

$$(\Box - m^2)\phi_{\mu\nu} = 0\,, \quad \partial^\mu \phi_{\mu\nu} = 0\,, \quad \phi^\mu_\mu = 0 \tag{20}$$

The number of fields does not match the number of equations. Action for this system:

$$S = \int d^n x(-\frac{1}{2}\partial_\lambda \phi^{\mu\nu}\partial^\lambda \phi_{\mu\nu} + \partial^\mu \phi_{\mu\nu}\partial_\lambda \phi^{\lambda\nu} + \frac{1}{2}\partial_\mu \phi^\nu_\nu \partial^\mu \phi^\lambda_\lambda - \partial^\lambda \phi_{\lambda\mu}\partial^\mu \phi^\nu_\nu - \\ - \frac{1}{2}m^2(\phi^{\mu\nu}\phi_{\mu\nu} - \phi^\mu_\mu \phi^\nu_\nu)) \tag{21}$$

Here, $\phi_{\mu\nu}$ is traceful. EoM for $\phi_{\mu\nu}$:

$$(\Box - m^2)\phi_{\mu\nu} - (\Box - m^2)\phi^\tau_\tau \eta_{\mu\nu} - \\ - \partial_\mu \partial^\lambda \phi_{\lambda\nu} - \partial_\nu \partial^\lambda \phi_{\lambda\mu} + \partial_\mu \partial_\nu \phi^\lambda_\lambda + \partial_\tau \partial_\rho \phi^{\tau\rho}\eta_{\mu\nu} = 0 \tag{22}$$

# Massive spin-2 field

$x^\mu$, $\varphi_{\mu\nu}$, $\varphi_{\mu\nu|\lambda}$, $\varphi_{\mu\nu|\lambda\rho}$, ... can be chosen as coordinate system on $\mathcal{M}$. Here $\varphi_{\mu\nu}$ and $\varphi_{\mu\nu|\lambda}$ are traceless parts of $\phi_{\mu\nu}$ and $\phi_{\mu\nu|\lambda}$ respectively.

$$\hat{\chi} = d_v\phi^{\mu\nu}(-\phi_{\mu\nu}{}^{|\lambda} + 2\phi_{\rho\nu}{}^{|\rho}\eta^\lambda_\mu + \eta_{\mu\nu}\phi^{\rho|\lambda}_\rho - \frac{1}{2}\phi^\rho_{\rho|\nu}\eta^\lambda_\mu - \frac{1}{2}\phi^\rho_{\rho|\mu}\eta^\lambda_\nu - \eta_{\mu\nu}\phi^{\rho\lambda}{}_{|\rho})(dx)^{n-1}_\lambda \quad (23)$$

Pullback to $\mathcal{M}$ gives:

$$\chi = -d_v\varphi^{\mu\nu}\varphi_{\mu\nu}{}^{|\lambda}(dx)^{n-1}_\lambda \quad (24)$$

# Massive spin-2 field

$$\mathcal{H} = (-\frac{1}{2}\varphi^{\mu\nu}{}_{|\lambda}\varphi_{\mu\nu}{}^{|\lambda} + \frac{1}{2}m^2\varphi^{\mu\nu}\varphi_{\mu\nu})(dx)^n \tag{25}$$

$$S^C[\varphi_{\mu\nu}, \varphi_{\mu\nu|\lambda}] = \int d^n x(-\partial_\lambda\varphi^{\mu\nu}\varphi_{\mu\nu}{}^{|\lambda} + \frac{1}{2}\varphi^{\mu\nu}{}_{|\lambda}\varphi_{\mu\nu}{}^{|\lambda} - \frac{1}{2}m^2\varphi^{\mu\nu}\varphi_{\mu\nu}) \tag{26}$$

This action does not reproduce equation $\partial^\nu\phi_{\mu\nu} = 0$.

# Minimal multisymplectic formulation

Parent action for system with Lagrangian $\mathcal{L} = \mathcal{L}(\phi^i, \phi^i_{|\alpha}, \phi^i_{|\alpha\beta})$ has the following form:

$$S^P[\phi, \phi_\alpha, \phi_{\alpha\beta}, \pi_i^{|\alpha}, \pi_i^{|\alpha\beta}] = \int d^n x(\mathcal{L} - \pi_i^{|\alpha}(\partial_\alpha \phi^i - \phi^i_{|\alpha}) - \pi_i^{|\alpha\beta}(\partial_\alpha \phi^i_{|\beta} - \phi^i_{|\alpha\beta})) \quad (27)$$

Massive spin-2 field case:

$$S^P = \int d^n x(-\frac{1}{2}\phi_{\alpha\beta|\gamma}\phi^{\alpha\beta|\gamma} + \phi_{\alpha\beta}{}^{|\alpha}\phi^{\gamma\beta}{}_{|\gamma} + \frac{1}{2}\phi_\alpha^{\alpha|\gamma}\phi^\beta_{\beta|\gamma} - \phi^{\gamma\beta}{}_{|\gamma}\phi^\alpha_{\alpha|\beta} - \\ - \frac{1}{2}m^2(\phi_{\alpha\beta}\phi^{\alpha\beta} - \phi^\alpha_\alpha\phi^\beta_\beta) + \pi_{\alpha\beta|\gamma}(\partial^\gamma\phi^{\alpha\beta} - \phi^{\alpha\beta|\gamma}))\,. \quad (28)$$

# Minimal multisymplectic formulation

Varying this action with respect to ${\phi^{\mu\nu}}_{|\lambda}$ one can obtain

$$
{\pi_{\mu\nu}}^{|\lambda} = -{\phi_{\nu\mu}}^{|\lambda} + {\phi_{\gamma\mu}}^{|\gamma}\eta^{\lambda}_{\nu} + {\phi_{\gamma\nu}}^{|\gamma}\eta^{\lambda}_{\mu} + \phi^{\gamma|\lambda}_{\gamma}\eta_{\mu\nu} - {\phi^{\gamma\lambda}}_{|\gamma}\eta_{\mu\nu} - \frac{1}{2}\phi^{\gamma}_{\gamma|\mu}\eta^{\lambda}_{\nu} - \frac{1}{2}\phi^{\gamma}_{\gamma|\nu}\eta^{\lambda}_{\mu}\,. \quad (29)
$$

This equation can be solved with respect to ${\phi^{\mu\nu}}_{|\lambda}$. Substitution (29) into (28) gives

$$
\begin{aligned}
S = \int d^n x(&-\phi^{\mu\nu|\lambda}\partial_\lambda\phi_{\mu\nu} + 2{\phi^{\lambda\mu}}_{|\lambda}\partial^\nu\phi_{\nu\mu} + \phi^{\mu|\lambda}_{\mu}\partial_\lambda\phi^{\nu}_{\nu} - {\phi^{\lambda\mu}}_{|\lambda}\partial_\mu\phi^{\nu}_{\nu} - \phi^{\mu|\lambda}_{\mu}\partial^\nu\phi_{\nu\lambda} + \\
&+ \frac{1}{2}\phi_{\alpha\beta|\gamma}\phi^{\alpha\beta|\gamma} - {\phi_{\alpha\beta}}^{|\alpha}{\phi^{\gamma\beta}}_{|\gamma} - \frac{1}{2}\phi^{\alpha|\gamma}_{\alpha}\phi^{\beta}_{\beta|\gamma} + {\phi^{\gamma\beta}}_{|\gamma}\phi^{\alpha}_{\alpha|\beta} - \frac{1}{2}m^2(\phi_{\alpha\beta}\phi^{\alpha\beta} - \phi^{\alpha}_{\alpha}\phi^{\beta}_{\beta})) \quad (30)
\end{aligned}
$$

# Minimal multisymplectic formulation

We can decompose $\phi^{\alpha\beta}{}_{|\gamma}$ into $\varphi^{\alpha\beta}{}_{|\gamma}$, $\phi^{\lambda\mu}{}_{|\lambda} = \psi^{\mu}$, $\phi^{\lambda}_{\lambda|\mu} = \xi_{\mu}$ and $\phi^{\alpha\beta}$ into $\varphi^{\alpha\beta}$ and $\phi^{\lambda}_{\lambda} = \rho$.

$$S = \int d^n x\Big(-\varphi^{\alpha\beta|\gamma}\partial_\gamma\varphi_{\alpha\beta} + \frac{1}{2}\varphi^{\alpha\beta|\gamma}\varphi_{\alpha\beta|\gamma} - \frac{1}{2}m^2\varphi_{\alpha\beta}\varphi^{\alpha\beta} +$$
$$+ \frac{2(n^2-2)}{n^2+n-2}\psi^{\beta}\partial^{\gamma}\varphi_{\gamma\beta} - \frac{n^2+n-4}{n^2+n-2}\xi^{\beta}\partial^{\gamma}\varphi_{\gamma\beta} + \frac{n-2}{n}\xi_{\gamma}\partial^{\gamma}\rho - \frac{n-2}{n}\psi_{\gamma}\partial^{\gamma}\rho -$$
$$- \frac{n^2-2}{n^2+n-2}\psi^{\alpha}\psi_{\alpha} - \frac{n^2-3}{2(n^2+n-2)}\xi^{\alpha}\xi_{\alpha} + \frac{n^2+n-4}{n^2+n-2}\psi^{\alpha}\xi_{\alpha} + \frac{1}{2}m^2\frac{n-1}{n}\rho^2\Big) \quad (31)$$

# Stueckelberg formulation

There is a way to construct complete intrinsic action for massive spin-2 field. If the starting point is Stueckelberg spin-2 system the intrinsic action reads as

$$
\begin{aligned}
S[\phi_{\mu\nu}, {\phi_{\mu\nu}}^{|\lambda}, A_\mu, A_{\mu|\nu}, C, C_\nu, \ldots] = \int d^n x (-{\phi_{\mu\nu}}^{|\lambda}\partial_\lambda \phi^{\mu\nu} + 2{\phi_{\lambda\mu}}^{|\lambda}\partial_\nu \phi^{\nu\mu} + \phi^{\mu|\lambda}_{\mu}\partial_\lambda \phi^{\nu}_{\nu} - \\
-\phi^{\lambda}_{\lambda|\nu}\partial_\mu \phi^{\mu\nu} - {\phi^{\mu\lambda}}_{|\mu}\partial_\lambda \phi^{\nu}_{\nu} + C_\mu \partial_\nu \phi^{\nu\mu} - C^\mu \partial_\mu \phi^\nu_\nu + 2A^{\mu|\lambda}\partial_\lambda A_\mu - 2A^{|\nu}_{\nu}\partial_\mu A^\mu + 2m\phi^{\mu\nu}\partial_\mu A_\nu - \\
-2m\phi^\nu_\nu \partial_\mu A^\mu + {\phi^{\mu\nu}}_{|\mu}\partial_\nu C - \phi^{\mu|\nu}_{\mu}\partial_\nu C + \frac{1}{2}{\phi^{\mu\nu}}_{|\lambda}{\phi_{\mu\nu}}^{|\lambda} - {\phi_{\lambda\nu}}^{|\lambda}{\phi^{\mu\nu}}_{|\mu} - \frac{1}{2}\phi^{\mu}_{\mu|\lambda}\phi^{\nu|\lambda}_{\nu} + {\phi_{\lambda\nu}}^{|\lambda}\phi^{\mu|\nu}_{\mu} - \\
- \frac{1}{2}m^2(\phi_{\mu\nu}\phi^{\mu\nu} - \phi^\mu_\mu \phi^\nu_\nu) + A_{\mu|\nu}A^{\mu|\nu} - A^{|\mu}_{\mu}A^{|\nu}_{\nu} - C^\nu {\phi_{\lambda\nu}}^{|\lambda} + C^\nu \phi^{\lambda}_{\lambda|\nu}) \quad (32)
\end{aligned}
$$

Here $C$ and $A_\mu$ are Stueckelberg fields.

# Stueckelberg formulation

By gauge-fixing $C$ and $A_\mu$ and eliminating $A_{\mu|\nu}$ one can obtain:

$$S^C[\phi_{\mu\nu}, \phi_{\mu\nu}{}^{|\lambda}, C_\nu] = \int d^n x(-\phi_{\mu\nu}{}^{|\lambda}\partial_\lambda\phi^{\mu\nu} + 2\phi_{\lambda\mu}{}^{|\lambda}\partial_\nu\phi^{\nu\mu} + \phi_\mu^{\mu|\lambda}\partial_\lambda\phi_\nu^\nu - \phi_{\lambda|\nu}^\lambda\partial_\mu\phi^{\mu\nu} -$$
$$-\phi^{\mu\lambda}{}_{|\mu}\partial_\lambda\phi_\nu^\nu + C_\mu\partial_\nu\phi^{\nu\mu} - C^\mu\partial_\mu\phi_\nu^\nu + \frac{1}{2}\phi^{\mu\nu}{}_{|\lambda}\phi_{\mu\nu}{}^{|\lambda} - \phi_{\lambda\nu}{}^{|\lambda}\phi^{\mu\nu}{}_{|\mu} - \frac{1}{2}\phi^\mu_{\mu|\lambda}\phi_\nu^{\nu|\lambda} + \phi_{\lambda\nu}{}^{|\lambda}\phi_\mu^{\mu|\nu} -$$
$$-\frac{1}{2}m^2(\phi_{\mu\nu}\phi^{\mu\nu} - \phi_\mu^\mu\phi_\nu^\nu) - C^\nu\phi_{\lambda\nu}{}^{|\lambda} + C^\nu\phi_{\lambda|\nu}^\lambda) \quad (33)$$

This action is equivalent to Fierz-Pauli one (21).

# Conclusion

- For natural systems (Einstein-Hilbert gravity, massive spin-1 field. Massless field of arbitrary spin and other examples are described in article) intrinsic action reproduces all equation of motion.
- For systems with differential consequences of zeroth order (massive spin-2 field, massive spin 3 is described in article) intrinsic action is not complete.
- For massive spin-2 field (and massive spin-3 field) minimal extenson of intrinsic action is constructed as an equivalent reduction of the multidimensional version of the Ostrogradsky Lagrangian (parent action).
- If the starting point is Stueckelberg spin-2 system the intrinsic action is complete.

# Thank you for attention.

# R

$$R = g^{\mu\nu}(\partial_\lambda \Gamma^\lambda{}_{\mu\nu} - \frac{1}{2}\partial_\mu \Gamma^\lambda{}_{\nu\lambda} - \frac{1}{2}\partial_\nu \Gamma^\lambda{}_{\mu\lambda} + \Gamma^\gamma{}_{\mu\nu}\Gamma^\lambda{}_{\gamma\lambda} - \Gamma^\gamma{}_{\mu\lambda}\Gamma^\lambda{}_{\nu\gamma}) \tag{34}$$

$$\Gamma^\gamma_{\alpha\beta} = g^{\gamma\lambda}(\partial_\alpha g_{\lambda\beta} + \partial_\beta g_{\lambda\alpha} - \partial_\lambda g_{\alpha\beta})\,, \tag{35}$$

# EoM of massive spin-2 field from intrinsic action

$$\varphi^{\alpha\beta}{}_{|\gamma} = \partial_\gamma \varphi^{\alpha\beta} - \frac{n}{n^2+n-2}(\partial_\lambda \varphi^{\lambda\beta}\eta^\alpha_\gamma + \partial_\lambda \varphi^{\lambda\alpha}\eta^\beta_\gamma) + \frac{2}{n^2+n-2}\partial^\lambda \varphi_{\lambda\gamma}\eta^{\alpha\beta} \tag{36}$$

$$\partial^\gamma \varphi^{\alpha\beta}{}_{|\gamma} - m^2 \varphi^{\alpha\beta} = 0\,. \tag{37}$$

# Massless field of arbitrary spin

$$S^C[\phi^{\mu(s)}, \phi^{\mu(s)}{}_{|\rho}] = \int d^n x(-\partial^\rho \phi^{\mu(s)} \phi_{\mu(s)|\rho} + s\partial_\nu \phi^{\nu\mu(s-1)} \phi^{\lambda}_{\mu(s-1)|\lambda} + \\ + \frac{s(s-1)}{2} \partial_\rho \phi^{\nu}_{\nu\mu(s-2)} \phi^{\lambda\mu(s-2)|\rho}_{\lambda} - \frac{s(s-1)}{2} \partial^\lambda \phi_{\rho\lambda\mu(s-2)} \phi^{\nu\mu(s-2)|\rho}_{\nu} - \\ - \frac{s(s-1)}{2} \partial_\rho \phi^{\nu}_{\nu\mu(s-2)} \phi^{\rho\mu(s-2)|\lambda}_{\lambda} + \frac{s(s-1)(s-2)}{4} \partial^\rho \phi^{\nu}_{\nu\rho\mu(s-3)} \phi^{\tau\mu(s-3)|\lambda}_{\tau\lambda} - \mathcal{H}), \quad (38)$$

where

$$\mathcal{H} = (-\frac{1}{2} \phi^{\mu(s)|\rho} \phi_{\mu(s)|\rho} + \frac{s}{2} \phi^{\mu(s-1)|\nu}_{\nu} \phi^{\lambda}_{\mu(s-1)|\lambda} + \frac{s(s-1)}{4} \phi^{\nu\mu(s-2)|\rho}_{\nu} \phi^{\lambda}_{\lambda\mu(s-2)|\rho} - \\ - \frac{s(s-1)}{2} \phi^{\nu\mu(s-2)|\rho}_{\nu} \phi^{\lambda}_{\rho\mu(s-2)|\lambda} + \frac{s(s-1)(s-2)}{8} \phi^{\nu\rho}_{\nu\mu(s-3)|\rho} \phi^{\tau\mu(s-3)|\lambda}_{\tau\lambda})(dx)^n \quad (39)$$

# Massive spin-3 field

$$\mathcal{L}[\phi,\rho] = (-\frac{1}{2}\partial_\gamma\phi^{\mu\nu\lambda}\partial^\gamma\phi_{\mu\nu\lambda} + \frac{3}{2}\partial_\lambda\phi^{\mu\nu\lambda}\partial^\gamma\phi_{\mu\nu\gamma} + \frac{3}{4}\partial_\gamma\phi^{\mu\gamma}_{\mu}\partial^\lambda\phi^{\nu}_{\nu\lambda} + \frac{3}{2}\partial_\nu\phi^{\gamma}_{\gamma\mu}\partial^\nu\phi^{\lambda\mu}_{\lambda} -$$
$$- 3\partial^\mu\phi^{\lambda\gamma}_{\lambda}\partial^\nu\phi_{\mu\nu\gamma} - \frac{1}{2}m^2\phi_{\mu\nu\gamma}\phi^{\mu\nu\gamma} + \frac{3}{2}m^2\phi^{\nu}_{\nu\mu}\phi^{\lambda\mu}_{\lambda} + \frac{9}{4}m^2\rho^2 +$$
$$+ \frac{3(n-1)(n-2)}{2n^2}\partial_\mu\rho\partial^\mu\rho - \frac{3(n-2)}{2n}m\rho\partial_\mu\phi^{\nu\mu}_{\nu})(dx)^n \quad (40)$$

$$S^C = \int d^n x(-\partial_\gamma\varphi^{\mu\nu\lambda}\varphi_{\mu\nu\lambda}{}^{|\gamma} + \frac{1}{2}\varphi^{\mu\nu\lambda}{}_{|\gamma}\varphi_{\mu\nu\lambda}{}^{|\gamma} - \frac{1}{2}m^2\varphi^{\mu\nu\lambda}\varphi_{\mu\nu\lambda}) \quad (41)$$

# Massive spin-3 field

$$S = \int d^n x \Big( -\phi^{\mu\nu\lambda}{}_{|\gamma}\partial^\gamma \phi_{\mu\nu\lambda} + 3\phi^{\lambda\mu\nu}{}_{|\lambda}\partial^\gamma \phi_{\mu\nu\gamma} + \frac{3}{2}\phi_\mu^{\mu\gamma}{}_{|\gamma}\partial_\lambda \phi_\nu^{\nu\lambda} + 3\phi^\gamma_{\gamma\mu|\nu}\partial^\nu \phi_\lambda^{\lambda\mu} -$$
$$- 3\phi_\lambda^{\lambda\gamma|\mu}\partial^\nu \phi_{\mu\nu\gamma} - 3\phi^{\mu\nu\gamma}{}_{|\gamma}\partial_\mu \phi^\lambda_{\lambda\nu} + \frac{3(n-1)(n-2)}{n^2}\rho^{|\mu}\partial_\mu \rho + \frac{3(n-2)}{2n} m\phi^\nu_{\nu\mu}\partial^\mu \rho +$$
$$+ \frac{1}{2}\phi_{\mu\nu\lambda}{}^{|\gamma}\phi^{\mu\nu\lambda}{}_{|\gamma} - \frac{3}{2}\phi^{\lambda\mu\nu}{}_{|\lambda}\phi_{\gamma\mu\nu}{}^{|\gamma} - \frac{3}{4}\phi_\mu^{\mu\gamma}{}_{|\gamma}\phi^\nu_{\nu\lambda}{}^{|\lambda} - \frac{3}{2}\phi^\gamma_{\gamma\mu|\nu}\phi_\lambda^{\lambda\mu|\nu} + 3\phi_\lambda^{\lambda\gamma|\mu}\phi_{\gamma\mu\nu}{}^{|\nu} -$$
$$- \frac{1}{2}m^2\phi_{\mu\nu\lambda}\phi^{\mu\nu\lambda} + \frac{3}{2}m^2\phi^\nu_{\nu\mu}\phi_\lambda^{\lambda\mu} + \frac{9}{4}m^2\rho^2 - \frac{3(n-1)(n-2)}{2n^2}\rho^\mu\rho_\mu \Big) . \qquad (42)$$

# Stueckelberg action

$$\mathcal{L} = (-\frac{1}{2}\partial_\gamma\phi^{\alpha\beta}\partial^\gamma\phi_{\alpha\beta} + \partial^\alpha\phi_{\alpha\beta}\partial_\gamma\phi^{\gamma\alpha} + \frac{1}{2}\partial_\alpha\phi^\beta_\beta\partial^\alpha\phi^\gamma_\gamma - \partial^\gamma\phi_{\gamma\alpha}\partial^\alpha\phi^\beta_\beta -$$
$$-\frac{1}{2}m^2(\phi^{\alpha\beta}\phi_{\alpha\beta} - \phi^\alpha_\alpha\phi^\beta_\beta) - \partial_\alpha A_\beta\partial^\alpha A^\beta + \partial_\alpha A_\alpha\partial^\beta A^\beta + 2m\phi^{\beta\alpha}\partial_\beta A_\alpha - 2m\phi^\alpha_\alpha\partial_\beta A^\beta +$$
$$+ \partial_\beta\phi^{\beta\alpha}\partial_\alpha C - \partial_\beta C\partial^\beta\phi^\alpha_\alpha)(dx)^n \quad (43)$$

Gauge transformation:

$$\delta\phi_{\alpha\beta} = \partial_\alpha f_\beta + \partial_\beta f_\alpha \quad (44)$$

$$\delta A_\alpha = \partial_\alpha g + mf_\alpha \quad (45)$$

$$\delta C = -2mg \quad (46)$$

# Stueckelberg equations

$$\partial_\alpha \partial_\beta \phi^{\alpha\beta} - \partial_\beta \partial^\beta \phi^\alpha_\alpha = 0 \tag{47}$$

$$\partial_\beta \partial^\beta A^\alpha - \partial^\alpha \partial^\beta A_\beta - m\partial_\beta \phi^{\beta\alpha} + m\partial^\alpha \phi^\beta_\beta = 0 \tag{48}$$

$$\begin{aligned}(\partial_\gamma \partial^\gamma - m^2)\phi_{\alpha\beta} - (\partial_\gamma \partial^\gamma - m^2)\phi^\tau_\tau \eta_{\alpha\beta} - \partial_\alpha \partial^\gamma \phi_{\gamma\beta} - \partial_\beta \partial^\gamma \phi_{\gamma\alpha} + \partial_\alpha \partial_\beta \phi^\gamma_\gamma + \\ + \partial_\tau \partial_\rho \phi^{\tau\rho} \eta_{\alpha\beta} + m(\partial_\alpha A_\beta + \partial_\beta A_\alpha) - 2m\partial_\gamma A^\gamma \eta_{\alpha\beta} - \partial_\alpha \partial_\beta C + \partial_\gamma \partial^\gamma C \eta_{\alpha\beta} = 0\end{aligned} \tag{49}$$