Complete General Relativity

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Einstein Equation

$$R_{ij} - \frac{1}{2} g_{ij} R = T_{ij}$$

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The presentation is organized as follows.

Parallel displacement and the gravitational field

$$\delta V^i = dV^i + \Gamma^i_{jk} dx^j V^k = 0,$$

where

$$\Gamma^i_{jk} = \frac{1}{2} g^{il}(\partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij}).$$

The commutator

$$(\nabla_i \nabla_j - \nabla_j \nabla_i) V^k = R_{ijl}{}^k V^l$$

defines

$$R_{ijl}{}^k = \partial_i \Gamma^k_{jl} - \partial_j \Gamma^k_{il} + \Gamma^k_{in} \Gamma^n_{jl} - \Gamma^k_{jn} \Gamma^n_{il}.$$

The antisymmetrical tensor (trace) $R_{ijk}{}^k$ is trivial. But the symmetrical tensor $R_{jl} = R_{kjl}{}^k$ and the scalar $R = R_{jl} g^{jl}$ define absolutely the l. h. s. of the Einstein equation.

The general parallel displacement

$$\delta V^i = dV^i + P^i_{jk} dx^j V^k = 0,$$

where the connection P^i_{jk} is considered as a primary entity. To define the Principle of General Gauge Relativity, we start from the consideration of the linear operators in the space of vector fields V^i.
Non–degenerate linear transformation has the form

$$\overline{V}^i = S^i_j V^j, \quad \mathrm{Det}(S^i_j) \neq 0,$$

where S^i_j is a tensor field of the second rank. These local internal transformations form a group of General Gauge Relativity with an associative binary operation $P^i_j = S^i_k T^k_j$. This is the general gauge group.

From the law of parallel displacement we have

$$\overline{P}^i_{jk} = S^i_m P^m_{jn} {S^{-1}}^n_k + S^i_m \partial_j {S^{-1}}^m_k,$$

where ${S^{-1}}^k_j$ are the components of the operator S^{-1} inverse to the operator S, $S^i_k {S^{-1}}^k_j = \delta^i_j$.
For the gauge covariant derivative $\delta V^i = \mathrm{D}_j V^i dx^j$

$$\mathrm{D}_j V^i = \partial_j V^i + P^i_{jk} V^k$$

we have

$$\overline{\mathrm{D}}_j \overline{V}^i = S^i_k \mathrm{D}_j V^k.$$

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The commutator of the gauge covariant derivatives

$$(\mathrm{D}_i\mathrm{D}_j - \mathrm{D}_j\mathrm{D}_i)V^k = [\mathrm{D}_i, \mathrm{D}_j]V^k = H_{ijl}{}^k V^l,$$

gives the curvature tensor of the general gauge field

$$H_{ijl}{}^k = \partial_i P^k_{jl} - \partial_j P^k_{il} + P^k_{in} P^n_{jl} - P^k_{jn} P^n_{il}.$$

The antisymmetric tensor (trace of the curvature tensor)

$$F_{ij} = H_{ijk}{}^k = \partial_i P^k_{jk} - \partial_j P^k_{ik}$$

is non trivial here and should be considered separately from the irreducible (traceless) tensor of curvature

$$I_{ijl}{}^k = H_{ijl}{}^k - \frac{1}{4} H_{ijn}{}^n \delta^k_l, \quad I_{ijl}{}^l = 0.$$

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For brevity, in what follows we will use the matrix notation

$$\mathbf{S} = (S^k_l), \quad \mathbf{P}_i = (P^k_{il}), \quad \mathbf{E} = (\delta^k_l),$$

$$\mathbf{H}_{ij} = (H_{ijl}{}^k), \quad \mathrm{Tr}\, \mathbf{S} = S^k_k.$$

The transformations of gauge symmetry in question take the form

$$\overline{\mathbf{P}}_i = \mathbf{S}\mathbf{P}_i\mathbf{S}^{-1} + \mathbf{S}\partial_i\mathbf{S}^{-1} = \mathbf{P}_i + \mathbf{S}\mathrm{D}_i\mathbf{S}^{-1},$$

where D_i is the gauge covariant derivative

$$\mathrm{D}_i\mathbf{S} = \partial_i\mathbf{S} + \mathbf{P}_i\mathbf{S} - \mathbf{S}\mathbf{P}_i = \partial_i\mathbf{S} + [\mathbf{P}_i, \mathbf{S}].$$

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For the curvature tensor we have

$$\mathrm{D}_i \mathbf{H}_{jk} = \partial_i \mathbf{H}_{jk} + [\mathbf{P}_i, \mathbf{H}_{jk}]$$

and the general covariant identity

$$\mathrm{D}_i \mathbf{H}_{jk} + \mathrm{D}_j \mathbf{H}_{ki} + \mathrm{D}_k \mathbf{H}_{ij} = 0$$

holds.

Under the transformations of the general gauge group the curvature tensor of the connection P^i_{jk}

$$\mathbf{H}_{ij} = \partial_i \mathbf{P}_j - \partial_j \mathbf{P}_i + [\mathbf{P}_i, \mathbf{P}_j]$$

transforms homogeneously

$$\overline{\mathbf{H}}_{ij} = \mathbf{S}\mathbf{H}_{ij}\mathbf{S}^{-1}.$$

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The ground state of the general gauge field is defined by the equation

$$\mathbf{H}_{ij} = 0,$$

which has nontrivial solution

$$P^i_{jk} = L^i_{jk} = E^i_\mu \partial_j E^\mu_k$$

(a linear connection of the ground state).

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A transition from the ground state to the exited one is characterized by the tensor of transition

$$T^i_{jk} = P^i_{jk} - L^i_{jk},$$

with a simple (homogeneous) law of transformation

$$\overline{T}^i_{jk} = S^i_m Q^m_{jn} {S^{-1}}^k_j, \quad \overline{\mathbf{T}}_i = \mathbf{S}\mathbf{T}_i\mathbf{S}^{-1},$$

and the traceless tensor

$$Q^i_{jk} = T^i_{jk} - \frac{1}{4} T^l_{jl}\delta^i_k, \quad \mathbf{Q}_j = \mathbf{T}_j - \frac{1}{4}\mathrm{Tr}(\mathbf{T}_j)\mathbf{E}.$$

We put

$$\mathcal{L}_P = -\frac{1}{4}\mathrm{Tr}(\mathbf{I}_{ij}\mathbf{I}^{ij}) + \frac{\mu^2}{2}\mathrm{Tr}\,(\mathbf{Q}_i\mathbf{Q}^i), \quad \mathcal{L}_{em} = -\frac{1}{4}F_{ij}F^{ij}.$$

The general covariant and general gauge covariant Lagrangian $\mathcal{L}$ of the general gauge field takes the form

$$\mathcal{L} = \alpha\mathcal{L}_P + \beta\mathcal{L}_{em} =$$

$$\alpha[-\frac{1}{4}\mathrm{Tr}(\mathbf{I}_{ij}\mathbf{I}^{ij}) + \frac{\mu^2}{2}\mathrm{Tr}\,(\mathbf{Q}_i\mathbf{Q}^i)] + \beta[-\frac{1}{4}F_{ij}F^{ij}],$$

where μ is a constant of dimension of cm^{-1}, and α and β are dimensionless parameters, dimension of the P^i_{jk} is equal to cm^{-1} and the action is dimensionless, $\mathbf{I}^{ij} = g^{ik}g^{jl}\mathbf{I}_{kl}, \quad \mathbf{Q}^i = g^{ik}\mathbf{Q}_k.$

By varying the Lagrangian $\mathcal{L}$ with respect to $\mathbf{P}_i$, the following equation holds

$$\alpha[\frac{1}{\sqrt{g}}\mathrm{D}_i(\sqrt{g}\mathbf{I}^{ij})+\mu^2\mathbf{Q}^j]+\beta[\frac{1}{\sqrt{g}}\partial_i(\sqrt{g}F^{ij})\mathbf{E}]=0, \quad (1)$$

where $g=-\mathrm{Det}(g_{ij})$.
Taking trace from equation (1), we find that

$$\frac{1}{\sqrt{g}}\partial_i(\sqrt{g}F^{ij})=0, \quad (2)$$

since $\mathrm{Tr}(\mathbf{I}^{\mathrm{ij}})=\mathrm{Tr}(\mathbf{Q}^{\mathrm{j}})=0$. Hence,

$$\frac{1}{\sqrt{g}}\mathrm{D}_i(\sqrt{g}\mathbf{I}^{ij})+\mu^2\mathbf{Q}^j=0. \quad (3)$$

From (3) it follows that $\mathbf{Q}^i$ has to satisfy the equation

$$\frac{1}{\sqrt{g}}\mathrm{D}_i(\sqrt{g}\mathbf{Q}^i)=0, \quad (4)$$

because $\mathrm{D}_i \mathrm{D}_j(\sqrt{g}\mathbf{I}^{ij}) = 0$.
From the Lagrangian $\mathcal{L}$ it follows that in Complete General Relativity the r. h. s. of the Einstein equation (energy–momentum tensor) is defined by the curvature tensor of the general gauge field as follows:

$$T_{ij} = \alpha[-\mathrm{Tr}\,(\mathbf{I}_{ik}\mathbf{I}_j{}^k) - g_{ij}\mathcal{L}_P + +\mu^2\mathrm{Tr}\,(\mathbf{Q}_i\mathbf{Q}_j)] +$$

$$\beta[-F_{il}F_j{}^l + \frac{1}{4}F_{kl}F^{kl}g_{ij}],$$

where $\mathbf{I}_j{}^k = \mathbf{I}_{jl}g^{kl}$.
One can show that the energy–momentum tensor in question satisfies the equation

$$\nabla^i T_{ij} = 0,$$

where ∇_i denotes the covariant derivative with respect to the connection belonging to metric g_{ij}.

At last, we write the Lagrangian

$$\mathcal{L}_g = -\frac{l_g}{2} R,$$

where l_g is a constant of dimension cm^{-2} and the Einstein equation takes the form

$$l_g G_{ij} = \alpha[-\mathrm{Tr}\,(\mathbf{I}_{ik}\mathbf{I}_j{}^k) - g_{ij}\mathcal{L}_P + \mu^2 \mathrm{Tr}\,(\mathbf{Q}_i\mathbf{Q}_j)] +$$

$$\beta[-F_{il}F_j{}^l + \frac{1}{4}F_{kl}F^{kl}g_{ij}].$$

$$\mathcal{L}_P = -\frac{1}{4}\mathrm{Tr}(\mathbf{I}_{ij}\mathbf{I}^{ij}) + \frac{\mu^2}{2}\mathrm{Tr}\,(\mathbf{Q}_i\mathbf{Q}^i).$$

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From the Einstein equation it follows that the interactions between the different states of the general gauge field are realized by a graviton exchange. This interaction can be characterized by an angle of mixing. In accordance with the Einstein equation, we can put

$$\sin\varphi = \frac{\alpha}{\sqrt{\alpha^2+\beta^2}}, \quad \cos\varphi = \frac{\beta}{\sqrt{\alpha^2+\beta^2}}$$

and redefine l_g.

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Thus, the general gauge field has two states: the familiar electromagnetic field

$$F_{ij} = \mathrm{Tr}\mathbf{H}_{ij}$$

(which should be considered as its singlet state) and more general state

$$\mathbf{I}_{ij} = \mathbf{H}_{ij} - \frac{1}{4}\mathrm{Tr}\mathbf{H}_{ij}\mathbf{E}$$

that can be called the general electromagnetic field. On this reason we will call by mphoton a particle that corresponds to a definite state of the general electromagnetic field. We can treat μ as a mass of the mphoton(scaling invariance is broken). These particles are the only source of the gravitational field.

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To make a clear the spontaneous broken symmetry we first of all mention that the strength tensor $\mathbf{I}_{ij}$ can be written in terms of the irreducible tensor $\mathbf{Q}_i$ only, since

$$\mathbf{I}_{ij} = \overset{o}{\mathrm{D}}_i\, \mathbf{Q}_j - \overset{o}{\mathrm{D}}_j\, \mathbf{Q}_i + \mathbf{Q}_i \mathbf{Q}_j - \mathbf{Q}_j \mathbf{Q}_i.$$

Here $\overset{o}{\mathrm{D}}_i$ denotes the gauge covariant derivative with respect to the connection $\mathbf{L}_i$ of the ground state and, hence, $[\overset{o}{\mathrm{D}}_i, \overset{o}{\mathrm{D}}_j] = 0$. For the antisymmetric tensor F_{ij} we obtain

$$F_{ij} = \partial_i P^k_{jk} - \partial_j P^k_{ik} = \partial_i (L^k_{jk} + T^k_{jk}) - \partial_j (L^k_{ik} + T^k_{ik}) = \partial_i T^k_{jk} -$$

since $\mathrm{Tr}(\mathbf{L}_i) = \partial_i \ln |p|, \quad p = \mathrm{Det}(E^\mu_i)$. Thus, we can consider the tensor field Q^k_{il} with the constraints $Q^k_{ik} = 0$ and covariant vector field $A_i = T^k_{ik}$ as independent quantities, which obey the equations

$$\frac{1}{\sqrt{g}}\partial_i(\sqrt{g}F^{ij}) = 0, \quad F_{ij} = \partial_i A_j - \partial_j A_i, \qquad (5)$$

$$\frac{1}{\sqrt{g}} \overset{o}{\mathrm{D}}_i \left(\sqrt{g}\mathbf{I}^{ij}\right) + [\mathbf{Q}_i, \mathbf{I}^{ij}] + \mu^2 \mathbf{Q}^j = 0, \qquad (6)$$

and

$$\frac{1}{\sqrt{g}} \overset{o}{\mathrm{D}}_i \left(\sqrt{g}\mathbf{Q}^i\right) + [\mathbf{Q}_i, \mathbf{Q}^i] = 0. \qquad (7)$$

Since the trace of $\mathbf{I}_{ij}$ is equal to zero, it is clear why we need to consider an traceless tensor $\mathbf{Q}^i$. In our case, the trace of $\mathbf{Q}^i$ is trivial and equation (7) is compatible.

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It is interesting to pay attention to the following analogy between gravity and electromagnetism. In the electron theory of Lorentz, the right– hand side of the Maxwell equations was presented with continues phenomenological distributions of charge and current. With the discovery of quantum mechanics or more exactly the Schrodinger and Dirac equations the details of the right– hand side in this case were clarified. But the physical content of the Maxwell–Dirac equations was disclosed only in the framework of quantum electrodynamics. We see the same situation in the Einstein theory of the gravitational field. It is clear that the investigation of Complete General Relativity as a closed system in the framework of quantum field theory is an urgent problem. Hence we need to look for hidden possibilities to solve the renormalization problem.

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Whilst the need for invisible matter was established almost a century ago, only its gravitational interaction has been conformed so far. Hence, it is natural to suppose that mphotons can represent the invisible matter and there is no reason for a plethora of models for this matter often called dark matter . In the framework of Complete General Relativity the photons represent the Cosmic Microwave Background and the mphotons correspond to the so called Weakly Interacting Massive Particles (WIMPs). But from the Einstein equation it follows that the Cosmic Microwave Background and invisible matter are tightly connected and, hence, the investigations of CMB can provide the discovery of hidden properties of WIMPs (or mphotons).

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We believe that main equations of Complete General Relativity provide a justified basis for discovering realistic cosmological models and new insight into the problem of the gravitational waves.

THANK YOU FOR ATTENTION