Cubic interactions of 4D irreducible massless higher spin bosonic fields within BRST approach

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Plan

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Cubic interaction

- Cubic interactions are the first approximation in a Lagrangian theory of interacting fields.
- Their peculiarity is that the cubic interaction for given three fields does not depend on the presence or the absence of any other fields in a full nonlinear theory.
- Thus, they are model independent and can be classified.
- The complete classification of consistent cubic interaction vertices of massless and massive fields of arbitrary spin was obtained in the non-covariant light-cone formalism in the space dimensions $d \geq 4$ [Metsaev 2005]

Cubic interactions for massless higher spin fields

- In Minkowsky space cubic vertices are characterized by the number of derivatives k
- For given three massless symmetric fields with spins s_1, s_2, s_3

  $$k_{min} = s_1 + s_2 + s_3 - 2s_{min} \leq k \leq s_1 + s_2 + s_3 = k_{max}, \quad d > 4$$

  There are only two vertices in four dimensions [Bengtsson, Bengtsson, Brink 1983]

  $$k = \{k_{min}, k_{max}\}, \quad d = 4$$

- Already in this case the Lorentz-covariant realization of cubic interaction vertices for higher spins requires very cumbersome calculations initiated by [Berends, Burgers, Van Dam 1984]

Irreducible massless higher spin fields and its interaction

- Irreducible massless higher spin fields with integer spin s is described by metric-like symmetric tensor $\phi^{\mu_1 \cdots \mu_s}$ which satisfy

$$\partial^2 \phi^{\mu_1 \cdots \mu_s} = 0, \quad \partial_\nu \phi^{\mu_1 \cdots \mu_{s-1} \nu} = 0, \quad \phi^{\mu_1 \cdots \mu_{s-2} \nu}{}_\nu = 0$$

- In the case of interacting fields these equation and constraints are modified from some requirements (gauge symmetries).
- General problem is still to derive these modified equation and constraints from the Lagrangian principle.

Introduction

Cubic interactions of Irreducible massless higher spin fields

- At the cubic level of Lagrangian the task can be simplified if to impose some free constraints by hands focusing on traceless $\phi^{\mu_1\cdots\mu_{s-2}\nu}{}_{\nu}=0$ and/or transverse $\partial_\nu\phi^{\mu_1\cdots\mu_{s-1}\nu}=0$ part of cubic interaction [Manvelyan, Mkrtchyan, Ruehl, 2010; Sagnotti, Taronna 2011; Metsaev 2012; etc.]
- General cubic interaction has been consider by [Manvelyan, Mkrtchyan, Ruehl, 2011] with double tracelessness constraint $\phi^{\mu_1\cdots\mu_{s-4}\nu\sigma}{}_{\nu\sigma}=0$
- One can ignore the tracelessness $\phi^{\mu_1\cdots\mu_{s-2}\nu}{}_{\nu}=0$ constraint then interacting models for Reducible higher spin fields result. It is so-called higher spin triplet model [Fotopoulos, Tsulaia, 2009]

Cubic interactions of Irreducible massless higher spin fields

- Note the number of work where consistent cubic vertices for massless higher spin fields was constructed in the frame-like formalism [Vasiliev 2012; Boulanger, Ponomarev, Skvortsov 2013; Khabarov, Zinoviev 2020].
- Recently cubic vertices for unconstraint symmetric massless higher spin fields with integer spin were considered within BRST approach [Buchbinder, Reshetnyak 2021].

Introduction

Irreducible massless higher spin fields in 4D Minkowski

Due to isomorphism $so(3,1) \approx sp(2,C)$ Irreducible massless higher spin with spin s can be described by multispinors $\phi_{a(s)\dot{a}(s)} = \phi_{a_1 \ldots a_s \dot{a}_1 \ldots \dot{a}_s}$ subject two conditions

$$\partial^2 \phi_{a(s)\dot{a}(s)} = 0, \quad \partial^{b\dot{b}} \phi_{a(s-1)b\dot{a}(s-1)\dot{b}} = 0 \tag{1}$$

Goal is to construct cubic interation that reproduce modified 4D Irreducible massless higher spin fields from the Lagrangian

The higher spin fields appear as the coefficients of the states in a Fock space

$$|\phi\rangle = \sum_{s=0}^{\infty} |\phi_s\rangle, \quad |\phi_s\rangle = \frac{1}{s!} \phi_{a(s)}{}^{\dot{a}(s)} c^{a(s)} c_{\dot{a}(s)} |0\rangle \tag{2}$$

generated by creation $c^a, c^{\dot{a}}$ and annihilation operators $a^a, a^{\dot{a}}$

$$\langle 0|c^a = \langle 0|c^{\dot{a}} = 0, \qquad a^a|0\rangle = a^{\dot{a}}|0\rangle = 0, \qquad \langle 0|0\rangle = 1 \tag{3}$$

with following nonzero commutation relations

$$[a^a, c^b] = \varepsilon^{ab}, \qquad [a^{\dot{a}}, c^{\dot{b}}] = -\varepsilon^{\dot{a}\dot{b}}. \tag{4}$$

Free massless fields in BRST approach

In Fock space we introduce the conjugate closed set of operators

$$p^2 = \partial^2, \quad l = a^a a^{\dot{a}} p_{a\dot{a}}, \quad l^+ = -c^a c^{\dot{a}} p_{a\dot{a}}, \quad p_{a\dot{a}} = \partial_{a\dot{a}}. \qquad (5)$$

with only nonzero commutation relation

$$[l^+, l] = (N + \bar{N} + 2)p^2, \quad N = (\bar{N})^+ = c^\alpha a_\alpha \qquad (6)$$

The Irrep conditions in terms of (5) take form

$$p^2|\phi\rangle = 0, \qquad l|\phi\rangle = 0. \qquad (7)$$

Central object is BRST charge Q acting in an extended Fock space $|\Phi\rangle$

- Q is Hermitian nilpotent $Q^+ = Q, \quad Q^2 = 0$
- Equations $Q|\Phi\rangle = 0$ reproduce Irrep conditions (7)

Then Lagrangian and gauge transformations are constructed as

$$\mathcal{L} \sim \langle\Phi|Q|\Phi\rangle, \quad \delta|\Phi\rangle = Q|\Lambda\rangle$$

Free massless fields in BRST approach

We construct the Hermitian nilpotent BRST charge in a standard way

$$Q = \eta^a F_a - \frac{1}{2}\eta^a\eta^b f_{ab}{}^c \mathcal{P}_c, \quad Q^+ = Q, \quad Q^2 = 0 \tag{8}$$

where F_a is set operators forming closed algebra $[F_a, F_b] = f_{ab}{}^c F_c$, η_a and $\mathcal{P}_a$ are the corresponding fermionic ghost and their momenta (antighost) satisfying usual anticommutation relations $\{\eta_a, \mathcal{P}_b\} = \delta_{ab}$.

In our case

$$F_a = \{p^2, l, l^+\}, \quad \eta_a = \{\theta, c^+, c\}, \quad \mathcal{P}_a = \{\pi, b^+, b\}$$

Then

$$Q = \theta p^2 + c^+ l + c l^+ + c^+ c(N + \bar{N} + 2)\pi \tag{9}$$

Free massless fields in BRST approach

The operator Q acts in a Fock space $|\Phi\rangle$ extended by ghost variables. In order to reproduce only physical states (7) from equation

$$Q|\Phi\rangle = 0 \tag{10}$$

we define the vacuum of extended Fock space as

$$c|0\rangle = b|0\rangle = \pi|0\rangle = 0 \tag{11}$$

Then the most general state $|\Phi\rangle$ of extended Fock space has form

$$|\Phi\rangle = |\phi\rangle + \theta b^+|\phi_1\rangle + c^+ b^+|\phi_2\rangle \tag{12}$$

In what follows we dnote

$$\begin{aligned}
|\phi\rangle &= |H\rangle = \frac{1}{s!} H_{a(s)}{}^{\dot{a}(s)} c^{a(s)} c_{\dot{a}(s)}|0\rangle \\
|\phi_1\rangle &= |C\rangle = \frac{1}{(s-1)!} C_{a(s-1)}{}^{\dot{a}(s-1)} c^{a(s-1)} c_{\dot{a}(s-1)}|0\rangle \\
|\phi_2\rangle &= |D\rangle = \frac{1}{(s-2)!} D_{a(s-2)}{}^{\dot{a}(s-2)} c^{a(s-2)} c_{\dot{a}(s-2)}|0\rangle
\end{aligned}$$

Free massless fields in BRST approach

For given spin s Lagrangian take the form

$$\begin{aligned} 2\mathcal{L} &= H_{a(s)}{}^{\dot{a}(s)}(\partial^2 H^{a(s)}{}_{\dot{a}(s)} - 2s\partial^a{}_{\dot{a}}C^{a(s-1)}{}_{\dot{a}(s-1)}) \\ &\quad -2sC_{a(s-1)}{}^{\dot{a}(s-1)}C^{a(s-1)}{}_{\dot{a}(s-1)} \\ &\quad -D_{a(s-2)}{}^{\dot{a}(s-2)}(\partial^2 D^{a(s-2)}{}_{\dot{a}(s-2)} + 2(s-1)\partial_b{}^{\dot{b}}C^{a(s-2)b}{}_{\dot{a}(s-2)\dot{b}}) \end{aligned}$$

Gauge transformations

$$\begin{aligned} \delta H_{a(s)}{}^{\dot{a}(s)} &= \frac{1}{s}\partial_a{}^{\dot{a}}\lambda_{a(s-1)}{}^{\dot{a}(s-1)} \\ \delta C_{a(s-1)}{}^{\dot{a}(s-1)} &= \partial^2\lambda_{a(s-1)}{}^{\dot{a}(s-1)} \\ \delta D_{a(s-2)}{}^{\dot{a}(s-2)} &= -(s-1)\partial^b{}_{\dot{b}}\lambda_{a(s-1)b}{}^{\dot{a}(s-1)\dot{b}} \end{aligned}$$

One can impose gauge fixing

$$\partial^2\lambda_{a(s-1)}{}^{\dot{a}(s-1)} = 0, \quad \partial^b{}_{\dot{b}}\lambda_{a(s-1)b}{}^{\dot{a}(s-1)\dot{b}}$$

Cubic interation in BRST approach

In order to construct cubic interactions we take three copies of vectors in extended Fock space $|\Phi_i\rangle, i = 1, 2, 3$ and corresponding operators. The operators now satisfy commutation relations

$$[a_i^a, c_j{}^b] = \delta_{ij}\varepsilon^{ab}, \qquad [a_i^{\dot{a}}, c_j{}^{\dot{b}}] = -\delta_{ij}\varepsilon^{\dot{a}\dot{b}}. \tag{13}$$

$$\{\theta_i, \pi_i\} = \{c_i, b_i^+\} = \{c_i^+, b_i\} = \delta_{ij} \tag{14}$$

The full interacting Lagrangian in cubic level can be written

$$\mathcal{L} = \sum_i \int d\theta_i \langle\Phi_i|Q_i|\Phi_i\rangle + g\int d\theta_1 d\theta_2 d\theta_3 \langle\Phi_1|\langle\Phi_2|\langle\Phi_3||V\rangle + h.c. \tag{15}$$

where $|V\rangle$ is some cubic vertex and g is a coupling constant.

Cubic interation in BRST approach

Lagrangian is invariant under following gauge transformations up to g^2 (in what follows $i \simeq i+3$)

$$\delta|\Phi_i\rangle = Q_i|\Lambda_i\rangle - g\int d\theta_{i+1}d\theta_{i+2}(\langle\Phi_{i+1}|\langle\Lambda_{i+2}| + \langle\Phi_{i+2}|\langle\Lambda_{i+1}|)|V\rangle$$

if

$$\hat{Q}|V\rangle = \sum_i Q_i|V\rangle = (Q_1+Q_2+Q_3)|V\rangle = 0. \tag{16}$$

It is BRST invariance condition. We will looking for the vertex in the form

$$|V\rangle = V|\Omega\rangle, \quad |\Omega\rangle = \theta_1\theta_2\theta_3|0_1\rangle\otimes|0_2\rangle\otimes|0_3\rangle. \tag{17}$$

where the function V depends on operators $c_i^a, c_i^{\dot{a}}, c_i^+, b_i^+, \pi_i$ as well as momenta $p_i^{a\dot{a}}$ with the momenta conservation law

$$\sum_i p_i^{a\dot{a}} = 0. \tag{18}$$

Cubic interation in BRST approach

However equation (16) do not determine the vertex $|V\rangle$ uniquely. Indeed if vertex $|V\rangle$ satisfies equation (16) then vertex

$$|V\rangle = |V\rangle + \hat{Q}|W\rangle \tag{19}$$

also satisfies this equation and relates to field redefinition

$$|\Phi_i\rangle \to |\tilde{\Phi}_i\rangle = |\Phi_i\rangle + \int d\theta_{i+1} d\theta_{i+2} \langle\Phi_{i+1}|\langle\Phi_{i+2}|W\rangle \tag{20}$$

Our aim is to fined such function V in (17) which satisfies BRST invariance (16) and determined up to the field redefinition freedom (19). We will just call such vertices as BRST-closed.
For three given massless fields with spin s_1, s_2, s_3 let us parametrize function V as

$$V(s_1, s_2, s_3; k) \tag{21}$$

where k is number of derivatives.

Problem

There are two symplest BRST-closed forms

$$L_i = c_i^a c_i^{\dot{a}} (p_{i+1} - p_{i+2})_{a\dot{a}} - 2c_i^+ (\pi_{i+1} - \pi_{i+2}), \quad (22)$$

$$Z = \sum_{i=1}^{3} Q_{ii+1} L_{i+2} = Q_{12} L_3 + Q_{23} L_1 + Q_{31} L_2, \quad (23)$$

where Q_{ii+1} is auxiliary operators

$$Q_{ii+1} = \bar{C}_{ii+1} C_{ii+1} + \frac{1}{2} c_i^+ b_{i+1}^+ + \frac{1}{2} c_{i+1}^+ b_i^+ \quad (24)$$

here $C_{ii+1} = c_i^a c_{i+1\,a}$ and $\bar{C}_{ii+1} = c_{i+1\,\dot{a}} c_i^{\dot{a}}$. The forms L_i and Z by themselves correspond to vertices $V(1,0,0;1)$ and $V(1,1,1;1)$ respectively. The first one describes current interaction for spin 1 and second one - the Yang-Mills interaction for three diffenrent spin 1.

Cubic interation in BRST approach

Problem

The analogies BRST-closed forms L_i and Z are known from [Metsaev 2012] where cubic vertices are investigated for irreducible massless higher spin fields subject tracelessness constraints by hands. General solution for cubic vertices is presented as product of L_i, Z and have form

$$V(s_1, s_2, s_3; k) \quad = \quad Z^{\frac{1}{2}(\mathbf{s}-k)} \prod_{i=1}^{3} L_i^{s_i+\frac{1}{2}(k-\mathbf{s})}, \quad \mathbf{s} = s_1 + s_2 + s_3$$

The BRST-invariance of such vertex

$$\hat{Q}V(s_1, s_2, s_3; k)|\Omega\rangle = 0 \tag{25}$$

is obvious due to that $[\hat{Q}, L_i] = 0$ and $[\hat{Q}, Z] = 0$.

Cubic interation in BRST approach

Problem

In our case these commutators are not vanish

$$
\begin{aligned}
[\hat{Q}, L_i] &= 2L_i c_i^+ c_i \pi_i + c_i^+ (c_i^{\dot{a}} a_{i\dot{b}} p_i^{a\dot{b}} - c_i^a a_{ib} p_i^{b\dot{a}})(p_{i+1} - p_{i+2})_{a\dot{a}} \\
[\hat{Q}, Z] &= \sum_{i=1}^{3} (Q_{ii+1}[Q, L_{i+2}] + L_{i+2}[Q, Q_{ii+1}])
\end{aligned}
$$

where

$$
\begin{aligned}
[\hat{Q}, Q_{ii+1}] &= \frac{1}{2} c_{i+1}^+ L_i - \frac{1}{2} c_i^+ L_{i+1} \\
&+ \bar{C}_{ii+1} (c_i^+ c_{i+1\,a} p_i^{a\dot{b}} a_{i\dot{b}} - c_{i+1}^+ c_{i\,a} p_{i+1}^{a\dot{b}} a_{i+1\,\dot{b}}) \\
&+ 2C_{ii+1} \bar{C}_{ii+1} (c_i^+ c_i \pi_i + c_{i+1}^+ c_{i+1} \pi_{i+1}) \\
&+ C_{ii+1} (c_i^+ c_{i+1\,\dot{a}} a_{ib} p_i^{b\dot{a}} - c_{i+1}^+ c_{i\,\dot{a}} a_{i+1\,b} p_{i+1}^{b\dot{a}}) \\
&+ \frac{1}{2} c_i^+ c_{i+1}^+ (N_{i+1} + \bar{N}_{i+1}) \pi_{i+1} + \frac{1}{2} c_{i+1}^+ c_i^+ (N_i + \bar{N}_i) \pi_i
\end{aligned}
$$

Cubic interation in BRST approach

Problem

As a consequence the double commutators do not vanish

$$[[\hat{Q}, L_i], L_j] \neq 0, \quad [[\hat{Q}, Z], L_j] \neq 0, \quad [[\hat{Q}, Z], Z] \neq 0$$

They are proportional to only creation operators therefore all triple commutators do vanish

$$[[[\hat{Q}, L_i], L_j], L_k] = [[[\hat{Q}, Z], L_i], L_j] = [[[\hat{Q}, Z], Z], L_i] = [[[\hat{Q}, Z], Z], Z] = 0$$

Problem

So if take the naive general vertex as a product of L_i and Z like []

$$V(s_1, s_2, s_3; k) \;=\; Z^{n_0} \prod_{i=1}^{3} L_i^{n_i} \tag{26}$$

where n_0, n_i are the same

$$n_0 = \frac{1}{2}(\mathbf{s} - k), \quad n_i = s_i + \frac{1}{2}(k - \mathbf{s}); \quad \mathbf{s} = s_1 + s_2 + s_3 \tag{27}$$

The action of $\hat{Q}$ on vertex (26) gives

$$\hat{Q}V(s_1, s_2, s_3; k)|\Omega\rangle = \frac{1}{2} n_0 (n_0 - 1) Z^{n_0 - 2} \prod_{i=1}^{3} L_i^{n_i} [[\hat{Q}, Z], Z]|\Omega\rangle$$

$$+ n_0 Z^{n_0 - 1} \sum_{i=1}^{3} n_i L^{n_i - 1} \prod_{j \neq i, j=1}^{3} L_j^{n_j} [[\hat{Q}, Z], L_i]|\Omega\rangle$$

$$+ \frac{1}{2} Z^{n_0} \sum_{i=1}^{3} n_i (n_i - 1) L^{n_i - 2} \prod_{j \neq i, j=1}^{3} L_j^{n_j} [[\hat{Q}, L_i], L_i]|\Omega\rangle$$

Cubic interation in BRST approach

Problem

Evidently it is not equal zero and one need to add some corrections to vertex $\Delta V(s_1, s_2, s_3; k)$ to compensate it

$$\hat{Q}(V(s_1, s_2, s_3; k) + \Delta V(s_1, s_2, s_3; k))|\Omega\rangle \quad = \quad 0$$

In general this task is very complicated so let us consider some particular cases.

Cubic interation in BRST approach

Solution $V(s_1, s_2, s_3; k)$**,** $k = k_{max} = s_1 + s_2 + s_3$

This case is at $n_0 = 0$

$$V(s_1, s_2, s_3; k) \quad = \quad \prod_{i=1}^{3} L_i^{n_i} = L_1^{n_1} L_2^{n_2} L_3^{n_3}, \quad n_i = s_i \tag{28}$$

This case

$$\hat{Q} V(s_1, s_2, s_3; k)|\Omega\rangle \quad = \quad \frac{1}{2} \sum_{i=1}^{3} n_i(n_i - 1) L^{n_i - 2} \prod_{j \neq i, j=1}^{3} L_j^{n_j} [[\hat{Q}, L_i], L_i]|\Omega\rangle$$

Note that

$$[[\hat{Q}, L_i^{n_i}], L_j^{n_j}] = 0, \quad i \neq j$$

that is why our strategy is to fined operator $\mathbb{L}_i^{(n_i)}$ which deforms product $L_i^{n_i}$ in such way that $\hat{Q}\mathbb{L}_i^{(n_i)}|\Omega\rangle = 0$ then the BRST-closed vertex will have form

$$\mathbb{V}(s_1, s_2, s_3; k) \quad = \quad \prod_{i=1}^{3} \mathbb{L}_i^{(n_i)} \tag{29}$$

Solution $V(s_1, s_2, s_3; k)$, $k = k_{max} = s_1 + s_2 + s_3$

Solution for the operator $\mathbb{L}_i^{(n_i)}$ is

$$\mathbb{L}_i^{(n_i)} = L_i^{n_i} + \Delta L_i^n$$

with the correction

$$\Delta L_i^{n_i} \quad = \quad n_i(n_i - 1)L_i^{n_i-2}c_i^+[l_i^+(2\pi_{i+1} + 2\pi_{i+2} - \pi_i) - 2L_i(\pi_{i+1} - \pi_{i+2})]$$

Cubic interation in BRST approach

Solution $V(s_1, s_2, s_3; k)$, $k = s_1 + s_2 + s_3 - 2$

This case is at $n_0 = 1$

$$V(s_1, s_2, s_3; k) \;\; = \;\; Z \prod_{i=1}^{3} L_i^{n_i} = Z L_1^{n_1} L_2^{n_2} L_3^{n_3}, \quad n_i = s_i - 1 \tag{30}$$

First of all by use explicit form for $Z = \sum_i Q_{ii+1} L_{i+2}$ the vertex can be rewritten as

$$V(s_1, s_2, s_3; k) \;\; = \;\; \sum_{i=1}^{3} Q_{ii+1} L_i^{n_i} L_{i+1}^{n_{i+1}} L_{i+2}^{n_{i+2}+1} \tag{31}$$

Secondly we replace the products $L_j^{n_j}$ on BRST-closed operators $L_j^{n_j} \to \mathbb{L}_j^{(n_j)}$

$$\sum_{i=1}^{3} Q_{ii+1} \mathbb{L}_i^{(n_i)} \mathbb{L}_{i+1}^{(n_{i+1})} \mathbb{L}_{i+2}^{(n_{i+2}+1)} \tag{32}$$

Solution $V(s_1, s_2, s_3; k)$, $k = s_1 + s_2 + s_3 - 2$

We present the solution in the form

$$\mathbb{V}(s_1, s_2, s_3; k) \;\; = \;\; \sum_{i=1}^{3} \mathbb{Q}_{ii+1}^{n_i n_{i+1}} \mathbb{L}_{i+2}^{n_{i+2}+1} \tag{33}$$

where operator $\mathbb{Q}_{ii+1}^{n_i n_{i+1}}$ is deformation of product $Q_{ii+1}\mathbb{L}_i^{(n_i)}\mathbb{L}_{i+1}^{(n_{i+1})}$ such that

$$\hat{Q}\mathbb{Q}_{ii+1}^{n_i n_{i+1}}|\Omega\rangle \;\; = \;\; (\frac{n_{i+1}+1}{2} c_{i+1}^{+} \mathbb{L}_i^{n_i+1} \mathbb{L}_{i+1}^{n_{i+1}} - \frac{n_i+1}{2} c_i^{+} \mathbb{L}_i^{n_i} \mathbb{L}_{i+1}^{n_{i+1}+1})|\Omega\rangle$$

then the vertex (33) is BRST-closed

Cubic interation in BRST approach

Solution $V(s_1, s_2, s_3; k)$, $k = s_1 + s_2 + s_3 - 2$

Solution for operator $\mathbb{Q}^{n_i n_{i+1}}_{ii+1}$ has form

$$
\begin{aligned}
\mathbb{Q}^{nm}_{ii+1} = & \, Q_{ii+1}\mathbb{L}^n_i \mathbb{L}^m_{i+1} + \\
& + n[-2c^+_i Q_{ii+1}\mathbb{L}^{n-1}_i \mathbb{L}^m_{i+1}(\pi_{i+1} - \pi_{i+2}) - \frac{1}{2} c^+_i b^+_i \mathbb{L}^{n-1}_i \mathbb{L}^{m+1}_{i+1} \\
& \quad + \frac{1}{2} c^+_i b^+_{i+1} \mathbb{L}^n_i \mathbb{L}^m_{i+1} \\
& - \frac{3}{2} \mathbb{L}^{n-1}_i \mathbb{L}^m_{i+1} c^+_i b^+_i l^+_{i+1} - c^+_i c^+_{i+1} b^+_i \mathbb{L}^{n-1}_i \mathbb{L}^m_{i+1}(2\pi_{i+1} + \pi_{i+2})] \\
& + m[-2c^+_{i+1} Q_{ii+1}\mathbb{L}^n_i \mathbb{L}^{m-1}_{i+1}(\pi_{i+2} - \pi_i) - \frac{1}{2} c^+_{i+1} b^+_{i+1} \mathbb{L}^{n+1}_i \mathbb{L}^{m-1}_{i+1} \\
& \quad + \frac{1}{2} c^+_{i+1} b^+_i \mathbb{L}^n_i \mathbb{L}^m_{i+1} \\
& + \frac{3}{2} c^+_{i+1} b^+_{i+1} l^+_i \mathbb{L}^n_i \mathbb{L}^{m-1}_{i+1} + c^+_{i+1} c^+_i b^+_{i+1} \mathbb{L}^n_i \mathbb{L}^{m-1}_{i+1}(\pi_{i+2} + 2\pi_i)] \\
& + 4nm c^+_i c^+_{i+1} \mathbb{L}^{n-1}_i \mathbb{L}^{m-1}_{i+1} Q_{ii+1}(\pi_{i+2} - \pi_i)(\pi_{i+1} - \pi_{i+2}) \\
& - 3nm c^+_i c^+_{i+1} \mathbb{L}^{n-1}_i \mathbb{L}^{m-1}_{i+1}(b^+_i \mathbb{L}_{i+1}\pi_{i+1} + b^+_{i+1}\mathbb{L}_i \pi_i)
\end{aligned}
$$

Cubic interation in BRST approach

Solution $V(s_1, s_2, s_3; k)$, $k = s_1 + s_2 + s_3 - 2$

It is known that in $d = 4$ such vertices generate the total derivative so they are valid only if to put at least one of n_1, n_2, n_3 to zero. For example putting $n_1 = 0$ we have vertex

$$\mathbb{V}(1, s_2, s_3; k_{min}) \quad = \quad \mathbb{Q}_{12}^{0n_2}\mathbb{L}_3^{n_3+1} + \mathbb{Q}_{23}^{n_2 n_3}\mathbb{L}_1 + \mathbb{Q}_{31}^{n_3 0}\mathbb{L}_2^{n_2+1}$$

which describe cubic interaction of one massless field with spin 1 and two massless fields with higher spins s_2, s_3. This vertex contain $k_{min} = s_2 + s_3 - 1$ number of derivative. Note that for the same set of fields there exist cubic vertex with $k_{max} = s_2 + s_3 + 1$ number of derivatives that have been constructed in previous section

$$\mathbb{V}(1, s_2, s_3; k_{max}) = \mathbb{L}_1\mathbb{L}_2^{n_2}\mathbb{L}_3^{n_3}$$

Example of interaction with the scalars

Let us consider explicit example of cubic interaction of two massless scalar fields

$$|\Phi_1\rangle = \varphi_1|0\rangle, \quad |\Phi_2\rangle = \varphi_2|0\rangle \tag{34}$$

and one massless field with arbitrary spin s (see section 1)

$$|\Phi_3\rangle = |H\rangle + \theta_3 b_3^+|C\rangle + c_3^+ b_3^+|D\rangle \tag{35}$$

The total Lagrangian has form

$$\mathcal{L} = \mathcal{L}_{free} + \mathcal{L}_{int} \tag{36}$$

here $\mathcal{L}_{free}$ is free Lagrangian for our system of fields. The interacting Lagrangian $\mathcal{L}_{int}$ correspond to the vertex $V(0,0,s;s) = \mathbb{L}_3^{(s)}$ and has form

$$\mathcal{L}_{int} \quad = \quad g\int d\theta_1 d\theta_2 d\theta_3 \langle\Phi_1|\langle\Phi_2|\langle\Phi_3|\mathbb{L}_3^{(s)}|\Omega\rangle + h.c. \tag{37}$$

Example of interaction with the scalars

where

$$
\begin{aligned}
\mathbb{L}_i^{(s)} &= L_i^s + s(s-1)L_i^{s-2}c_i^+[l_i^+(2\pi_{i+1}+2\pi_{i+2}-\pi_i)-2L_i(\pi_{i+1}-\pi_{i+2})] \\
L_i &= c_i^a c_i^{\dot a}(p_{i+1}-p_{i+2})_{a\dot a} - 2c_i^+(\pi_{i+1}-\pi_{i+2})
\end{aligned}
$$

In components it is rewritten

$$
\frac{(-1)^{s+1}}{gs!}\mathcal{L}_{int} = H^{a(s)}{}_{\dot a(s)} j_{a(s)}{}^{\dot a(s)} - (s-1)\partial_a{}^{\dot a} C^{a(s-1)}{}_{\dot a(s-1)} j_{a(s-2)}{}^{\dot a(s-2)}
$$

where $j_{a(s)}{}^{\dot a(s)}$ are the higher spin currents constructed from two scalars

$$
j_{a(s)}{}^{\dot a(s)} = \varphi_1(\overrightarrow{\partial}_a{}^{\dot a} - \overleftarrow{\partial}_a{}^{\dot a})^s\varphi_2 = \sum_{k=0}^{s} C_s^k(-\partial_a{}^{\dot a})^{s-k}\varphi_1(\partial_a{}^{\dot a})^k\varphi_2
$$

$$
C_s^k = \frac{s!}{k!(s-k)!}
$$

Example of interaction with the scalars

The relevant gauge transformations for higher spin fields remain as in free theory

$$\begin{aligned}
\delta H_{a(s)}{}^{\dot a(s)} &= \frac{1}{s}\partial_a{}^{\dot a}\lambda_{a(s-1)}{}^{\dot a(s-1)} \\
\delta C_{a(s-1)}{}^{\dot a(s-1)} &= \partial^2\lambda_{a(s-1)}{}^{\dot a(s-1)} \\
\delta D_{a(s-2)}{}^{\dot a(s-2)} &= -(s-1)\partial^b{}_{\dot b}\lambda_{a(s-1)b}{}^{\dot a(s-1)\dot b}
\end{aligned} \tag{38}$$

Example of interaction with the scalars

For scalars we have following gauge transformations

$$\delta\varphi_1 = (-1)^s g_l 2s(s-1)![s\sum_{k=0}^{s-1} C_{s-1}^k (\partial_a{}^{\dot{a}})^k \lambda^{a(s-1)}{}_{\dot{a}(s-1)} (2\partial_a{}^{\dot{a}})^{s-k-1}\varphi_2$$
$$-(s-1)\sum_{k=0}^{s-2} C_{s-2}^k (\partial_a{}^{\dot{a}})^{k+1} \lambda^{a(s-1)}{}_{\dot{a}(s-1)} (2\partial_a{}^{\dot{a}})^{s-k-2}\varphi_2]$$

$$\delta\varphi_2 = g_l 2s(s-1)![s\sum_{k=0}^{s-1} C_{s-1}^k (\partial_a{}^{\dot{a}})^k \lambda^{a(s-1)}{}_{\dot{a}(s-1)} (2\partial_{2a}{}^{\dot{a}})^{s-k-1}\varphi_1$$
$$-(s-1)\sum_{k=0}^{s-2} C_{s-2}^k (\partial_a{}^{\dot{a}})^{k+1} \lambda^{a(s-1)}{}_{\dot{a}(s-1)} (2\partial_{2a}{}^{\dot{a}})^{s-k-2}\varphi_1]$$

From the gauge transformations for scalar fields one can see if spin s is even then we can identify scalars $\varphi_1 = \varphi_2 = \varphi$ and obtain interaction of massless fields with integers spin and single scalar. In turn if spin s is odd then we cannot identify φ_1 and φ_2 but can them combine in one complex scalar $\varphi = \varphi_1 + i\varphi_2$

Summary

- We have analyzed and constructed the cubic interactions for totally unconstrained massless higher spin fields in $4D$ Minkowski space. The construction is realized in the framework of the BRST approach for higher spin fields adopted to multispinor formalism.
- In BRST approach the problem of constructing the cubic interaction vertices is reduced to finding the vector $|V\rangle$ depending on three copies of operators (17) acting on a vacuum of the extended Fock space. Such the vector $|V\rangle$ should be BRST-invariant $\hat{Q}V = 0$ up to the field redefinition, $|V\rangle = \hat{Q}|W\rangle$.
- We have found the solution for cubic vertices as deformation of a product of the simplest BRST-closed forms L_i and Z. For three given massless fields with spins s_1, s_2, s_3 we have constructed cubic vertices with $k_{max} = s_1 + s_2 + s_3$ and $k = s_1 + s_2 + s_3 - 2$ number of derivatives. They correspond to deformations of products $L_1^{s_1} L_2^{s_2} L_3^{s_3}$ and $Z L_1^{s_1-1} L_2^{s_2-1} L_3^{s_3-1}$ respectively.
- There remains open the construction of deformation for cubic vertices containing Z in power 2 and higher.