Chiral effects in classical spinning gas

Dmitry S. Kaparulin

in collaboration with M.A. Bubenchikov, O.D. Nosyrev and N.N. Levin

Faculty of Mechanics and Mathematics,
Tomsk State University

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General remarks

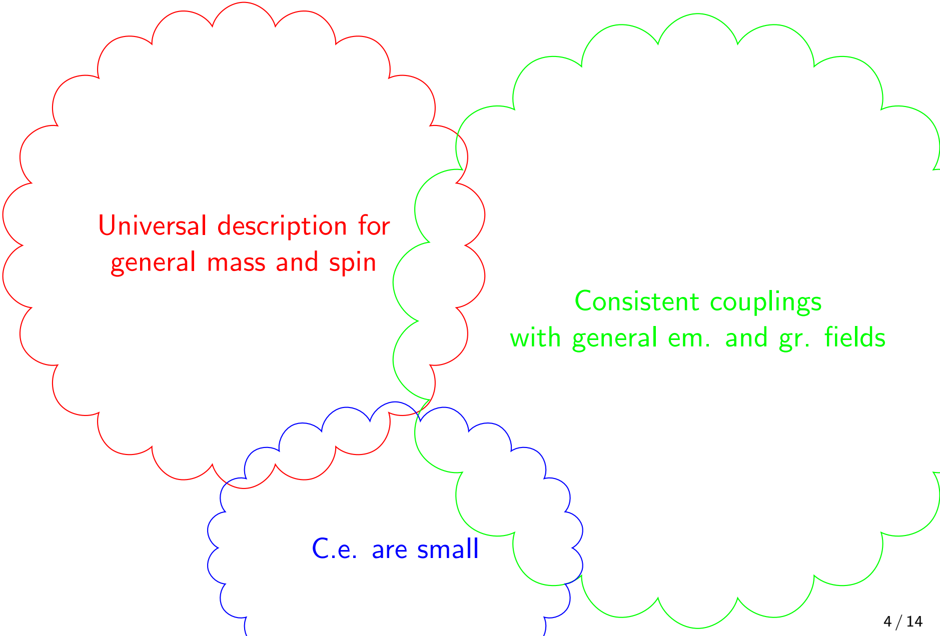
- 1878, J. Maxwell, distribution function for rotating gas
- 1902, J. Gibss, statistical mechanics of rotating ensembles

$$\rho = \frac{1}{Z} \exp\left(- \frac{E - (\boldsymbol{\omega}, \boldsymbol{J})}{kT} \right), \quad Z = \int \exp\left(- \frac{E - (\boldsymbol{\omega}, \boldsymbol{J})}{kT} \right) d\Gamma\,; \tag{1}$$

$$-kT \ln Z = \Phi = U - TS + (\boldsymbol{\omega}, \boldsymbol{J})\,. \tag{2}$$

- 1915, S. Barnett, A. Einstein and S. de Haas discover chiral effects
- 1925, G. Uhlenbeck and S.Goudsmit postulate existence of spin
- 1926, J. Frenkel, classical spinning particle concept
- Modern studies: quantum theory of chiral effects
  [Becattini, Tinti'10, Chernodub, Gongyo'17; Fukushima'19]

Motivation: three reasons



Problem setting

We consider a system of non-interacting non-relativistic particles with spin in a rotating cylinder;

- the radius of cylinder is r;
- the height of cylinder is h;
- the angular velocity of cylinder is $\boldsymbol{\omega} = (0, 0, \omega)$.

Distribution function

$$\rho = \rho_0 \rho_1 \dots \rho_{N-1}\,, \quad \rho_a = \frac{1}{Z_0} \exp\left(- \frac{\epsilon_a - (\boldsymbol{\omega}, \boldsymbol{j}_a)}{kT} \right), \quad a = 0, \dots, N-1; \tag{3}$$

Partition function

$$Z = \frac{1}{N!} (Z_0)^N\,, \qquad Z_0 = \int \exp\left(- \frac{\epsilon_0 - (\boldsymbol{\omega}, \boldsymbol{j}_0)}{kT} \right) d\Gamma_0. \tag{4}$$

Our aim is to find ρ_0 and Z_0.

Model of non-relativistic spinning particle

We consider a non-relativistic point particle with the mass m and spin s. The state of particle is determined by the eight variables:

- the particle position vector $\boldsymbol{r}$;
- the particle linear momentum $\boldsymbol{p}$;
- two angular variables describing spin α, β.

Total angular momentum

$$\epsilon_0 = p^2/2m\,, \qquad \boldsymbol{j}_0 = [\boldsymbol{r}, \boldsymbol{p}] + \boldsymbol{S}\,, \qquad (\boldsymbol{S}, \boldsymbol{S}) = \hbar^2 s(s+1); \tag{5}$$

$$\boldsymbol{S} = \hbar\sqrt{s(s+1)}(\cos\alpha\sin\beta, \sin\alpha\sin\beta, \cos\beta)\,. \tag{6}$$

Phase-space invariant measure

$$d\Gamma_0 = \frac{(2s+1)d\boldsymbol{p}d\boldsymbol{r}\sin\beta d\beta d\alpha}{4\pi(2\pi\hbar)^3} \tag{7}$$

[Gorbunov et al.'99;Ramírez et al'14]

Generalized Maxwell-Boltzmann distribution

One-particle distribution function (by positions, momenta, and directions of spin) has the following form

$$\varrho_0 = \frac{1}{Z_0} \exp\left(-\frac{\bar{p}^2}{2mkT} + \frac{1}{2} m\omega^2 (x^2 + y^2) + \frac{\hbar\omega}{kT} \sqrt{s(s+1)} \cos\beta \right). \quad (8)$$

Here, $\bar{\boldsymbol{p}} = \boldsymbol{p} - [\boldsymbol{\omega}, \boldsymbol{r}]$ is the particle momentum in the rotating frame.

It includes:

- the Maxwell distribution for linear momenta;
- the Boltzmann distribution for coordinates;
- the (new) distribution function for spin.

As the one-particle distribution function factorizes, the behaviour of translational and spinning degrees of freedom is statistically independent.

This justifies consideration of statistical mechanics of spin.

Distribution function for α, β

Integrating by the coordinates and momenta, we obtain the one-particle distribution function by states of spin

$$\varrho_0(sp) = \frac{\frac{\hbar\omega\sqrt{s(s+1)}}{kT}}{\sinh\left(\frac{\hbar\omega\sqrt{s(s+1)}}{kT}\right)} \exp\left(\frac{\hbar\omega}{kT}\sqrt{s(s+1)}\cos\beta\right). \qquad (9)$$

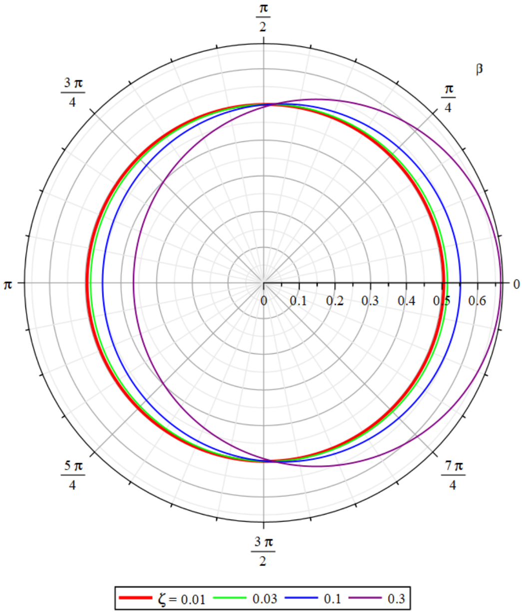
It has the following properties:

- it does not depend on the azimutal angle α;
- it predicts polarization of spinning degree of freedom;
- the value of chiral effects is controlled by the dimensionless parameter

$$\zeta = \frac{\hbar\omega}{kT}. \qquad (10)$$

Remark. Usually, ζ is a very small quantity (order 10^{-12} or smaller).

Indicatrix for $\zeta = 0.01, 0.03, 0.1, 0.3$



Thermodynamics of classical spin

Partition function factorizes into the product of translational and spinning contributions, so

$$Z_0 = Z_{tr} Z_{sp} \qquad \Rightarrow \qquad \Phi = \Phi_{tr} + \Phi_{sp}, \Phi_{sp} = -kT \ln \Phi_{sp} \,. \tag{11}$$

Thermodynamic potential

$$\Phi_{sp} = -kT \left[\ln \frac{2s+1}{\sqrt{s(s+1)}} + \ln \sinh \frac{\sqrt{s(s+1)}\hbar\omega}{kT} - \ln \frac{\hbar\omega}{2kT} \right]. \tag{12}$$

Entropy

$$S_{sp} = -k \left[\frac{\Phi_{sp}}{kT} + \frac{\sqrt{s(s+1)}\hbar\omega}{kT} \coth \frac{\sqrt{s(s+1)}\hbar\omega}{kT} \right]. \tag{13}$$

Heat capacity (at constant ω)

$$C_{sp} = k - \frac{s(s+1)\hbar^2\omega^2}{kT^2} \sinh^{-2} \left(\frac{\sqrt{s(s+1)}\hbar\omega}{kT} \right). \tag{14}$$

High temperature expansion

Thermodynamic potential*

$$\Phi_{sp} = -kT\left(\ln(2s+1) + \frac{s(s+1)}{6}\frac{\hbar^2\omega^2}{k^2T^2} - \frac{s^2(s+1)^2}{180}\frac{\hbar^2\omega^2}{k^2T^2} + \ldots \right). \quad (15)$$

Entropy

$$S_{sp} = k\left(\ln(2s+1) - \frac{s(s+1)}{6}\frac{\hbar^2\omega^2}{k^2T^2} + \frac{s^2(s+1)^2}{60}\frac{\hbar^4\omega^4}{k^4T^4} + \ldots \right). \quad (16)$$

Heat capacity

$$C_{sp} = \frac{s(s+1)}{3}\frac{\hbar^2\omega^2}{kT^2}\left(1 - \frac{s(s+1)}{5}\frac{\hbar^2\omega^2}{k^2T^2} + \ldots \right). \quad (17)$$

Angular momentum

$$J_{sp} = \frac{s(s+1)}{3}\frac{\hbar^2\omega}{kT}\left(1 - \frac{s(s+1)}{15}\frac{\hbar^2\omega^2}{k^2T^2} + \ldots \right). \quad (18)$$

* The dots denote terms that are at least fifth order in ω.

Comparison with quantum case

Partition function in quantum case

$$Z_{sp} = \sum_{s_z=-S}^{S} \exp\left(-\frac{\hbar\omega s_z}{kT}\right) = \sinh^{-1}\left(\frac{1}{2}\frac{\hbar\omega}{kT}\right)\sinh\left(\frac{2s+1}{2}\frac{\hbar\omega}{kT}\right). \quad (19)$$

Thermodynamic potential

$$\Phi_{sp} = -kT\left(\ln\sinh\frac{(2s+1)\hbar\omega}{2kT} - \ln\sinh\frac{1}{2}\frac{\hbar\omega}{kT}\right). \quad (20)$$

Entropy

$$S_{sp} = k\left[\frac{\Phi_{sp}}{kT} + \frac{(2s+1)\hbar\omega}{2kT}\coth\frac{(2s+1)\hbar\omega}{2kT} - \frac{\hbar\omega}{2kT}\coth\frac{\hbar\omega}{2kT}\right]. \quad (21)$$

Heat capacity

$$C_{sp} = \frac{\hbar^2\omega^2}{4kT^2}\sinh^{-2}\frac{\hbar\omega}{2kT} - \frac{(2s+1)^2}{4}\frac{\hbar^2\omega^2}{kT^2}\sinh^{-2}\frac{(2s+1)\hbar\omega}{2kT}. \quad (22)$$

Comparison with quantum case - high temp.

Thermodynamic potential*

$$\Phi_{sp} = -kT\left(\ln(2s+1) + \frac{s(s+1)}{6}\frac{\hbar^2\omega^2}{k^2T^2} - \frac{s^2(s+1)^2 + 1/2s(s+1)}{180}\frac{\hbar^2\omega^2}{k^2T^2} + \ldots \right) \tag{23}$$

Entropy

$$S_{sp} = k\left(\ln(2s+1) - \frac{s(s+1)}{6}\frac{\hbar^2\omega^2}{k^2T^2} + \frac{s^2(s+1)^2 + 1/2s(s+1)}{60}\frac{\hbar^4\omega^4}{k^4T^4} + \ldots \right). \tag{24}$$

Heat capacity

$$C_{sp} = \frac{s(s+1)}{3}\frac{\hbar^2\omega^2}{kT^2}\left(1 - \frac{s(s+1) + 1/2}{5}\frac{\hbar^2\omega^2}{k^2T^2} + \ldots\right). \tag{25}$$

Angular momentum

$$J_{sp} = \frac{s(s+1)}{3}\frac{\hbar^2\omega}{kT}\left(1 - \frac{s(s+1) + 1/2}{15}\frac{\hbar^2\omega^2}{k^2T^2} + \ldots\right). \tag{26}$$

* The dots denote terms that are at least fifth order in ω.

Conclusion

Results.

- Statistical mechanics of classical rotating non-relativistic ideal gas of spinning particles is developed.
- Generalised Maxwell-Boltzmann distribution (by positions, momenta and spin) for classical particle with spin is obtained.
- It is shown that the thermodynamic parameters depend on the angular velocity. This confirms the presence of chiral effects in the system.

Further research.

- Generalized Maxwell-Juttner distribution for a particle with spin.
- Massless particles with continuous helicity.
- Systems with self-interaction.

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