Secret symmetry of instantonic Schwinger-Keldysh path integral

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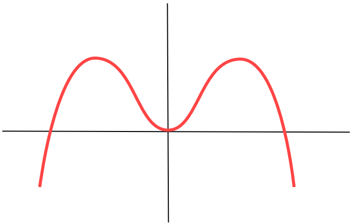
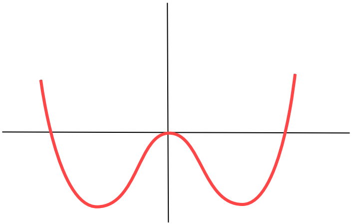
What is a problem?

We want to calculate real-time correlation functions in instantonic systems

$$\langle \phi(t_1) \dots \phi(t_n) \rangle = \frac{1}{Z} \operatorname{tr}\left[e^{-\beta H} \phi(t_1) \dots \phi(t_n)\right]$$





Plan of the talk

- Instantonic correlation functions in **imaginary time**
- Instantonic correlation functions in **real time**
- Further comments and applications

Instantons

Arise in different places in QM and QFT

1. Tunneling processes
2. False vacuum decay
3. Structure of vacuum (periodic potentials, Yang-Mills, etc)
   (Callan, Dashen, Gross; ...)
4. Initial states of Universe
   (Halliwell, Myers; Barvinsky, Kamenschik)

Instantonic correlation functions

Examples:

1. False vacuum decay in real time
   - standard way — imaginary part of partition function
   - (a bit) new way — S-matrix and poles of real-time correlation functions **(Ai, Garbrecht, Tamarit 2019)**
2. CMB for "thermal states"

Imaginary time correlation functions I

- Consider Euclidean quantum mechanics, described by the partition function

$$Z = \mathrm{Tr}\, e^{-\beta \hat{H}} = \int\limits_{\text{periodic}} \mathcal{D}x\ e^{-S[x]}, \qquad S[x] = \int_0^\beta d\tau\ \left[\frac{\dot{x}^2}{2} - V(x)\right]$$

- Zero mode issue! Expand the action about a saddle point x_c

$$\ddot{x}_c(\tau) + V'(x_c) = 0, \qquad x(0) = x(\beta),$$

and go to the integration over perturbation η, i.e. $x = x_c + \eta$

$$Z = e^{-S[x_c]} \int\limits_{\text{periodic}} \mathcal{D}\eta\ \exp\left\{-S^{(2)}[x_c, \eta] - S^{\text{int}}[x_c, \eta]\right\}$$

Imaginary time correlation functions II

where $S^{(2)}[x_c, \eta]$ is quadratic part of $S[x_c + \eta]$

$$S^{(2)}[x_c, \eta] = \frac{1}{2} \int d\tau \, d\tau' \, \eta(\tau) \, K(\tau, \tau') \, \eta(\tau'),$$
$$K(\tau, \tau') = [-\partial_\tau^2 - V''(x_c)] \, \delta(\tau - \tau').$$

Taking derivative of e.o.m.

$$[-\partial_\tau^2 - V''(x_c)] \dot{x}_c(\tau) = 0,$$

we observe that $\dot{x}_c$ — zero mode of K.

- Reason — translation invariance of the action in Euclidean time

$$S[x^{\tau_0}] = S[x], \qquad x^{\tau_0}(\tau) = x(\tau + \tau_0).$$

Integration over zero-mode is non-Gaussian!

Imaginary time correlation functions III

- Solution — gauge-fixing

$$1 = \frac{1}{\sqrt{2\pi\xi}} \int_0^\beta d\tau_0 \, \frac{d\chi[x^{\tau_0}]}{d\tau_0} \, \exp\Big\{-\frac{1}{2\xi}\chi[x^{\tau_0}]^2\Big\},$$

where

$$\chi[x] = \frac{1}{\|\dot{x}_c\|} \int_0^\beta d\tau \, \dot{x}_c(\tau) x(\tau)$$

- Partition function takes the form

$$Z = \frac{\beta}{\sqrt{2\pi\xi}} \int \mathcal{D}x \; J[x] \; e^{-S_\xi[x]},$$

$$S_\xi[x] = S[x] + \frac{1}{2\xi}\chi[x]^2, \qquad J[x] = \frac{d\chi[x^{\tau_0}]}{d\tau_0}\Big|_{\tau_0=0}$$

Imaginary time correlation functions IV

- Perturbative expansion for Z originates from

$$Z = \frac{\beta \|\dot{x}_c\|}{\sqrt{2\pi\xi}} e^{-S[x_c]} \int \mathcal{D}\eta \Big[1 + \frac{1}{\|\dot{x}_c\|} \int_0^\beta d\tau \, \eta_0(\tau) \dot{\eta}(\tau) \Big] e^{-S_\xi^{(2)}[x_c,\eta] - S^{\text{int}}[x_c,\eta$$

$$S_\xi^{(2)}[x_c, \eta] = \frac{1}{2} \int d\tau \, d\tau' \; \eta(\tau) \, K_\xi(\tau, \tau') \, \eta(\tau'),$$

$$K_\xi(\tau, \tau') = \left[-\partial_\tau^2 - V''(x_c) \right] \delta(\tau - \tau') + \frac{1}{\xi \|\dot{x}_c\|^2} \dot{x}_c(\tau) \dot{x}_c(\tau').$$

- Inserting $x(t)$ to path integral, we get a puzzle!

$$\langle x(t) \rangle = x_c(t) + \text{ corrections.}$$

Imaginary time correlation functions V

- Generalize the partition function Z to the generating functional

$$Z[j] = \mathrm{Tr}\Big[e^{-\beta \hat{H}} T_\tau \exp\Big(\int_0^\beta d\tau\, j(\tau)\hat{x}(\tau)\Big)\Big],$$

in terms of which the n-point correlation function reads

$$D(\tau_1, \dots \tau_n) = \mathrm{Tr}\Big[e^{-\beta \hat{H}} T_\tau \Big(\hat{x}(\tau_1) \dots \hat{x}(\tau_n)\Big)\Big] = \frac{1}{Z} \frac{\delta^n Z[j]}{\delta j(\tau_1) \dots \delta j(\tau_n)}\Big|_{j=}$$

Path integral representation of $Z[j]$ has the form

$$Z[j] = \mathrm{Tr}\, e^{-\beta \hat{H}} = \int_{\text{periodic}} \mathcal{D}x\; e^{-S[x] + \int_0^\beta d\tau\, j(\tau) x(\tau)}$$

Imaginary time correlation functions VI

- The calculation of $Z[j]$ repeats those of Z, except that the integration over τ_0, originated from the partition of unity

$$Z[j] = \frac{1}{\sqrt{2\pi\xi}\|\dot{x}_c\|} \int \mathcal{D}x \; J[x] \; e^{-S_\xi[x]} \times \int_0^\beta d\tau_0 \, e^{\int_0^\beta d\tau \, j(\tau) x(\tau - \tau_0)},$$

- Now, observables are gauge-invariant!

$$D(\tau_1, \ldots \tau_n) \propto \int \mathcal{D}x \left[\int_0^\beta d\tau_0 \, x(\tau_1 - \tau_0) \ldots x(\tau_n - \tau_0) \right] J[x] \; e^{-S_\xi[x]}$$

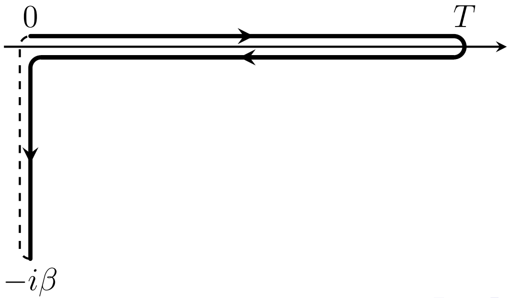
Some comments

- In general, obtained correlators cannot be analytically continued to real times **(Evans 1992; Baier, Niegawa 1994)**
- Imaginary time instantonic correlators first arise in QCD context **(Polyakov 1976; Callan, Dashen, Gross 1976)**
- Loop calculations a rarely known (for partition function only!) **(Lowe, Stone 1978; Bezoglov, Onischenko 2017; Shuryak, Turbiner 2018)**
- Recent interest in resurgence methods community **(Dunne, Unsal et al)**

Schwinger-Keldysh path integral I

- Schwinger-Keldysh correlation function generating functional

$$Z[j_+, j_-, j_e] = \mathrm{Tr}\Big[e^{-\beta \hat{H}} T_C \exp\Big(\int_0^\beta d\tau\, j_e(\tau)\hat{x}(\tau) + i\int_0^T d\tau\, j_+(\tau)\hat{x}(\tau) - i\int_0^T d\tau\, j_-(\tau)\hat{x}(\tau)\Big)\Big]$$



Schwinger-Keldysh path integral II

- Path integral form

$$Z[j_e, j_+, j_-] = \int\limits_{x_e(0)=x_e(\beta)} \mathcal{D}x_e \int\limits_{\substack{x_+(0)=x_e(-0)\\ x_-(0)=x_e(+0)\\ x_+(T)=x_-(T)}} \mathcal{D}x_+\,\mathcal{D}x_-$$

$$\exp\Big\{-S_e[x_e] + iS[x_+] - iS[x_-]\Big\}\times$$

$$\times\exp\Big\{\int_0^\beta d\tau\, j_e x_e + i\int_0^T dt\, j_+ x_+ + i\int_0^T dt\, j_- x_-\Big\}$$

- Real-time contour breaks the symmetry $\tau \mapsto \tau + \tau_0$! However, zero-mode is still present

$$\eta_0(z) \propto \partial_z x_c(z), \qquad z \in C$$

Schwinger-Keldysh path integral III

- We cannot perform a finite shift along imaginary axis! What is the corresponding symmetry?

Real-time correlation functions I

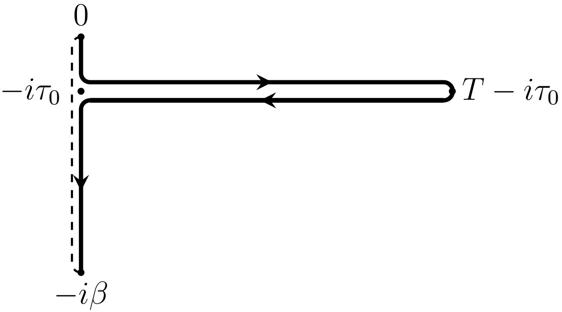
- Path integral is independent of gluing point!

$$Z_{\tau_0}[j_e, j_+, j_-] = \int\limits_{x_e(0)=x_e(\beta)} \mathcal{D}x_e \int\limits_{\substack{x_+(0)=x_e(\tau_0-0)\\ x_-(0)=x_e(\tau_0+0)\\ x_+(T)=x_-(T)}} \mathcal{D}x_+\,\mathcal{D}x_-$$

$$\exp\left\{-S_e[x_e] + iS[x_+] - iS[x_-]\right\}\times$$

$$\times\exp\left\{\int_0^\beta d\tau\, j_e x_e + i\int_0^T dt\, j_+ x_+ + \int_0^T dt\, j_- x_-\right\}$$

Real-time correlation functions II



- Independence of τ_0. Then just average over it!

$$Z = Z_{\tau_0} = \frac{1}{\beta} \int_0^{\beta} d\tau_0 \, Z_{\tau_0}$$

Real-time correlation functions III

- Treating the integration over τ_0 on the same footing as x_e, x_+, x_-, we observe that Z

$$x_e(\tau) \mapsto x_e(\tau+\tau_1), \quad x_\pm(t) \mapsto x_\pm(t), \quad \tau_0 \mapsto \tau_0 - \tau_1.$$

- Repeating a gauge-fixing procedure, one obtains

$$Z[j_e, j_+, j_-] = \frac{1}{\beta\sqrt{2\pi\xi}} \int_0^\beta d\tau_0 \int \mathcal{D}[x]_{\tau_0} \, J[x]$$
$$\exp\left\{-S_e[x_e] + iS[x_+] - iS[x_-] - \frac{1}{2\xi}(\chi)^2\right\}\times$$
$$\times \int_0^\beta d\tau_1 \exp\left\{\int_0^\beta d\tau \, j_e(\tau) x_e(\tau-\tau_1)\right.$$
$$\left. + i\int_0^T dt \, j_+ x_+ + i\int_0^T dt \, j_- x_-\right\}.$$

Discussion and TODO

Done

- Consistent perturbation theory for real-time correlators is constructed
- Dependence on difference of real-time points can be proven
- Complex backgrounds contributes to real-time correlation functions!

ToDo

- Examples: in QM and Cosmology.
- Schwinger-Keldysh superfield formalism


Thank you for your attention!