Invariant traces of flat space chiral higher-spin algebra as scattering amplitudes

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Introduction

Basics

Definition

Higher-spin theories are theories involving massless fields of spin greater than 2. It is usually assumed that graviton is in the spectrum, so these are extensions of gravity.

Why interesting?

Massless fields of spin >1/2 inevitably lead to *symmetries*.
Symmetries are important in physics and mathematics. In particular, these lead to *improved quantum behaviour*. Higher-spin theories – a promising approach to *quantum gravity*.

Basics

Problems

Interacting higher-spin theories are very hard to construct. For some *natural assumptions*, no-go theorems show that interacting higher-spin theories in flat space do not exist.

[Weinberg '64; Coleman, Mandula '67]

Roughly speaking, there are too many symmetries for any interactions to be possible.

Basics

Positive results

Interacting higher-spin theories in AdS were suggested by Vasiliev.

[Vasiliev '90, '03]

Holography gives solid support for the existence of higher-spin theories in AdS

[Sezgin, Sundell '02, Klebanov, Polyakov '02]

Via holography, the boundary theory defines «the AdS space S matrix» of higher-spin theories. This S-matrix can then be used to reconstruct the action

[Petkou '03, Bekaert, Erdmenger, DP, Sleight '15, Sleight, Taronna '16]

Basics

Positive results

Chiral (self-dual) higher-spin theories can be constructed in 4d flat space

[Metsaev '90, DP, Skvortsov '16]

There is no contradiction with no-go theorems, as scattering in self-dual theories is (almost) trivial

This talk

We will make an extension with self-dual higher-spin theories in a way that *scattering becomes more non-trivial*.

Symmetries is the main guiding principle

We will define the theory via its *S-matrix*

We will learn the recipe from the AdS case

The S-matrix is expected to be rather exotic, not to contradict the no-go theorems.

No-go theorems

The Coleman-Mandula theorem

If the following assumptions are satisfied:

There are finitely many particles with mass below any M

The S-matrix is non-trivial at almost all energies

The S-matrix is analytic at almost all energies

Then, the symmetry of the S-matrix may only be a direct product of the Poincare group and internal symmetry.

[Coleman, Mandula '67]

The Coleman-Mandula theorem

If the following assumptions are satisfied:

The S-matrix is *non-trivial at almost all energies*

The S-matrix is *analytic at almost all energies*

Then, the symmetry of the S-matrix may only be a direct product of the Poincare group and internal symmetry.

Higher-spin holography

Boundary theory

In the *simplest case*, the boundary theory is a theory of N *free massless scalars*

$$S = \frac{1}{2}\int d^3x \phi^a \Box \phi_a$$

Higher-spin fields are dual to higher-spin conserved currents

$$J_{\mu_1 \ldots \mu_s} =: \phi^a \partial_{\mu_1} \ldots \partial_{\mu_s} \phi_a : + \ldots$$

Higher-spin S-matrix is computed by the correlators of these currents

$$\langle J_{\mu_1 \ldots \mu_{s_1}}(x_1) \ldots J_{\mu_1 \ldots \mu_{s_n}}(x_n) \rangle$$

[Sezgin, Sundell '02, Klebanov, Polyakov '02]

Higher-spin symmetry

Higher-spin symmetry is defined as a *symmetry of the free equation of motion*

$$\Box\varphi = 0$$

It is generated by differential operators L such that

$$L: \quad \Box\varphi = 0 \qquad \Rightarrow \qquad \Box(L\varphi) = 0$$

$$\Box L = L'\Box$$

with trivial symmetries

$$L = M\Box$$

factored out.

Higher-spin symmetry alone allows one to fix the n-point correlator up to an overall factor.

Employing SL(2,C) spinors

SL(2,C) spinors

Four dimensional Lorentz algebra is isomorphic to

$$so(3,1) \sim sl(2,\mathbb{C}).$$

Accordingly Lorentz vectors can be converted to sl(2,C) bispinors and back

$$p_{\alpha\dot{\alpha}} \equiv p_a(\sigma^a)_{\alpha\dot{\alpha}}, \qquad p_a = -\frac{1}{2}(\sigma_a)^{\dot{\alpha}\alpha}p_{\alpha\dot{\alpha}}.$$

Here sigma are the Pauli matrices. For light-like vectors (massless momenta) one has

$$p^a p_a = 0 \qquad \Leftrightarrow \qquad \det(p_{\alpha\dot{\alpha}}) = 0 \qquad \Leftrightarrow \qquad p_{\alpha\dot{\alpha}} = -\lambda_\alpha \bar{\lambda}_{\dot{\alpha}}.$$

For real positive energy momenta

$$\bar{\lambda}_{\dot{\alpha}} = (\lambda_\alpha)^*$$

We will relax this condition: *lambda's are independent, hence, momenta are complex*.

Massless on-shell fields

In terms of sl(2,C) spinors massless representations are realised by

$$
\begin{aligned}
J_{\alpha\beta} &= i\left(\lambda_\alpha \frac{\partial}{\partial\lambda^\beta} + \lambda_\beta \frac{\partial}{\partial\lambda^\alpha}\right), \\
\bar{J}_{\alpha\beta} &= i\left(\bar{\lambda}_{\dot{\alpha}} \frac{\partial}{\partial\bar{\lambda}^{\dot{\beta}}} + \bar{\lambda}_{\dot{\beta}} \frac{\partial}{\partial\bar{\lambda}^{\dot{\alpha}}}\right), \\
P_{\alpha\dot{\alpha}} &= -\lambda_\alpha \bar{\lambda}_{\dot{\alpha}},
\end{aligned}
$$

which act on functions $\Phi(\lambda, \bar{\lambda})$ on $\mathbb{C}^2/\{0\}$. One can introduce the *helicity* operator

$$
H \equiv \frac{1}{2}\left(\bar{N} - N\right), \qquad \bar{N} \equiv \bar{\lambda}^{\dot{\alpha}} \frac{\partial}{\partial\bar{\lambda}^{\dot{\alpha}}}, \qquad N \equiv \lambda^\alpha \frac{\partial}{\partial\lambda^\alpha}
$$

Its eigenspaces

$$
H\Phi_h = h\Phi_h
$$

are irreducible helicity h massless representations. Spin s = helicity +s and helicity -s. For bosonic fields

$$
h \in \mathbb{Z}, \qquad \Phi(-\lambda, -\bar{\lambda}) = \Phi(\lambda, \bar{\lambda})
$$

Massless on-shell fields

In summary, we use lambda, bar *lambda* spinors to encode *momenta*, and the homogeneity degree *operator H* to encode *spin*. This is very efficient when dealing with amplitudes!

$$A^{+1,+1,-1} = \frac{[12]^4}{[12][23][31]}\delta^4(\lambda_1\bar{\lambda}_1 + \lambda_2\bar{\lambda}_2 + \lambda_3\bar{\lambda}_3)$$

where

$$[ij] \equiv \bar{\lambda}^i_{\dot{\alpha}}\bar{\lambda}^j_{\dot{\beta}}\epsilon^{\dot{\alpha}\dot{\beta}}, \qquad \langle ij\rangle \equiv \lambda^i_\alpha\lambda^j_\beta\epsilon^{\alpha\beta}$$

Practical convenience

Instead of a multiplet of fields $\varphi^{a(s)}(p)$ with trace, divergence, on-shell constraints and gauge invariance, now we have a single field $\Phi(\lambda, \bar{\lambda})$.

Massless on-shell fields in AdS

Massless fields in AdS can be realised as

$$
\begin{aligned}
J_{\alpha\beta} &= i\left(\lambda_\alpha \frac{\partial}{\partial \lambda^\beta} + \lambda_\beta \frac{\partial}{\partial \lambda^\alpha}\right), \\
\bar{J}_{\alpha\beta} &= i\left(\bar{\lambda}_{\dot{\alpha}} \frac{\partial}{\partial \bar{\lambda}^{\dot{\beta}}} + \bar{\lambda}_{\dot{\beta}} \frac{\partial}{\partial \bar{\lambda}^{\dot{\alpha}}}\right), \\
P_{\alpha\dot{\alpha}} &= -\lambda_\alpha \bar{\lambda}_{\dot{\alpha}} + \frac{\partial}{\partial \lambda^\alpha} \frac{\partial}{\partial \bar{\lambda}^{\dot{\alpha}}}.
\end{aligned}
$$

The rest remains the same except that helicity +s and helicity -s are equivalent representations.

[Vasiliev theory, twistor literature]

Higher-spin invariant amplitudes in AdS

[Colombo, Sundell '12; Didenko, Skvortsov '12; Gelfond, Vasiliev '13]

Higher-spin algebra

Higher-spin algebra in AdS space is defined in terms of the *associative* star product (Weyl-Moyal)

$$(\Psi_1 \star \Psi_2)(\lambda_3, \bar{\lambda}_3) \equiv \int d^2\lambda_1 d^2\bar{\lambda}_1 d^2\lambda_2 d^2\bar{\lambda}_2 \Psi_1(\lambda_1, \bar{\lambda}_1)\Psi_2(\lambda_2, \bar{\lambda}_2) e^{i([21]+[13]+[32])} e^{i(\langle 21\rangle+\langle 13\rangle+\langle 32\rangle)}.$$

The Lie algebra commutator is just

$$[\Psi_1, \Psi_2]_\star = \Psi_1 \star \Psi_2 - \Psi_2 \star \Psi_1.$$

The AdS isometries so(3,2) are generated by commutators with quadratic polynomials

$$P_{\alpha\dot{\alpha}} \sim \lambda_\alpha \bar{\lambda}_{\dot{\alpha}}, \qquad J_{\alpha\alpha} \sim \lambda_\alpha \lambda_\alpha, \qquad \bar{J}_{\dot{\alpha}\dot{\alpha}} \sim \bar{\lambda}_{\dot{\alpha}} \bar{\lambda}_{\dot{\alpha}}.$$

[Fradkin, Vasiliev '87]

On-shell fields

The representation of this algebra, which carries on-shell fields is constructed as

$$\delta_\Psi \Phi = -\Psi \star \Phi + \Phi \star \tilde{\Psi}, \qquad \tilde{\Psi}(\lambda, \bar{\lambda}) \equiv \Psi(-\lambda, \bar{\lambda}) = \Psi(\lambda, -\bar{\lambda}).$$

It can then be checked that for Psi that correspond to the so(3,2) generators, Phi, indeed, transform as massless on-shell fields.

Invariants of the higher-spin algebra

The star product features a *trace*, which is *cyclic* for bosonic fields

$$\mathrm{tr}(\Psi_1 \star \Psi_2) = \mathrm{tr}(\Psi_2 \star \Psi_1), \qquad \mathrm{tr}(\Psi) \equiv \int d^2\lambda d^2\bar{\lambda} \Psi(\lambda, \bar{\lambda}) \delta^2(\lambda) \delta^2(\bar{\lambda}) = \Psi(0, 0).$$

Together with associativity, this implies that

$$G_n \equiv \mathrm{tr}(\Psi_1 \star \Psi_2 \star \cdots \star \Psi_n)$$

Is invariant under higher-spin algebra transformations

$$\delta_\xi \Psi = [\Psi, \xi]_\star.$$

Thus one constructs invariants of the higher-spin algebra. Here, however, Psi does not transform as fields, but as algebra parameters (adjoint representation).

Invariant scattering amplitudes

One can show that if

$$\delta_\xi \Phi = -\xi \star \Phi + \Phi \star \tilde{\xi}$$

then $\Psi = \Phi \star \delta^2(\lambda)$ transforms as $\delta_\xi \Psi = [\Psi, \xi]_\star$.

[Didenko, Vasiliev '09]

Accordingly,

$$G_n \equiv \mathrm{tr}(\Phi_1 \star \delta^2(\lambda) \star \Phi_2 \star \delta^2(\lambda) \star \cdots \star \ \Phi_n \star \delta^2(\lambda)),$$

where Phi's now transform as on-shell fields is HS-invariant.

These give candidate higher-spin amplitudes, which have been *checked holographically*.

[Colombo, Sundell '12; Didenko, Skvortsov '12; Gelfond, Vasiliev '13]

Invariant scattering amplitudes

More explicitly, for 3-point functions one finds

$$G_3 = \int d^2\lambda_1 d^2\bar{\lambda}_1 d^2\lambda_2 d^2\bar{\lambda}_2 d^2\lambda_3 d^2\bar{\lambda}_3 \Phi_1(\lambda_1, \bar{\lambda}_1)\Phi_2(\lambda_2, \bar{\lambda}_2)\Phi_3(\lambda_3, \bar{\lambda}_3)$$
$$e^{i[12]}\delta^2(\bar{\lambda}_1 + \bar{\lambda}_2 + \bar{\lambda}_3)e^{i(\langle 21\rangle + \langle 13\rangle + \langle 32\rangle)}.$$

The kernel of this integral can be regarded as an amplitude

$$A_3 = e^{i[12]}\delta^2(\bar{\lambda}_1 + \bar{\lambda}_2 + \bar{\lambda}_3)e^{i(\langle 21\rangle + \langle 13\rangle + \langle 32\rangle)}.$$

Chiral higher-spin theory

Chiral higher-spin theory

In 4d Minkowski flat space there exists the so-called *chiral higher-spin theory*. It is constructed in the light-cone gauge, by requiring Poincare invariance of the action.

[Metsaev '91; DP, Skvortsov '16]

In a well-defined sense it can be regarded as the *higher-spin generalisation* of *self-dual Yang-Mills* theory and *self-dual gravity*. It is also chiral, the action is not real in the (3,1) signature.

[DP '17]

Chiral higher-spin theory

Other properties carry over from self-dual theories: *integrability, vanishing of tree-level n-point amplitudes with n>3.*

The three-point amplitude is

$$M_3^{h_1,h_2,h_3} = g\frac{\ell^{h-1}}{(h-1)!}[12]^{h_1+h_2-h_3}[23]^{h_2+h_3-h_1}[31]^{h_3+h_1-h_2}, \qquad h \equiv h_1+h_2+h_3.$$

To be non-trivial it requires complex momenta (feature of massless 3-pt amplitudes)

Chiral higher-spin theory

Chiral higher-spin theories have also been studied at quantum level: finite at one loop

[Skvortsov, Tran, Tsulaia '18'20]

Twistor space and free differential algebra reformulations are available

[Krasnov, Skvortsov, Tran '21; Skvortsov, Van Dongen '22;
Sharapov, Skvortsov, Sukhanov, Van Dongen '22]

Chiral theory

Direct analysis in the light-cone gauge shows that there is *no local parity-invariant completion*. The same, however, applies to theories in AdS as well.

This is why we attempt here to go beyond the self-dual sector using higher-spin symmetries – at least this works in AdS.

Higher-point amplitudes in flat space

Chiral theory

What we will do: consider 2-pt and 3-pt functions in the chiral theory and try to identify *the associative HS product* and *the cyclic trace*, which will enable us to construct HS invariant higher-point amplitudes

2-point amplitudes

By two-point amplitudes in flat space we understand the Wightman functions. For scalar fields one has

$$G_2^0 = \int d^4p_1 d^4p_2 \theta(p_1^0)\delta(p_1^2)\delta^4(p_1+p_2)\Phi_1(p_1)\Phi_2(p_2).$$

Converting this to the spinor-helicity representation, using regularisation

$$\sum_{h=-\infty}^{\infty} z^h = \delta(1-z).$$

to sum over helicities, we obtain

$$A_2 = \delta^2(\lambda_1-\lambda_2)\delta^2(\bar{\lambda}_1+\bar{\lambda}_2).$$

3-point amplitudes

We need to sum

$$A_3^{h_1,h_2,h_3} = g\frac{\ell^{h-1}}{(h-1)!}[12]^{h_1+h_2-h_3}[23]^{h_2+h_3-h_1}[31]^{h_3+h_1-h_2}\delta^4(\lambda_1\bar{\lambda}_1+\lambda_2\bar{\lambda}_2+\lambda_3\bar{\lambda}_3)$$

over helicities on each leg. With the previous regularisation this gives

$$A_3 = g[12]^3 e^{\ell[12]}\delta([12]-[23])\delta([12]-[31])\delta^4(\lambda_1\bar{\lambda}_1+\lambda_2\bar{\lambda}_2+\lambda_3\bar{\lambda}_3).$$

One can further simplify this expression by changing arguments of delta functions

$$A_3 = g e^{\ell[12]}\delta^2(\bar{\lambda}_1+\bar{\lambda}_2+\bar{\lambda}_3)\delta^2(\lambda_2-\lambda_3)\delta^2(\lambda_1-\lambda_3).$$

It is very reminiscent of the result that we have in AdS!

Algebraic structures

Following the AdS setup, we introduce the *associative product*

$$(\Phi_1 \ltimes \Phi_2)(\lambda_3,\bar\lambda_3) \equiv \int d^2\lambda_1 d^2\bar\lambda_1 d^2\lambda_2 d^2\bar\lambda_2 \Phi_1(\lambda_1,\bar\lambda_1)\Phi_2(\lambda_2,\bar\lambda_2)e^{\ell[12]}\delta^2(\bar\lambda_1+\bar\lambda_2-\bar\lambda_3)\delta^2(\lambda_2-\lambda_3)\delta^2(\lambda_1-\lambda_3)$$

and trace, which is *cyclic* with respect to it

$$\mathrm{tr}_\ltimes(\Phi(\lambda,\bar\lambda)) \equiv \int d^2\lambda d^2\bar\lambda \Phi(\lambda,\bar\lambda)\delta^2(\bar\lambda), \qquad \mathrm{tr}_\ltimes(\Phi_1 \ltimes \Phi_2) = \mathrm{tr}_\ltimes(\Phi_2 \ltimes \Phi_1).$$

These are chosen so that the kernels of

$$G_2 = \mathrm{tr}_\ltimes(\Phi_1 \ltimes \Phi_2), \qquad G_3 = \mathrm{tr}_\ltimes(\Phi_1 \ltimes \Phi_2 \ltimes \Phi_3)$$

reproduce amplitudes that we have just computed

Higher-spin algebra in flat space

Associativity of the product and cyclicity of the trace implies that A_2 and A_3 are invariant under

$$\bar{\delta}_\varepsilon \Phi \equiv [\Phi, \varepsilon]_\ltimes \equiv \Phi \ltimes \varepsilon - \varepsilon \ltimes \Phi.$$

In this way we find that chiral higher-spin theories have some global higher-spin symmetry. This was not built in!

Relevance of this algebra was seen before when reformulating the chiral higher-spin theory as the self-dual theory, in terms of twistors and free differential algebras

[DP '17; Krasnov, Skvortsov, Tran '21; Skvortsov, Van Dongen '22; Sharapov, Skvortsov, Sukhanov, Van Dongen '22]

Higher-point amplitudes

In the same way as in AdS, one can construct higher point amplitudes

$$G_n \equiv \mathrm{tr}_{\ltimes}(\Phi_1 \ltimes \Phi_2 \ltimes \cdots \ltimes \Phi_n),$$

which are manifestly higher-spin invariant.

Properties

Computing explicitly we find

$$G_n = \int \prod_{i=1}^{n} d^2\lambda_i d^2\bar{\lambda}_i \Phi_i(\lambda_i, \bar{\lambda}_i) \prod_{n\geq i>j\geq 2} e^{\ell [ji]} \delta^2(\sum_{i=1}^{n} \bar{\lambda}_i) \prod_{i=2}^{n} \delta^2(\lambda_1 - \lambda_i).$$

For four-point function one gets

$$A_4 = e^{\ell([23]+[24]+[34])} \delta^2(\bar{\lambda}_1 + \bar{\lambda}_2 + \bar{\lambda}_3 + \bar{\lambda}_4) \delta^2(\lambda_1 - \lambda_2) \delta^2(\lambda_1 - \lambda_3) \delta^2(\lambda_1 - \lambda_4).$$

It has interesting features:

1) Scattering occurs at all lambda equal
2) Barred lambda is conserved separately
3) This means that scattering is non-trivial only for p_i p_j =0. That is all Mandelstam variables are vanishing
4) Chiral, relies on complex momenta

Distributions in HS occurred before

[Joung, Nakach, Tseytlin '15;Taronna '16; Bekaert, Erdmenger, DP, Sleight '16; Beccaria, Nakach, Tseytlin '16; Sleight, Taronna '16]

Contractions from AdS

Contractions at the level of star product

$$e^{i\bar{R}([12]+[23]+[31])}e^{iR(\langle 12\rangle+\langle 23\rangle+\langle 31\rangle)} \quad \to \quad e^{i\bar{R}([12]+[23]+[31])}\delta^2(\lambda_1-\lambda_2)\delta^2(\lambda_1-\lambda_3), \qquad R\to\infty.$$

Product becomes commutative in lambdas.

Contraction at the level of 3-pt amplitudes

$$e^{-\frac{i}{\bar{R}}[12]}\delta^2(\bar{\lambda}_1+\bar{\lambda}_2+\bar{\lambda}_3)e^{iR(\langle 12\rangle+\langle 23\rangle+\langle 31\rangle)} \quad \to \quad e^{-\frac{i}{\bar{R}}[12]}\delta^2(\bar{\lambda}_1+\bar{\lambda}_2+\bar{\lambda}_3)\delta^2(\lambda_1-\lambda_2)\delta^2(\lambda_1-\lambda_3), \qquad R\to\infty.$$

We actually reproduce the flat space amplitude!

‘Global symmetry’

Consider quadratic generators

$$\bar{J}_{\dot\alpha\dot\alpha} \sim \bar\lambda_{\dot\alpha}\bar\lambda_{\dot\alpha}, \qquad P_{\alpha\dot\alpha} \sim \lambda_\alpha\bar\lambda_{\dot\alpha}, \qquad L_{\alpha\alpha} \sim \lambda_\alpha\lambda_\alpha.$$

They commute as

$$\begin{aligned} &[\bar J, \bar J] \sim \bar J, \qquad &[\bar J, P] \sim P, \qquad &[P, P] \sim L, \\ &[L, \bar J] = 0, \qquad &[L, P] = 0, \qquad &[L, L] = 0. \end{aligned}$$

This is not Poincare! It is some central extension of the chiral part of the Poincare algebra.

Can be obtained as a chiral contraction of so(3,2):

$$\bar J \to \bar J, \qquad P \to \frac{1}{R}P, \qquad J \to \frac{1}{R^2}L \qquad \text{where} \qquad R \to \infty.$$

How do we get the Poincare algebra?

‘Global symmetry’

The missing part of the Lorentz algebra – J – is the symmetry of amplitudes, but not part of the chiral flat space higher-spin algebra.

$$
\begin{aligned}
&[\bar{J},\bar{J}] \sim \bar{J}, \qquad [\bar{J},P] \sim P, \qquad [P,P] \sim L,\\
&[L,\bar{J}] = 0, \qquad [L,P] = 0, \qquad [L,L] = 0,\\
&[J,P] \sim P, \qquad [J,\bar{J}] = 0, \qquad [J,L] \sim L, \qquad [J,J] \sim J.
\end{aligned}
$$

Moreover, fields transform in the representation with L = 0. Thus, translations commute and we indeed reproduce massless representation of the Poincare algebra.

Conclusion

Conclusion

1) We find that amplitudes in the chiral higher-spin theory quite manifestly have the form of invariant traces of a certain associative algebra. This pattern closely mimics the one in AdS, which was confirmed holographically.
2) This ensures that the chiral higher-spin theory has a certain global higher-spin algebra as a symmetry.
3) Using the associative product and the respective cyclic trace extracted from 2-pt and 3-pt functions, one can construct manifestly higher-spin invariant higher-point amplitudes
4) This gives us *first flat space amplitudes in higher-spin gauge theories, which are non-vanishing beyond 3-point level*
5) Amplitudes involve distributions

Further directions

1) Restoring parity-invariance. Unlike in AdS, naive addition of parity-conjugate amplitudes breaks higher-spin symmetry. So, in the current form, amplitudes are chiral. This means, at least, that these crucially rely on complex momenta
2) What is the theory (action) these amplitudes correspond to? Is it local?
3) Fix undetermined relative factors for each n-point amplitude. This may require developing the holographic description of this theory.

Thank you!

External lines

As usual, on the external lines of the S-matrix one has the on-shell states, which are solutions to the free equations of motion. For massless fields in flat space EOM's in the covariant form read

$$\begin{aligned} \eta_{aa}\varphi^{a(s)} &= 0, \\ \Box\varphi^{a(s)} &= 0, \\ \partial_a\varphi^{a(s)} &= 0 \end{aligned}$$

Gauge transformations are given by

$$\delta\varphi^{a(s)} = \partial^a\xi^{a(s-1)}$$

$$\begin{aligned} \eta_{aa}\xi^{a(s-1)} &= 0, \\ \Box\xi^{a(s-1)} &= 0, \\ \partial_a\xi^{a(s-1)} &= 0 \end{aligned}$$

These are usually solved in the Fourier space.

Constraints from gauge invariance

Solutions from the previous slide define massless representations of the Poincare algebra. Amplitudes are Poincare invariant forms on these representations

$$A_{a_1(s_1),\dots a_n(s_n)}(p_1,\dots,p_n) = M_{a_1(s_1),\dots a_n(s_n)}(p_1,\dots,p_n)\delta^d(p_1+\dots+p_n)$$

Gauge invariance leads to the familiar Ward identities in massless theories

$$p_i^{a_i} M_{a_1(s_1),\dots a_n(s_n)}(p_1,\dots,p_n) = 0, \qquad \forall i.$$

The Ward identities are, however, *approach-dependent*. In particular, one can use instead of phi their gauge-fixed counterparts. Then, there will be no gauge symmetries and no Ward identities. *Global symmetries*, in turn, are more universal

Global symmetries

Global symmetries in gauge theories occur as follows. One should look into the kernel of the free gauge transformation

$$\delta\varphi^{a(s)} = \partial^a \tilde{\xi}^{a(s-1)} = 0.$$

Parameters that solve eqn above generate global symmetry transformations. In the non-linear theory this happens as follows

$$\delta^{nl}_{\tilde{\xi}} \varphi^{a(s)} = \partial^a \tilde{\xi}^{a(s-1)} + T(\tilde{\xi}, \varphi) + \dots$$

where T is linear in phi and xi and gives the first non-linear correction to the gauge transformation law. Global symmetries are generated by

$$\delta^{gl}_{\tilde{\xi}} \varphi^{a(s)} = T(\tilde{\xi}, \varphi).$$

They still survive in a gauge-fixed theory.

Examples

The Yang-Mills theory. Gauge transformations in the free theory are

$$\delta A^a(x) = \partial^a \xi(x).$$

So, the global symmetry parameters are x-independent. In the non-linear theory they generate

$$\delta_{\tilde{\xi}} A^a(x) = \partial^a \tilde{\xi} + [A(x), \tilde{\xi}] = [A(x), \tilde{\xi}]$$

which are, indeed, the global transformations in internal space.

Examples

Gravity. Gauge transformations in the free theory are

$$\delta g^{aa}(x) = \partial^a \xi^a(x).$$

Global parameters are just the Killing vectors

$$\tilde{\xi}^a(x) = a^a + \omega^{a,b} x_b, \qquad \omega_{a,b} = -\omega_{b,a}.$$

In the non-linear theory, these generate the flat space isometries, that is the global Poincare algebra

$$\delta_{\tilde{\xi}} g^{aa}(x) = \mathcal{L}_{\tilde{\xi}} g^{aa}(x).$$

Higher-spin case

In the *general spin case* global symmetry parameters

$$\partial^a \tilde{\xi}^{a(s-1)} = 0$$

are given by the traceless Killing tensors of the Minkowski space.

This defines the spectrum of the global higher-spin algebra.

Further consistency conditions

Global symmetry transformations should close into themselves

$$[\delta_{\tilde{\xi}_1}, \delta_{\tilde{\xi}_2}]\varphi = \delta_{\tilde{\xi}_3}\varphi \equiv \delta_{[\tilde{\xi}_1, \tilde{\xi}_2]}\varphi,$$

which defines the commutator of global symmetries. It should satisfy the Jacobi identity, that is global symmetries form a Lie algebra. If we want to have gravity as spin-2, it should have the Poincare subalgebra

Finally,

$$\delta_{\tilde{\xi}}\varphi \to \varphi$$

should be a representation of this algebra. Moreover, under the Poincare subalgebra, fields should transform in the massless higher-spin representations that we started from.

2-point amplitudes

By two-point amplitudes in flat space we understand the Wightman functions. For scalar fields one has

$$G_2^0 = \int d^4 p_1 d^4 p_2 \theta(p_1^0)\delta(p_1^2)\delta^4(p_1 + p_2)\Phi_1(p_1)\Phi_2(p_2).$$

Converting this to the spinor-helicity representation, we obtain

$$A_2^0 = \langle 1\mu\rangle[\mu 1]\delta(\langle 1\mu\rangle[\mu 1] + \langle 2\mu\rangle[\mu 2])\delta(\langle 12\rangle)\delta([12]).$$

Note that it is not manifestly Lorentz covariant due to the presence of the reference spinor.

Analogously, for helicity-h two-point function one finds

$$A_2^h = \left(-\frac{[1\mu]\langle\mu 2\rangle}{[2\mu]\langle\mu 1\rangle}\right)^h \langle 1\mu\rangle[\mu 1]\delta(\langle 1\mu\rangle[\mu 1] + \langle 2\mu\rangle[\mu 2])\delta(\langle 12\rangle)\delta([12]).$$

2-point amplitudes

To bring it to the form, which is reminiscent of that in AdS, we sum it over spins

$$A_2 = \sum_{h=-\infty}^{\infty} \left(-\frac{\langle 1\mu\rangle[\mu 2]}{\langle 2\mu\rangle[\mu 1]} \right)^h \langle 1\mu\rangle[\mu 1]\delta(\langle 1\mu\rangle[\mu 1] + \langle 2\mu\rangle[\mu 2])\delta(\langle 12\rangle)\delta([12]).$$

To perform the sum, we use the following standard regularisation

$$\sum_{h=-\infty}^{\infty} z^h = \delta(1-z).$$

This gives

$$A_2 = \delta\left(\langle 2\mu\rangle[\mu 1] + \langle 1\mu\rangle[\mu 2]\right)\langle 2\mu\rangle[\mu 1]\langle 1\mu\rangle[\mu 1]\delta(\langle 1\mu\rangle[\mu 1] + \langle 2\mu\rangle[\mu 2])\delta(\langle 12\rangle)\delta([12]).$$

By going to new arguments of delta-functions, this can be written as

$$A_2 = \delta^2(\lambda_1 - \lambda_2)\delta^2(\bar{\lambda}_1 + \bar{\lambda}_2).$$

Restoring parity-invariance

Amplitude

$$G_n \equiv \mathrm{tr}(\Phi_1 \star \delta^2(\lambda) \star \Phi_2 \star \delta^2(\lambda) \star \cdots \star \ \Phi_n \star \delta^2(\lambda)),$$

is superficially chiral (delta-functions on lambda but not on lambda bar).

One can show that

$$\bar{G}_n \equiv \mathrm{tr}(\Phi_1 \star \delta^2(\bar{\lambda}) \star \Phi_2 \star \delta^2(\bar{\lambda}) \star \cdots \star \ \Phi_n \star \delta^2(\bar{\lambda})),$$

is invariant with respect to higher-spin symmetries as well. By adding these, we obtain a parity-invariant amplitude

Properties

One may try to cure chirality of amplitudes by adding

$$G_n \equiv \mathrm{tr}_{\rtimes}(\Phi_1 \rtimes \Phi_2 \rtimes \cdots \rtimes \Phi_n),$$

where

$$(\Phi_1 \rtimes \Phi_2)(\lambda_3, \bar{\lambda}_3) \equiv \int d^2\lambda_1 d^2\bar{\lambda}_1 d^2\lambda_2 d^2\bar{\lambda}_2 \Phi_1(\lambda_1, \bar{\lambda}_1)\Phi_2(\lambda_2, \bar{\lambda}_2) e^{\ell\langle 12\rangle}\delta^2(\lambda_1 + \lambda_2 - \lambda_3)\delta^2(\bar{\lambda}_2 - \bar{\lambda}_3)\delta^2(\bar{\lambda}_1 - \bar{\lambda}_3)$$

is parity conjugate to the original $\ltimes$ product. Unlike in AdS space, however, amplitudes above are not invariant with respect to the original symmetry

$$\bar{\delta}_\varepsilon \Phi \equiv [\Phi, \varepsilon]_{\ltimes} \equiv \Phi \ltimes \varepsilon - \varepsilon \ltimes \Phi.$$

So, the naive way of curing parity by adding parity-conjugate amplitudes, unlike in AdS, breaks the original symmetry of the theory.