Example of the 4-pt non-vacuum $\mathcal{W}_3$ classical block

Pavlov Mikhail (LPI RAS)

August 10, Dubna, SQS-2022

Russian Science Foundation

Introduction and motivation

Based on work arXiv: 2205.06208 [hep-th]

Motivation:

1. Explicit results for non-vacuum 4-pt blocks associated with the $\mathcal{W}_3$ symmetry beyond known [Fateev, Litvinov, '07], [Fateev, Ribault, '11]
2. AdS/CFT in terms of Wilson lines

Plan

1. $\mathcal{W}_3$ algebra and conformal blocks
2. Classical and heavy-light (HL) approximations
3. Monodromy method: BPZ equation
4. Monodromy method: fusion rules
5. Accessory parameter, functional arbitrariness fixing and the 4-pt block

$\mathcal{W}_3$ algebra

The mode expansion of spin-2 and spin-3 currents is

$$T(z) = \sum_{-\infty}^{\infty} L_m z^{-m-2} , \qquad W(z) = \sum_{-\infty}^{\infty} W_n z^{-n-3} . \tag{2.1}$$

The modes satisfy commutation relations

$$[L_n,L_m] = (n-m)L_{n+m} + \frac{c}{12}(n^3-n)\delta_{n+m,0} ,$$

$$[L_n,W_m] = (2n-m)W_{n+m} ,$$

$$[W_n,W_m] = \frac{c}{3\cdot 5!}(n^2-1)(n^2-4)n\delta_{n+m,0} + \frac{16}{22+5c}(n-m)\Lambda_{n+m} +$$

$$\frac{(n-m)}{30}\left(2m^2+2n^2-mn-8\right)L_{n+m} , \tag{2.2}$$

$$\Lambda_m = \sum_{p\le -2} L_p L_{m-p} + \sum_{p\ge -1} L_{m-p}L_p - \frac{3(m+2)(m+3)}{10}L_m . \tag{2.3}$$

The highest-weight vectors

A $\mathcal{W}_3$ highest-weight vector $|\Delta, Q\rangle$ is given

$$L_n|\Delta, Q\rangle = \delta_{n,0}\,\Delta|\Delta, Q\rangle\,, \qquad W_n|\Delta, Q\rangle = \delta_{n,0}\,Q|\Delta, Q\rangle\,, \quad \text{where} \quad n \geq 0\,, \tag{3.1}$$

where Δ stands for the conformal dimension, Q denotes the spin-3 charge. The $\mathcal{W}_3$ module is spanned by the basis states

$$\mathcal{L}_{-I}|\Delta, Q\rangle \equiv L_{-i_1}...L_{-i_k}W_{-j_1}...W_{-j_p}|\Delta, Q\rangle, \qquad I \equiv \{i_1, ..., i_k; j_1, ..., j_p\},$$

$$1 \leq i_1 \leq ... \leq i_k, \qquad 1 \leq j_1 \leq ... \leq j_p\,. \tag{3.2}$$

The primary operator $\mathcal{O}_{\Delta,Q}$ is defined as follows

$$|\Delta, Q\rangle = \lim_{z\to 0} \mathcal{O}_{\Delta,Q}(z)|0\rangle. \tag{3.3}$$

In what follows, we will use a notation $\mathcal{O}_1$ instead of $\mathcal{O}_{\Delta_1,Q_1}$.

2, 3-pt correlation functions

Similarly to the Virasoro algebra case, the global subalgebra of $\mathcal{W}_3$ algebra (which is $sl(3)$) imposes restrictions on correlation functions of primary operators. For example, the Ward identities associated with $sl(3)$ fix 2-pt correlation function to be

$$\langle \mathcal{O}_1(z_1)\mathcal{O}_2(z_2)\rangle = \frac{\delta_{\Delta_1,\Delta_2}\delta_{Q_1,-Q_2}}{(z_1-z_2)^{2\Delta_1}} \ . \tag{4.1}$$

In contrast, the Ward identities do not completely determine the 3-point correlation function which depends on an arbitrary function. However, the coordinate dependence is known

$$\langle \mathcal{O}_1(z_1)\mathcal{O}_2(z_2)\mathcal{O}_3(z_3)\rangle \sim z_{12}^{(\Delta_3-\Delta_3-\Delta_1)} z_{13}^{(\Delta_2-\Delta_3-\Delta_1)} z_{23}^{(\Delta_2+\Delta_3-\Delta_1)}, \tag{4.2}$$

wherein $z_{ij} \equiv z_i - z_j$.

The 4-pt conformal block

The conformal blocks decomposition of the 4-pt correlation function reads

$$\langle\mathcal{O}_1(z_1)\mathcal{O}_2(z_2)\mathcal{O}_3(z_3)\mathcal{O}_4(z_4)\rangle = \sum_{p\in S} C_{12}^p C_{34}^p \mathcal{F}_W\left(z_i|\Delta_i, Q_i, \tilde{\Delta}_p, \tilde{Q}_p, c\right) ,$$

$$\mathcal{F}_W\left(z_i|\Delta_i, Q_i, \tilde{\Delta}_p, \tilde{Q}_p, c\right) \equiv$$

$$\sum_I \frac{\langle\mathcal{O}_1(z_1)\mathcal{O}_2(z_2)\mathcal{L}_{-I}|\tilde{\Delta}_p, \tilde{Q}_p\rangle\langle\tilde{\Delta}_p, \tilde{Q}_p|\mathcal{L}_I\mathcal{O}_3(z_3)\mathcal{O}_4(z_4)\rangle}{\langle\tilde{\Delta}_p, \tilde{Q}_p|\mathcal{L}_I\mathcal{L}_{-I}|\tilde{\Delta}_p, \tilde{Q}_p\rangle\ C_{12}^p C_{34}^p} ,$$

$$C_{12}^p = \langle\Delta_1, Q_1|\mathcal{O}_2(1)|\tilde{\Delta}_p, \tilde{Q}_p\rangle , \qquad C_{34}^p = \langle\tilde{\Delta}_p, \tilde{Q}_p|\mathcal{O}_3(1)|\Delta_4, Q_4\rangle , \tag{5.1}$$

where $\mathcal{F}_W\left(z_i|\Delta_i, Q_i, \tilde{\Delta}, \tilde{Q}, c\right)$ denotes a 4-pt $\mathcal{W}_3$ block and C_{12}^p, C_{34}^p are the structure constants. The block depends on two arbitrary functions which are governed by

$$C_n = \frac{\langle\Delta_1, Q_1|\mathcal{O}_2(1)(W_{-1})^n|\tilde{\Delta}, \tilde{Q}\rangle}{\langle\Delta_1, Q_1|\mathcal{O}_2(1)|\tilde{\Delta}, \tilde{Q}\rangle} , \qquad \tilde{C}_n = \frac{\langle\tilde{\Delta}, \tilde{Q}|(W_1)^n\mathcal{O}_3(1)|\Delta_4, Q_4\rangle}{\langle\tilde{\Delta}, \tilde{Q}|\mathcal{O}_3(1)|\Delta_4, Q_4\rangle} , \tag{5.2}$$

where $n = 1, 2, \ldots.$

The classical $\mathcal{W}_3$ block and HL approximation

In the classical limit $c \to \infty$, assuming all $\Delta, Q \sim c$, it has been argued that the block exponentiates

$$\lim_{c\to\infty} \mathcal{F}_W(z_i) = \exp\left[\frac{c}{6} f_W(z_i)\right] , \tag{6.1}$$

where $f_W(z_i)$ stands for the 4-pt classical block. One can define classical dimensions/spin-3 charges ϵ, q

$$\Delta = \frac{c\epsilon}{6}, \qquad Q = \frac{cq}{6}, \tag{6.2}$$

which are finite at the classical limit. For the further needs, the additional approximation of heavy and light operators (HL) is considered. We assume $\epsilon_1 = \epsilon_2 \equiv \epsilon$ and $q_1 = -q_2 \equiv q$ and $\epsilon_3 = \epsilon_4$ and $q_3 = -q_4$ along with

$$\epsilon, \tilde{\epsilon} \ll \epsilon_H , \qquad q \ll q_H . \tag{6.3}$$

The degenerate representation and BPZ equation

Consider a null-vector

$$L_1|\tilde{\psi}_3\rangle = L_2|\tilde{\psi}_3\rangle = W_1|\tilde{\psi}_3\ \rangle = 0\ . \tag{7.1}$$

Let $\Psi_W(y, z_i)$ is a 5-pt block with the degenerate operator $\psi_3(y)$ corresponding to the null-vector $|\psi_3\rangle$

$$\Psi_W(y, z_i)\Big|_{c\to\infty} \to \psi(y,z_i)\exp\left[\frac{c}{6} f_W(z_i)\right]\ . \tag{7.2}$$

$\psi(y,z_i)$ satisfies the BPZ differential equation

$$\left[\frac{d^3}{dy^3} + 4T(y,z_i)\frac{d}{dy} + \frac{2dT(y,z_i)}{dy} - 4W(y, z_i)\right]\psi(y, z_i) = 0\ , \tag{7.3}$$

$$\begin{gathered} T(y, z_i) = \sum_{i=1}^{4}\left(\frac{\epsilon_i}{(y - z_i)^2} + \frac{c_i}{(y - z_i)}\right)\ , \quad c_i = \partial_i f(z_i)\ , \\ W(y, z_i) = \sum_{i=1}^{4}\left(\frac{q_i}{(y - z_i)^3} + \frac{a_i}{(y - z_i)^2} + \frac{b_i}{(y - z_i)}\right)\ . \end{gathered} \tag{7.4}$$

A monodromy matrix is

$$\psi_a(\Gamma \circ y, z) = \psi_b(y, z) M_{ab}(\Gamma | z) \ , \tag{8.1}$$

where $\Gamma \circ y$ denotes a traversal along the contour Γ. One can be shown that solutions of the BPZ equation above must have the monodromy matrix (assuming for the sake of simplicity $\tilde{q} = 0$)

$$\tilde{M}_{ab}(\Gamma) = \exp[2\pi i \gamma_a]\delta_{ab} \ , \qquad \gamma_1 = 1 \ , \qquad \gamma_{2,3} = \left(1 \pm \sqrt{1 - 4\tilde{\epsilon}}\right) \ . \tag{8.2}$$

when bypassing the contour Γ enclosing points z_1 and z_2.

The monodromy problem is to find solutions to the BPZ equation with the monodromy matrix above. This gives algebraic equations for the parameters a_i, b_i, c_i, solving which with respect to accessory parameters, one can obtain the block. Of course, this problem will be solved within the framework of the HL approximation.

HL approximation: zeroth order

Here and after we consider slightly different form of the BPZ equation obtained by fixing $(z_1, z_2, z_3, z_4) \to (1, z, 0, \infty)$, so Γ encircles points $(1, z)$ It also leaves 4 parameters (c_2, a_1, a_2, b_2) independent. Hence the zeroth order BPZ equation has the form

$$D^{(0)}\psi^{(0)}(y) = 0 \,, \qquad D^{(0)} = \left[\frac{d^3}{dy^3} + 4T^{(0)}\frac{d}{dy} + \frac{2dT^{(0)}}{dy} - 4W^{(0)}\right] \,,$$

where

$$T^{(0)} = \frac{\epsilon_H}{y^2} \,, \qquad W^{(0)} = \frac{q_H}{y^3} \,. \tag{9.1}$$

There are three branches of solutions

$$\psi_a^{(0)}(y) = y^{1+p_a}, \qquad a = 1, 2, 3 \,, \tag{9.2}$$

$$p_a^3 - \alpha^2 p_a - 4q_H = 0 \,, \qquad \alpha = \sqrt{1 - 4\epsilon_H} \,. \tag{9.3}$$

Monodromy matrices have the form

$$M_{ab}^{(0)}(\Gamma) = \tilde{M}_{ab}^{(0)}(\Gamma) = \delta_{ab}. \tag{9.4}$$

HL approximation: first order

In the first order of the HL approximation, we have

$$D^{(0)}\psi^{(1)}(y) = -D^{(1)}\psi^{(0)}(y)\,, \qquad D^{(1)} = \left[4T^{(1)}\frac{d}{dy} + \frac{2dT^{(1)}}{dy} - 4W^{(1)}\right]\,, \tag{10.1}$$

$$T^{(1)} = \frac{c_2(z-1)+2\epsilon}{y-1} - \frac{c_2 z + 2\epsilon}{y} + \frac{c_2}{y-z} + \frac{\epsilon}{(z-y)^2} + \frac{\epsilon}{(y-1)^2}\,,$$

$$W^{(1)} = \frac{q}{(y-1)^3} - \frac{q}{(y-z)^3} + \frac{a_1}{(y-1)^2} + \frac{a_2}{(y-z)^2} + \frac{b_2}{y-z}$$

$$\frac{(2a_2 z - a_2 - a_1 + b_2 z(z-1))\,(z-1)}{y^2} - \frac{2a_2\,(z-1) - 2a_1 + b_2\,(z-1)^2}{y-1}\,.$$

Note that the BPZ equation in the first order is the inhomogeneous differential equation with the same left hand side as the zeroth order BPZ equation.

HL approximation: first order - monodromy equations

The first-order solutions can be expressed in terms of zeroth-order solutions

$$\psi_a^{(1)}(y,z) = \psi_b^{(0)}(y)\, T_{ab}(y,z)\;, \tag{11.1}$$

where elements of matrix T_{ab} are expressed in terms of functions $T^{(1)}$ and $W^{(1)}$. Hence the monodromy matrix $M_{ab}^{(1)}$ has the form

$$M_{ab}^{(1)} = \int_\Gamma dy T_{ab} \equiv I_{ab}\;. \tag{11.2}$$

Under the HL approximation, the eigenvalues of this matrix must be equal to the eigenvalues of the matrix $M_{ab}^{(1)}$ which are given by

$$\tilde{M}_{ab}^{(1)} = \gamma_a^{(1)}\delta_{ab}\;, \qquad \gamma_{1,2}^{(1)} = \pm 4\pi i\tilde{\epsilon}\;, \qquad \gamma_3^{(1)} = 0\;. \tag{11.3}$$

Hence we have monodromy equations for the 4-pt non-vacuum block

$$I_{12}I_{23}I_{31} + I_{13}I_{21}I_{32} = 0\;, \qquad I_{12}I_{21} + I_{13}I_{31} + I_{23}I_{32} = -16\pi^2\tilde{\epsilon}^2\;. \tag{11.4}$$

Fixation of arbitrariness and accessory parameter

The case $\tilde{\epsilon} = 0$ was considered before [deBoer, et. al.]. For $\tilde{\epsilon} \neq 0$ we have two equations for 4 parameters c_2, a_1, a_2, b_1 which is a reflection of functional arbitrariness. Indeed, we remember that the block depends on two arbitrary functions. We fix functional arbitrariness

$$c_2 = c_2^{(v)} + c_2^{(n)} \,, \qquad a_{1,2} = a_{1,2}^{(v)} \,, \qquad b_2 = b_2^{(v)} \,, \tag{12.1}$$

where parameters with the superscript (v) correspond to the case $\tilde{\epsilon} = 0$ and are known. Then, we find

$$c_2^{(n)} = \frac{\tilde{\epsilon}}{2z} \prod_{1 \leq i \leq j \leq 3} p_{ij}^{-1/2} \Big(\frac{p_3^2 \left(z^{-p_{12}/2} - z^{p_{12}/2}\right)^2}{p_{12}} -$$

$$\frac{p_2^2 \left(z^{-p_{13}/2} - z^{p_{13}/2}\right)^2}{p_{13}} - \frac{p_1^2 \left(z^{-p_{23}/2} - z^{p_{23}/2}\right)^2}{p_{23}} \Big)^{-\frac{1}{2}}.$$

The block

One can find the block expanding the accessory parameter in $q_H/\alpha^2 \ll 1$.
The block function has the form

$$f_W^{(n)} = \tilde{\epsilon}\left(f_0 + \frac{q_H^2}{4\alpha^4} f_2 + \ldots\right) , \tag{13.1}$$

where

$$f_0 = 2\mathrm{Arcth}\left(z^{\alpha/2}\right) ,$$

$$f_2 = z^{a/2}\left(3\alpha\log(z)\left(\frac{28(1-z^\alpha)+3\alpha\log z}{(1-z^\alpha)^2} - 36\,{}_3F_2\left(\frac{1}{2},\frac{1}{2},1;\frac{3}{2},\frac{3}{2};z^\alpha\right)\right)\right) +$$

$$27\Phi\left(z^\alpha,3,\frac{1}{2}\right) + 8 + 27\alpha^2\log^2 z\ \mathrm{Arcth}\left(z^{\alpha/2}\right) + 8z^{-\frac{\alpha}{2}} + \frac{9\alpha^2 z^{\frac{3\alpha}{2}}\log^2 z}{(z^\alpha-1)^2} .$$

Here $\Phi(x,s,a)$ denotes the Lerch transcendent function

$$\Phi(x,s,a) = \sum_{k=0}^{\infty} \frac{x^k}{(a+k)^s} .$$

Thank you for your attention!

Towards arbitrariness in the case of the 3-pt correlation function

The differential realization of $\mathcal{W}_3$ includes two additional operators $\hat{W}_{-1}\mathcal{O}(z)$ and $\hat{W}_{-2}\mathcal{O}(z)$ [Watts:'94]

$$\mathcal{L}_m\mathcal{O}(z) = \left(z^{m+1}\partial_z + \Delta(m+1)z^m\right)\mathcal{O}(z),$$

$$\mathcal{W}_n\mathcal{O}(z) = \left(\frac{Q}{2}(m+2)(m+1)z^m + (m+2)z^{m+1}\hat{W}_{-1} + z^{m+2}\hat{W}_{-2}\right)\mathcal{O}(z). \tag{15.1}$$

Then, the 3-pt correlation function depends on nine variables: coordinates z_j and the operators $\hat{W}_{-1}^{(j)}, \hat{W}_{-2}^{(j)}$, $j = 1, 2, 3$. So 8 Ward identities are not enough to completely eliminate functional arbitrariness.

Vacuum block

The 4-pt vacuum conformal block has the form [deBoer '14, Hegde '15]

$$f_W^{(v)}(z) = \frac{(3q-\epsilon)}{2}\log\left(p_3\left(2z^{2p_2+p_3} - z^{p_2+2p_3} - 1\right) + p_2\left(z^{2p_2+p_3} - 2z^{p_2+2p_3} + \right.\right.$$

$$-\frac{(\epsilon+3q)}{2}\left(-\log\left(p_3\left(z^{p_2} - 2z^{p_3} + z^{2(p_2+p_3)}\right) - p_2\left(z^{2(p_2+p_3)-2z^{p_2}+z^{p_3}}\right)\right)\right)$$

$$-\epsilon(1+p_1)\log z\ .$$