A parafermionic hypergeometric function and supersymmetric 6j-symbols .

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Fusion matrix

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}(n^3 - n)\delta_{n,-m} + \text{right copy}$$

Bulk OPE has the form

$$\Phi_i(z_1, \bar{z}_1)\Phi_j(z_2, \bar{z}_2) = \sum_k \frac{C_{ij}^k}{|z_1 - z_2|^{\Delta_i + \Delta_j - \Delta_k}} \Phi_k(z_2, \bar{z}_2) + \ldots .$$

By the BPZ arguments we have for 4-point correlation function $\langle \Phi_i(\infty)\Phi_k(1)\Phi_j(x,\bar{x})\Phi_l(0)\rangle$ in s and t channels

$$\sum_p C_{jl}^p C_{kp}^i \mathcal{F}_p^s \begin{bmatrix} k & j \\ i & l \end{bmatrix} (x) \bar{\mathcal{F}}_p^s \begin{bmatrix} k & j \\ i & l \end{bmatrix} (\bar{x})$$

$$\sum_q C_{kj}^q C_{ql}^i \mathcal{F}_q^t \begin{bmatrix} k & j \\ i & l \end{bmatrix} (x) \bar{\mathcal{F}}_q^t \begin{bmatrix} k & j \\ i & l \end{bmatrix} (\bar{x}),$$

where $\mathcal{F}_p^s \left[\begin{array}{cc} k & j \\ i & l \end{array} \right] (x)$ and $\mathcal{F}_q^t \left[\begin{array}{cc} k & j \\ i & l \end{array} \right] (x)$ are s and t channels conformal blocks correspondingly. Conformal blocks in s and t channels are related by the fusing matrix

$$\mathcal{F}_p^s \left[\begin{array}{cc} k & j \\ i & l \end{array} \right] (x) = \sum_q F_{p,q} \left[\begin{array}{cc} k & j \\ i & l \end{array} \right] \mathcal{F}_q^t \left[\begin{array}{cc} k & j \\ i & l \end{array} \right] (x)\,, \qquad (0.1)$$

and hence one has:

$$\sum_p C_{jl}^p C_{kp}^i F_{pq} \left[\begin{array}{cc} k & j \\ i & l \end{array} \right] F_{pq}^* \left[\begin{array}{cc} k & j \\ i & l \end{array} \right] = C_{kj}^q C_{ql}^i\,. \qquad (0.2)$$

Liouville field theory

Let us review basic facts on the Liouville field theory. Liouville field theory is defined on a two-dimensional surface with metric g_{ab} by the local Lagrangian density

$$\mathcal{L} = \frac{1}{4\pi} g_{ab} \partial_a \varphi \partial_b \varphi + \mu e^{2b\varphi} + \frac{Q}{4\pi} R\varphi \,, \tag{0.3}$$

where R is associated curvature. This theory is conformal invariant if the coupling constant b is related with the background charge Q as

$$Q = b + \frac{1}{b} \,. \tag{0.4}$$

The symmetry algebra of this conformal field theory is the Virasoro algebra

$$[L_m, L_n] = (m-n) L_{m+n} + \frac{c_L}{12}(n^3 - n)\delta_{n,-m} \tag{0.5}$$

with the central charge

$$c_L = 1 + 6Q^2 \,. \tag{0.6}$$

Primary fields V_α in this theory, which are associated with exponential fields $e^{2\alpha\varphi}$, have conformal dimensions

$$\Delta_\alpha = \alpha(Q - \alpha)\,. \tag{0.7}$$

$$F_{\alpha_s,\alpha_t}\left[\begin{array}{cc} \alpha_3 & \alpha_2 \\ \alpha_4 & \alpha_1 \end{array}\right] = \frac{N(\alpha_4,\alpha_3,\alpha_s)N(\alpha_s,\alpha_2,\alpha_t)}{N(\alpha_4,\alpha_t,\alpha_1)N(\alpha_t,\alpha_3,\alpha_2)}\left\{\begin{array}{cc|c} \alpha_2 & \alpha_1 & \alpha_s \\ \alpha_4 & \alpha_3 & \alpha_t \end{array}\right\}_b \tag{0.8}$$

6j-symbols

6j-symbols of the Faddeev modular quantum double:
$U_q(sl(2,\mathbb{R})\oplus U_{\tilde{q}}(sl(2,\mathbb{R}),\ q=e^{\pi i b^2}$ and $\tilde{q}=e^{\pi i b^{-2}}$.

$$\left\{\begin{smallmatrix}\alpha_1 & \alpha_2 \\ \alpha_3 & \alpha_4\end{smallmatrix}\middle| \begin{smallmatrix}\alpha_s \\ \alpha_t\end{smallmatrix}\right\}_b=\frac{S_b(\alpha_s+\alpha_2-\alpha_1)S_b(\alpha_1+\alpha_t-\alpha_4)}{S_b(\alpha_t+\alpha_2-\alpha_3)S_b(\alpha_3+\alpha_s-\alpha_4)}|S_b(2\alpha_t)|^2 J_h(\beta_a^\circ,\gamma_a^\circ)$$

$$J_h(\beta_a^\circ,\gamma_a^\circ)=\int_{-i\infty}^{i\infty}\prod_{a=1}^{4}S_b(z+\gamma_a^\circ)S_b(-z+\beta_a^\circ)\frac{dz}{i}$$

$$S_b(x)=\gamma^{(2)}(x,b,b^{-1}) \tag{0.9}$$

where $\gamma^{(2)}(x,\omega_1,\omega_2)$ is hyperbolic gamma function.

Ponsot-Teschner parametrization

$$\begin{aligned}
\gamma_1^\circ &= -Q/2 + \alpha_3 - \alpha_4\,, & \beta_1^\circ &= Q/2 + \alpha_s\,,\\
\gamma_2^\circ &= -Q/2 + \alpha_1 - \alpha_2\,, & \beta_2^\circ &= Q/2 - \alpha_t + \alpha_4 + \alpha_2\,,\\
\gamma_3^\circ &= Q/2 - \alpha_3 - \alpha_4\,, & \beta_3^\circ &= -Q/2 + \alpha_t + \alpha_4 + \alpha_2\,,\\
\gamma_4^\circ &= Q/2 - \alpha_1 - \alpha_2\,, & \beta_4^\circ &= 3Q/2 - \alpha_s\,.
\end{aligned} \tag{0.10}$$

$$\sum_{a=1}^{4}(\gamma_a^\circ + \beta_a^\circ) = 2Q\,. \tag{0.11}$$

The function $\gamma^{(2)}(y;\omega_1,\omega_2)$ has the integral representation

$$\gamma^{(2)}(y;\omega_1,\omega_2)=\exp\left(-\int_0^\infty\left(\frac{\sinh(2y-\omega_1-\omega_2)x}{2\sinh(\omega_1 x)\sinh(\omega_2 x)}-\frac{2y-\omega_1-\omega_2}{2\omega_1\omega_2 x}\right)\right.$$

and obeys the equations:

$$\frac{\gamma^{(2)}(y+\omega_1;\omega_1,\omega_2)}{\gamma^{(2)}(y;\omega_1,\omega_2)}=2\sin\frac{\pi y}{\omega_2}\,,\quad \frac{\gamma^{(2)}(y+\omega_2;\omega_1,\omega_2)}{\gamma^{(2)}(y;\omega_1,\omega_2)}=2\sin\frac{\pi y}{\omega_1}.$$

The function $\gamma^{(2)}(y;\omega_1,\omega_2)$ has the following asymptotics :

$$\lim_{y\to\infty} e^{\frac{i\pi}{2}B_{2,2}(y;\omega_1,\omega_2)}\gamma^{(2)}(y;\omega_1,\omega_2)=1,\ \arg\,\omega_1<\arg\,y<\arg\,\omega_2+\pi,$$

$$\lim_{y\to\infty} e^{-\frac{i\pi}{2}B_{2,2}(y;\omega_1,\omega_2)}\gamma^{(2)}(y;\omega_1,\omega_2)=1,\ \arg\,\omega_1-\pi<\arg\,y<\arg\,\omega_2,$$

where $B_{2,2}(y;\omega_1,\omega_2)$ is the second order Bernoulli polynomial:

$$B_{2,2}(y;\omega_1,\omega_2)=\frac{y^2}{\omega_1\omega_2}-\frac{y}{\omega_1}-\frac{y}{\omega_2}+\frac{1}{6}\left(\frac{\omega_1}{\omega_2}+\frac{\omega_2}{\omega_1}\right)+\frac{1}{2}.$$

Define

$$J_h(\underline{\beta},\underline{\gamma}) = \int_{-\mathrm{i}\infty}^{\mathrm{i}\infty} \prod_{a=1}^{4} \gamma^{(2)}(\beta_a - z;\omega_1,\omega_2)\gamma^{(2)}(\gamma_a + z;\omega_1,\omega_2)\frac{dz}{\mathrm{i}\sqrt{\omega_1\omega_2}} \tag{0.12}$$

with the parameters β_a, γ_a satisfying the balancing condition

$$\sum_{a=1}^{4}(\gamma_a + \beta_a) = 2(\omega_1 + \omega_2)\,. \tag{0.13}$$

$W(E_7)$-group transformation laws: η-symmetry

Consider the function $I_h(\underline{s})$ defined by the integral

$$I_h(\underline{s}) = \int_{-i\infty}^{i\infty} \frac{\prod_{j=1}^{8} \gamma^{(2)}(s_j \pm z;\omega)}{\gamma^{(2)}(\pm 2z;\omega)} \frac{dz}{2i\sqrt{\omega_1\omega_2}}, \qquad (0.14)$$

with s_j satisfying the conditions $\mathrm{Re}(s_j) > 0$ and

$$\sum_{j=1}^{8} s_j = 2Q, \quad Q := \omega_1 + \omega_2. \qquad (0.15)$$

$$I_h(\underline{s}) = \prod_{1\leq j<k\leq 4} \gamma^{(2)}(s_j + s_k;\omega) \prod_{5\leq j<k\leq 8} \gamma^{(2)}(s_j + s_k;\omega)\, I_h(\underline{\tilde{s}}),$$

where

$$\tilde{s}_j = s_j + \eta, \quad \tilde{s}_{j+4} = s_{j+4} - \eta, \quad j = 1,2,3,4, \quad \eta = \frac{1}{2}(\omega_1 + \omega_2 - \sum_{j=1}^{4} s_j).$$

Let us parametrize

$$s_{1,2,5,6} = \gamma_{1,2,3,4} + \mathrm{i}\mu, \qquad s_{3,4,7,8} = \beta_{1,2,3,4} - \mathrm{i}\mu \tag{0.16}$$

in (0.14), shift the integration variable $z \to z - \mathrm{i}\mu$ on both sides of this relation, and take the limit $\mu \to -\infty$.

$$J_h(\underline{\beta}, \underline{\gamma}) = \prod_{j,k=1}^{2} \gamma^{(2)}(\beta_j + \gamma_k; \omega) \prod_{j,k=3}^{4} \gamma^{(2)}(\beta_j + \gamma_k; \omega) \tag{0.17}$$

$$\times J_h(\beta_1 + \eta, \beta_2 + \eta, \beta_3 - \eta, \beta_4 - \eta, \gamma_1 + \eta, \gamma_2 + \eta, \gamma_3 - \eta, \gamma_4 - \eta),$$

$$\eta = \tfrac{1}{2}(\omega_1 + \omega_2 - (\beta_1 + \beta_2 + \gamma_1 + \gamma_2)).$$

$W(E_7)$-group transformation laws: reflection

$$I_h(\underline{s}) = \prod_{1\leq j<k\leq 8} \gamma^{(2)}(s_j+s_k;\omega) I_h\left(\underline{\lambda}\right), \qquad \lambda_j = \frac{\omega_1+\omega_2}{2} - s_j. \tag{0.18}$$

For deriving another symmetry transformation, we replace in (0.18)

$$s_j \to \gamma_j + \mathrm{i}\mu, \qquad s_{j+4} = \beta_j - \mathrm{i}\mu, \qquad j=1,\ldots,4, \qquad z \to z-\mathrm{i}\mu.$$

The balancing condition takes the form $\sum_{j=1}^{4}(\beta_j+\gamma_j) = 2Q$. After taking the limit $\mu \to -\infty$, we come to the identity

$$\int_{-\mathrm{i}\infty}^{\mathrm{i}\infty}\prod_{j=1}^{4}\gamma^{(2)}(\gamma_j+z;\omega)\gamma^{(2)}(\beta_j-z;\omega)\mathrm{d}z=\prod_{j,k=1}^{4}\gamma^{(2)}(\gamma_j+\beta_k;\omega)$$

$$\times\int_{-\mathrm{i}\infty}^{\mathrm{i}\infty}\prod_{j=1}^{4}\gamma^{(2)}\left(\tfrac{1}{2}Q-\beta_j+z;\omega\right)\gamma^{(2)}\left(\tfrac{1}{2}Q-\gamma_j-z;\omega\right)\mathrm{d}z. \quad (0.19)$$

$W(E_7)$-group transformation laws: η+reflection or Teschner-Vartanov

$$I_h(\underline{s}) = \prod_{j,k=1}^{4} \gamma^{(2)}(s_j + s_{k+4}; \omega) I_h(G - s_1, \ldots, G - s_4, \qquad (0.20)$$
$$Q - G - s_5, \ldots, Q - G - s_8),$$

where $G := \frac{1}{2}\sum_{j=1}^{4} s_j$ and $Q = \omega_1 + \omega_2$. Now, for the parametrization (0.16), the $\mu \to 0$ limiting procedure yields

$$J(\underline{\beta}, \underline{\gamma}) = \prod_{j,k=1}^{2} \gamma^{(2)}(\gamma_j + \beta_{k+2}; \omega)\gamma^{(2)}(\gamma_{j+2} + \beta_k; \omega) J(G - \gamma_1, G - \gamma_2,$$

$$Q - G - \gamma_3, Q - G - \gamma_4; G - \beta_1, G - \beta_2, Q - G - \beta_3, Q - G - \beta_4),$$

where $G = \frac{1}{2}(\gamma_1 + \gamma_2 + \beta_1 + \beta_2)$.

$$J(\underline{\beta}^{\circ}, \underline{\gamma}^{\circ}) = \Omega(\underline{\alpha}) J(\underline{\beta}^{\diamond}, \underline{\gamma}^{\diamond}) \tag{0.21}$$

where

$$\begin{array}{llll} \gamma_1^{\diamond} = \alpha_{1234} \, , & \gamma_3^{\diamond} = 2Q \, , & \beta_1^{\diamond} = -\alpha_{23t} \, , & \beta_3^{\diamond} = -\alpha_{12s} \, , \\ \gamma_2^{\diamond} = \alpha_{13st} \, , & \gamma_4^{\diamond} = \alpha_{24st} \, , & \beta_2^{\diamond} = -\alpha_{14t} \, , & \beta_4^{\diamond} = -\alpha_{34s} \, , \end{array}$$

$$\alpha_{ijk} \equiv \alpha_i + \alpha_j + \alpha_k \, , \quad \alpha_{ijkl} \equiv \alpha_i + \alpha_j + \alpha_k + \alpha_l$$

and

and

$$\Omega(\underline{\alpha}) = \gamma^{(2)}(Q+\alpha_s-\alpha_3-\alpha_4;\omega)\gamma^{(2)}(Q+\alpha_s-\alpha_1-\alpha_2;\omega)$$
$$\times\gamma^{(2)}(Q-\alpha_t+\alpha_2-\alpha_3;\omega)\gamma^{(2)}(Q-\alpha_t+\alpha_4-\alpha_1;\omega)$$
$$\times\gamma^{(2)}(-Q+\alpha_t+\alpha_2+\alpha_3;\omega)\gamma^{(2)}(Q-\alpha_s+\alpha_3-\alpha_4;\omega)$$
$$\times\gamma^{(2)}(-Q+\alpha_t+\alpha_4+\alpha_1;\omega)\gamma^{(2)}(Q-\alpha_s+\alpha_1-\alpha_2;\omega).$$

The equality (0.21) was derived by Teschner and Vartanov in a more complicated way and it was used for derivation of the hyperbolic volume of a non-ideal tetrahedron in the quasi-classical limit of $6j$-symbols for the Faddeev modular double.

Comment on Regge symmetry

$$\left\{ \begin{matrix} \alpha_1 & \alpha_2 \\ \alpha_3 & \alpha_4 \end{matrix} \middle| \begin{matrix} \alpha_s \\ \alpha_t \end{matrix} \right\} = \left\{ \begin{matrix} S-\alpha_1 & S-\alpha_2 \\ S-\alpha_3 & S-\alpha_4 \end{matrix} \middle| \begin{matrix} \alpha_s \\ \alpha_t \end{matrix} \right\} \tag{0.22}$$

where

$$S = \frac{1}{2}(\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4) \tag{0.23}$$

Setting

$$\alpha_1 = Q/2 - \mathrm{i}(N_1/2 + \sigma_1\delta), \qquad \alpha_3 = Q/2 - \mathrm{i}(N_3/2 + \sigma_3\delta),$$
$$\alpha_2 = Q/2 - \mathrm{i}(N_2/2 + \sigma_2\delta), \qquad \alpha_4 = Q/2 + \mathrm{i}(N_4/2 + \sigma_4\delta),$$

$$\alpha_t = Q/2 - \mathrm{i}(M_1/2 + \rho_1\delta), \quad \alpha_s = Q/2 + \mathrm{i}(M_2/2 + \rho_2\delta),$$

$$\lim_{\delta\to i} \left\{ \begin{matrix} \alpha_1 & \alpha_2 \\ \alpha_3 & \alpha_4 \end{matrix} \middle| \begin{matrix} \alpha_s \\ \alpha_t \end{matrix} \right\}_{b=i+\delta} \propto \left\{ \begin{matrix} \sigma_1, N_1 & \sigma_2, N_2 \\ \sigma_3, N_3 & \sigma_4, N_4 \end{matrix} \middle| \begin{matrix} \rho_1, M_1 \\ \rho_2, M_2 \end{matrix} \right\}$$

where on the rhs we have $6j$-symbols for the group $SL(2,\mathbb{C})$.

$$\left\{ \begin{matrix} -j_1 b & -j_2 b \\ -j_3 b & -j_4 b \end{matrix} \middle| \begin{matrix} -j_s b \\ -j_t b \end{matrix} \right\}_b \propto \left\{ \begin{matrix} j_1 & j_2 \\ j_3 & j_4 \end{matrix} \middle| \begin{matrix} j_s \\ j_t \end{matrix} \right\}_q \tag{0.24}$$

where, $2j_i \in \mathbb{Z}$, $q = e^{i\pi b^2}$ and on the rhs we have $6j$-symbols for the $\mathrm{U_q(su(2))}$.

N=1 Super Liouville field theory

Let us review basic facts on the $N=1$ Super Liouville field theory. $N=1$ super Liouville field theory is defined on a two-dimensional surface with metric g_{ab} by the local Lagrangian density

$$\mathcal{L}=\frac{1}{2\pi}g_{ab}\partial_a\varphi\partial_b\varphi+\frac{1}{2\pi}(\psi\bar{\partial}\psi+\bar{\psi}\partial\bar{\psi})+2i\mu b^2\bar{\psi}\psi e^{b\varphi}+2\pi\mu^2b^2e^{2b\varphi}\,, \tag{0.25}$$

The energy-momentum tensor and the superconformal current are

$$T=-\frac{1}{2}(\partial\varphi\partial\varphi-Q\partial^2\varphi+\psi\partial\psi)\,, \tag{0.26}$$

$$G=i(\psi\partial\varphi-Q\partial\psi)\,. \tag{0.27}$$

The superconformal algebra is

$$[L_m,L_n]=(m-n)L_{m+n}+\frac{c}{12}m(m^2-1)\delta_{m+n}\,, \tag{0.28}$$

$$[L_m, G_k] = \frac{m-2k}{2} G_{m+k}\,, \tag{0.29}$$

$$\{G_k, G_l\} = 2L_{l+k} + \frac{c}{3}\left(k^2 - \frac{1}{4}\right)\delta_{k+l}\,, \tag{0.30}$$

with the central charge

$$c_L = \frac{3}{2} + 3Q^2\,. \tag{0.31}$$

where

$$Q = b + \frac{1}{b}\,. \tag{0.32}$$

Here k and l take integer values for the Ramond algebra and half-integer values for the Neveu-Schwarz algebra.

NS primary fields N_α in this theory, $N_\alpha = e^{\alpha\varphi}$, have conformal dimensions

$$\Delta_\alpha^{NS} = \frac{1}{2}\alpha(Q-\alpha)\,. \tag{0.33}$$

The physical states have $\alpha = \frac{Q}{2} + iP$.
Introduce also the field

$$\tilde{N}_\alpha = G_{-1/2}N_\alpha = -i\alpha\psi e^{\alpha\varphi}. \tag{0.34}$$

The R-R is defined as

$$R_\alpha^\pm = \sigma^\pm e^{\alpha\varphi}\,, \tag{0.35}$$

where σ is the spin field . The dimension of the R-R operator is

$$\Delta_\alpha^R = \frac{1}{16} + \frac{1}{2}\alpha(Q-\alpha)\,. \tag{0.36}$$

Supersymmetric Racah-Wigner symbols for the supergroup $\mathrm{U}_q(\mathrm{osp}(1|2))$

$$S_b\left(\frac{y}{2}\right)S_b\left(\frac{y}{2}+\frac{Q}{2}\right)\equiv S_{\mathrm{NS}}(y)\equiv S_1(y),$$

$$S_b\left(\frac{y}{2}+\frac{b}{2}\right)S_b\left(\frac{y}{2}+\frac{b^{-1}}{2}\right)\equiv S_{\mathrm{R}}(y)\equiv S_0(y).$$

The subscript a of $S_a(y)$ is defined mod 2: $S_{a+2}(y)\equiv S_a(y)$.

$$\left\{\begin{smallmatrix}\alpha_1 & \alpha_2 \\ \alpha_3 & \overline{\alpha}_4\end{smallmatrix}\middle|\begin{smallmatrix}\alpha_s \\ \alpha_t\end{smallmatrix}\right\}^{\nu_3\nu_4}_{\nu_1\nu_2}=$$
$$\frac{S_{\nu_4}(\alpha_s+\alpha_2-\alpha_1)S_{\nu_1}(\alpha_1+\alpha_t-\alpha_4)}{S_{\nu_2}(\alpha_t+\alpha_2-\alpha_3)S_{\nu_3}(\alpha_3+\alpha_s-\alpha_4)}I_{\alpha_s,\alpha_t}\left[\begin{array}{cc}\alpha_3 & \alpha_2\\ \alpha_4 & \alpha_1\end{array}\right]^{\nu_3\nu_4}_{\nu_1\nu_2},$$

where $\overline{\alpha}_4=Q-\alpha_4$, $\nu_i=0,1,\ i=1,2,3,4$, $\sum_{i=1}^4\nu_i=0 \mod 2$, and

$$I_{\alpha_s,\alpha_t}\left[\begin{array}{cc}\alpha_3 & \alpha_2\\ \alpha_4 & \alpha_1\end{array}\right]_{\nu_1\nu_2}^{\nu_3\nu_4} =$$

$$(-1)^{\nu_3\nu_2+\nu_4}\int_{-\mathrm{i}\infty}^{\mathrm{i}\infty}\sum_{\nu=0}^{1}(-1)^{\nu(\nu_2+\nu_4)}S_{1+\nu+\nu_3}(y+\gamma_1^\circ)S_{1+\nu+\nu_4}(y+\gamma_2^\circ)$$

$$\times \quad S_{1+\nu+\nu_3}(y+\gamma_3^\circ)S_{1+\nu+\nu_4}(y+\gamma_4^\circ)S_{\nu}(-y+\beta_1^\circ)S_{\nu+\nu_2+\nu_3}(-y+\beta_2^\circ)$$

$$\times \quad S_{\nu+\nu_2+\nu_3}(-y+\beta_3^\circ)S_{\nu}(-y+\beta_4^\circ)\frac{dy}{2\mathrm{i}}. \qquad (0.37)$$

$$F_{N_{\alpha_s},N_{\alpha_t}}\left[\begin{array}{cc} N_{\alpha_3} & N_{\alpha_2} \\ N_{\alpha_4} & N_{\alpha_1} \end{array}\right] \propto \left\{\begin{array}{cc|c} \alpha_1 & \alpha_2 & \alpha_s \\ \alpha_3 & \overline{\alpha}_4 & \alpha_t \end{array}\right\}^{11}_{11} \tag{0.38}$$

$$F_{N_{\alpha_s},\tilde{N}_{\alpha_t}}\left[\begin{array}{cc} N_{\alpha_3} & N_{\alpha_2} \\ N_{\alpha_4} & N_{\alpha_1} \end{array}\right] \propto \left\{\begin{array}{cc|c} \alpha_1 & \alpha_2 & \alpha_s \\ \alpha_3 & \overline{\alpha}_4 & \alpha_t \end{array}\right\}^{11}_{00} \tag{0.39}$$

$$F_{\tilde{N}_{\alpha_s},N_{\alpha_t}}\left[\begin{array}{cc} N_{\alpha_3} & N_{\alpha_2} \\ N_{\alpha_4} & N_{\alpha_1} \end{array}\right] \propto \left\{\begin{array}{cc|c} \alpha_1 & \alpha_2 & \alpha_s \\ \alpha_3 & \overline{\alpha}_4 & \alpha_t \end{array}\right\}^{00}_{11} \tag{0.40}$$

$$F_{\tilde{N}_{\alpha_s},\tilde{N}_{\alpha_t}}\left[\begin{array}{cc} N_{\alpha_3} & N_{\alpha_2} \\ N_{\alpha_4} & N_{\alpha_1} \end{array}\right] \propto \left\{\begin{array}{cc|c} \alpha_1 & \alpha_2 & \alpha_s \\ \alpha_3 & \overline{\alpha}_4 & \alpha_t \end{array}\right\}^{00}_{00} \tag{0.41}$$

General Supersymmetric Racah-Wigner symbols

$$\left\{ \begin{array}{ccc} \alpha_1^{a_1} & \alpha_3^{a_3} & \alpha_s^{a_s} \\ \alpha_2^{a_2} & \alpha_4^{a_4} & \alpha_t^{a_t} \end{array} \right\}_{\nu_1\nu_2}^{\nu_3\nu_4} =$$

$$\mathcal{P}(\alpha_i,\nu_i)\int_{\mathcal{C}} du \sum_{\nu=0}^{1}\Big((-1)^X S_{1+\nu_3+\nu_4+\nu}(u-\alpha_{12s})$$
$$\times S_{1+\nu}(u-\alpha_{s34})S_{1+\nu_1+\nu_4+\nu}(u-\alpha_{23t})S_{1+\nu_2+\nu_4+\nu}(u-\alpha_{1t4})$$
$$\times S_{\nu_4+\nu}(\alpha_{1234}-u)S_{\nu_1+\nu+a_1}(\alpha_{st13}-u)S_{\nu_2+\nu+a_2}(\alpha_{st24}-u)$$
$$S_{\nu_3+\nu+a_s}(2Q-u)\Big)\,,$$

where

$$(-1)^X = (-1)^{\nu(a_s\nu_1+a_1\nu_3+a_4\nu_4+a_1a_s+a_2a_4+a_s+a_t)}\,,$$

and

$$\sum_{i=1}^{4} \nu_i = a_s + a_t \mod 2\,, \qquad (0.42)$$

$$a_s = a_1 + a_2 = a_3 + a_4 \mod 2, \qquad a_t = a_1 + a_4 = a_2 + a_3 \mod 2.$$

- What is reason of all weird signs?
- If for the case when all $a_i = 0$ the expression for the susy 6j symbols in the Teschner-Vartanov parametrization is equivalent to the expression in the Ponsot-Teschner parametrization?
- What is expression for the susy 6j symbols in the Ponsot-Teschner parametrization for general values of a_i?

Lens hyperbolic gamma functions

$$\Lambda(y;m;\omega) = \gamma^{(2)}\left(\frac{y+m\omega_1}{r};\omega_1,\frac{\omega_1+\omega_2}{r}\right)$$
$$\times\gamma^{(2)}\left(\frac{y+(r-m)\omega_2}{r};\omega_2,\frac{\omega_1+\omega_2}{r}\right).$$

For the fundamental region $0 \leq m \leq r$ one has

$$\Lambda(y,m,\omega) = \prod_{k=0}^{m-1}\gamma^{(2)}\left(\frac{y}{r}+\omega_2\left(1-\frac{m}{r}\right)+(\omega_1+\omega_2)\frac{k}{r};\omega\right)$$
$$\times\prod_{k=0}^{r-m-1}\gamma^{(2)}\left(\frac{y}{r}+\frac{m}{r}\omega_1+(\omega_1+\omega_2)\frac{k}{r};\omega\right), \qquad (0.43)$$

$$\Lambda(y,m+kr;\omega) = (-1)^{mk+r\frac{k(k-1)}{2}}\Lambda(y,m;\omega). \qquad (0.44)$$

$$\lim_{y\to\infty}\Lambda(y;m;\omega) = e^{-\frac{\mathrm{i}\pi}{2}\left(\frac{1}{r}B_{2,2}(y;\omega)+\frac{m^2}{r}-m+\frac{r}{6}-\frac{1}{6r}\right)},$$

$$\text{for } \arg\omega_1 < \arg y < \arg\omega_2 + \pi,$$

$$\lim_{y\to\infty}\Lambda(y;m;\omega) = e^{\frac{\mathrm{i}\pi}{2}\left(\frac{1}{r}B_{2,2}(y;\omega)+\frac{m^2}{r}-m+\frac{r}{6}-\frac{1}{6r}\right)},$$

$$\text{for } \arg\omega_1 - \pi < \arg y < \arg\omega_2.$$

Recall the relation between $S_{\rm NS}(x)$, $S_{\rm R}(x)$ and the function $\Lambda(y,m;\omega)$ for $r=2$. Setting $\omega_2=b$ and $\omega_1=b^{-1}$, $Q=b+b^{-1}$, and using the notation accepted in conformal field theory literature $\gamma^{(2)}(z;b,1/b)=: S_b(z)$, we obtain

$$\Lambda(y,0;b^{-1},b)=S_b\left(\frac{y}{2}\right)S_b\left(\frac{y}{2}+\frac{Q}{2}\right)\equiv S_{\rm NS}(y)\equiv S_1(y), \tag{0.45}$$

$$\Lambda(y,1;b^{-1},b)=S_b\left(\frac{y}{2}+\frac{b}{2}\right)S_b\left(\frac{y}{2}+\frac{b^{-1}}{2}\right)\equiv S_{\rm R}(y)\equiv S_0(y). \tag{0.46}$$

The subscript a of $S_a(y)$ is defined mod 2: $S_{a+2}(y)\equiv S_a(y)$. Let us describe the relation between $\Lambda(y,m;b^{-1},b)$ and $S_a(y)$ for arbitrary m. Recalling formula (0.44), we see that for $r=2$ it takes a simple form:

$$\Lambda(y,m+2k;\omega)=(-1)^{mk}\Lambda(y,m;\omega)\,. \tag{0.47}$$

Formulae (0.45), (0.46), and (0.47) imply

$$\Lambda(y, K; b^{-1}, b) = S_{\mathrm{NS}}(y) \equiv S_1(y), \quad \text{if} \quad K \quad \text{is} \quad \text{even},$$

$$\Lambda(y, K; b^{-1}, b) = (-1)^{\frac{K-1}{2}} S_{\mathrm{R}}(y) \equiv S_0(y), \quad \text{if} \quad K \quad \text{is} \quad \text{odd},$$

or, combining these equalities, we have

$$\Lambda(y, K; b^{-1}, b) = (-1)^{F(K)} S_{K+1}(y), \qquad F(K) = \frac{1-(-1)^K}{2}\frac{K-1}{2}.$$

Parafermionic hypergeometric function

For some positive integer r we define the function

$$J(\underline{\beta},\underline{l};\underline{\gamma},\underline{t}) = \int_{-\mathrm{i}\infty}^{\mathrm{i}\infty} \sum_{m\in\mathbb{Z}_r} \prod_{a=1}^{4} \Lambda(-y+\beta_a; l_a - m;\omega) \tag{0.48}$$

$$\times \prod_{a=1}^{4} \Lambda(y+\gamma_a; t_a + m;\omega)\frac{dy}{\mathrm{i}r\sqrt{\omega_1\omega_2}} \ .$$

Here t_a , $l_a \in \mathbb{Z}$. Parameters γ_j, β_j and l_j, t_j satisfy the constraints:

$$\sum_{j=1}^{4}(\gamma_j+\beta_j) = 2Q, \quad \sum_{j=1}^{4}(l_j+t_j) = 0. \tag{0.49}$$

Define the function:

$$W_\epsilon(\underline{s},\underline{n};\omega)=\int_{-\mathrm{i}\infty}^{\mathrm{i}\infty}\sum_{m\in\mathbb{Z}_r+\epsilon}\frac{\prod_{a=1}^{8}\Lambda(s_a\pm y;n_a\pm m;\omega)}{\Lambda(\pm 2y;\pm 2m;\omega)}\frac{dy}{2\mathrm{i}r\sqrt{\omega_1\omega_2}}$$

where $\Lambda(x\pm y;n\pm m;\omega)=\Lambda(x+y;n+m;\omega)\Lambda(x-y;n-m;\omega)$ and $n_a\in\mathbb{Z}+\epsilon\,,\epsilon=0,\frac{1}{2}$. Also the following balancing constraints on the parameters s_j and n_j hold true:

$$\sum_{j=1}^{8}s_j=2Q,\qquad \sum_{j=1}^{8}n_j=0.$$

Parafermionic Symmetry I

$$W_\epsilon(\underline{s},\underline{n};\omega) = W_\delta(\underline{\tilde{s}},\underline{\tilde{n}};\omega) \tag{0.50}$$
$$\times \prod_{1\le j<k\le 4} \Lambda(s_j+s_k;n_j+n_k;\omega) \prod_{5\le j<k\le 8} \Lambda(s_j+s_k;n_j+n_k;\omega)\,,$$
$$\tilde{s}_j = s_j+\xi, \quad \tilde{s}_{j+4} = s_{j+4}-\xi, \quad j=1,2,3,4, \tag{0.51}$$
$$\xi = \frac{1}{2}(\omega_1+\omega_2-\sum_{j=1}^{4} s_j)$$

$$\begin{cases} \tilde{n}_a = n_a - \frac{1}{2}(\sum_{b=1}^4 n_b), & a=1,2,3,4, \\ \tilde{n}_a = n_a + \frac{1}{2}(\sum_{b=1}^4 n_b), & a=5,6,7,8, \end{cases} \tag{0.52}$$

Here $\delta = 0, \frac{1}{2}$, and one should take $\delta=\epsilon$, if $\frac{1}{2}\left(\sum_{b=1}^4 n_b\right)$ is an integer, or otherwise $\delta \neq \epsilon$.

Let us parametrize the variables n_a and s_a in (0.50) in the following way:

$$s_{1,2,5,6} = \gamma_{1,2,3,4} + \mathrm{i}\mu, \qquad n_{1,2,5,6} = t_{1,2,3,4}, \tag{0.53}$$
$$s_{3,4,7,8} = \beta_{1,2,3,4} - \mathrm{i}\mu, \qquad n_{3,4,7,8} = l_{1,2,3,4},$$

where γ_j, β_j and l_j, t_j satisfy the constraints (0.49).
Now shifting the integration variable $y \to y - \mathrm{i}\mu$ on both sides and taking the limit $\mu \to -\infty$, we obtain

$$J(\underline{\beta}, \underline{l}; \underline{\gamma}, \underline{t}) = e^{\mathrm{i}\pi[t_3+t_4+l_1+l_2]} J(\underline{\tilde{\beta}}, \underline{l^*}; \underline{\tilde{\gamma}}, \underline{t^*}) \tag{0.54}$$

$$\times \prod_{j,k=1}^{2} \Lambda(\gamma_j + \beta_k; t_j + l_k; \omega) \prod_{j,k=3}^{4} \Lambda(\gamma_j + \beta_k; t_j + l_k; \omega)\,, \tag{0.55}$$

$$\tilde{\beta}_a = \beta_a + \Theta(a)\eta\,, \quad \tilde{\gamma}_a = \gamma_a + \Theta(a)\eta\,,$$

$$l_a^* = l_a - (\Theta(a)+1)N, \qquad t_a^* = t_a - (\Theta(a)-1)N, \quad a = 1,2,3,4.$$

where $\Theta(a)$ is the sign function taking the values

$$\Theta(a) = 1\,, \quad a = 1,2 \quad \text{and} \quad \Theta(a) = -1\,, \quad a = 3,4 \tag{0.56}$$

and

$$N = \frac{1}{2}(t_1 + t_2 + l_1 + l_2), \quad \eta = \frac{1}{2}(Q - \gamma_1 - \gamma_2 - \beta_1 - \beta_2).$$

Parafermionic Symmetry -Reflection

$$W_\epsilon(\underline{s},\underline{n};\omega) = W_\epsilon(Q/2-\underline{s},-\underline{n};\omega)\prod_{1\le j<k\le 8}\Lambda(s_j+s_k;n_j+n_k;\omega).$$

Parameterize s_a and n_a as:

$$s_j=\gamma_j+\mathrm{i}\mu,\qquad s_{j+4}=\beta_j-\mathrm{i}\mu,\quad n_j=t_j,\quad n_{j+4}=l_j\qquad j=1,\ldots,4. \tag{0.57}$$

Now shifting the integration variables on both sides $y\to y-\mathrm{i}\mu$ of the latter identity and taking the limit $\mu\to-\infty$, we come to the relation

$$J_\epsilon(\underline{\beta},\underline{l};\underline{\gamma};\underline{t}) = e^{\mathrm{i}\pi\left[\sum_{i=1}^4 t_i\right]} \prod_{j,k=1}^{4} \Lambda(\gamma_j+\beta_k;t_j+l_k;\omega)$$

$$\times\int_{-\mathrm{i}\infty}^{\mathrm{i}\infty}\sum_{m\in\mathbb{Z}_r}\prod_{a=1}^{4}\Lambda(\frac{Q}{2}-y-\gamma_a;-t_a-m;\omega)$$

$$\times\Lambda(\frac{Q}{2}+y-\beta_a;-l_a+m;\omega)\frac{dy}{\mathrm{i}r\sqrt{\omega_1\omega_2}}.$$

Parafermionic Teschner-Vartanov formula

$$W_{\epsilon}(\underline{s},\underline{n};\omega)=W_{\rho}(\underline{\hat{s}},\underline{\hat{n}};\omega)\prod_{1\leq b,c\leq 4}\Lambda(s_b+s_{c+4};n_b+n_{c+4};\omega)\quad(0.58)$$

$$\hat{s}_j=G-s_j,\quad \hat{s}_{j+4}=Q-G-s_{j+4},\quad j=1,2,3,4,\quad G=\frac{1}{2}\left(\sum_{j=1}^{4}s_j\right),$$

$$\begin{cases}\hat{n}_a=-n_a+\frac{1}{2}\left(\sum_{\ell=1}^{4}n_\ell\right), & a=1,2,3,4,\\ \hat{n}_a=-n_a-\frac{1}{2}\left(\sum_{\ell=1}^{4}n_\ell\right), & a=5,6,7,8.\end{cases}\quad(0.59)$$

As before, the discrete parameter ρ should be taken $\rho=\epsilon$, if $\frac{1}{2}\left(\sum_{\ell=1}^{4}n_\ell\right)$ is an integer, or $\rho\neq\epsilon$ otherwise.

$$J(\underline{\beta},\underline{l};\underline{\gamma},\underline{t}) = e^{2\mathrm{i}\pi M} J(\underline{\hat{\beta}},\underline{l'};\underline{\hat{\gamma}},\underline{t'}) \times \prod_{j,k=1}^{2} \Lambda(\gamma_j+\beta_{k+2};t_j+l_{k+2};\omega)\Lambda(\gamma_{j+2}+\beta_k;t_{j+2}+l_k;\omega),$$

$$\hat{\beta}_a = Q/2+\Theta(a)(G-Q/2)-\gamma_a, \qquad l'_a = -t_a+(\Theta(a)-1)M,$$
$$\hat{\gamma}_a = Q/2+\Theta(a)(G-Q/2)-\beta_a, \qquad t'_a = -l_a+(\Theta(a)+1)M,$$
$$a=1,2,3,4.$$

$$M=\frac{1}{2}(t_1+t_2+l_1+l_2), \qquad G=\frac{1}{2}(\gamma_1+\gamma_2+\beta_1+\beta_2). \quad (0.60)$$

$$\Theta(a)=1\,, \quad a=1,2 \quad \text{and} \quad \Theta(a)=-1\,, \quad a=3,4 \qquad (0.61)$$

$$J(\underline{\beta}^{\circ},\underline{l};\underline{\gamma}^{\circ},\underline{t})=e^{2i\pi M}\Omega(\underline{\alpha},\underline{t},\underline{l})J(\underline{\beta}^{\diamond},\underline{l}',\underline{\gamma}^{\diamond},\underline{t}'), \qquad (0.62)$$

where $\beta^{\diamond}_a$ and $\gamma^{\diamond}_a$ are Ponsot-Teschner parameters in and

$$\begin{aligned}\Omega(\underline{\alpha},\underline{t},\underline{l}) &= \Lambda(Q+\alpha_s-\alpha_3-\alpha_4;l_1+t_3)\Lambda(Q+\alpha_s-\alpha_1-\alpha_2;l_1+t_4)\\ &\times\Lambda(Q-\alpha_t+\alpha_2-\alpha_3;l_2+t_3)\Lambda(Q-\alpha_t+\alpha_4-\alpha_1;l_2+t_4)\\ &\times\Lambda(-Q+\alpha_t+\alpha_2+\alpha_3;l_3+t_1)\Lambda(Q-\alpha_s+\alpha_3-\alpha_4;l_4+t_1)\\ &\times\Lambda(-Q+\alpha_t+\alpha_4+\alpha_1;l_3+t_2)\Lambda(Q-\alpha_s+\alpha_1-\alpha_2;l_4+t_2).\end{aligned}$$

$$I_{\alpha_s,\alpha_t}\left[\begin{array}{cc}\alpha_3 & \alpha_2\\ \alpha_4 & \alpha_1\end{array}\right]^{\nu_3\nu_4}_{\nu_1\nu_2} = J(\underline{\beta^\circ},\underline{\gamma^\circ},\underline{\nu})$$

where

$$J(\underline{\beta^\circ},\underline{\gamma^\circ},\underline{\nu}) = \int_{-\mathrm{i}\infty}^{\mathrm{i}\infty}\sum_{\nu=0}^{1}\Lambda(y+\gamma_1^\circ,\nu_3+\nu)\Lambda(y+\gamma_2^\circ,\nu+\nu_4)$$
$$\times\Lambda(y+\gamma_3^\circ,\nu+\nu_3)\Lambda(y+\gamma_4^\circ,\nu-\nu_4)\Lambda(-y+\beta_1^\circ,-\nu-1)$$
$$\times\Lambda(-y+\beta_2^\circ,-\nu+1+\nu_2-\nu_3)\Lambda(-y+\beta_3^\circ,-\nu+1-\nu_2-\nu_3)$$
$$\Lambda(-y+\beta_4^\circ,-\nu-1)\frac{dy}{2\mathrm{i}}.$$

Note that $\sum_{i=1}^{4}(t_i+l_i)=0$.

$$I_{\alpha_s,\alpha_t}\left[\begin{array}{cc}\alpha_3 & \alpha_2\\ \alpha_4 & \alpha_1\end{array}\right]^{\nu_3\nu_4}_{\nu_1\nu_2} = (-1)^{\nu_2+\nu_4}\Omega(\underline{\alpha},\underline{\nu})J(\underline{\alpha},\underline{\nu}). \qquad (0.63)$$

$$\begin{aligned}
&J(\underline{\alpha},\underline{\nu}) = \\
&(-1)^D\int_{-\mathrm{i}\infty}^{\mathrm{i}\infty}\sum_{\nu=0}^{1} S_{\nu_2-\nu_3-\nu+1}(-y-\alpha_{23t})S_{\nu_2-\nu_4-\nu+1}(-y-\alpha_{14t})\\
&\times S_{-\nu_4-\nu_3-\nu+1}(-y-\alpha_{12s})S_{-\nu+1}(-y-\alpha_{34s})S_{\nu_4+\nu}(y+\alpha_{1234})\\
&\times S_{-\nu_2+\nu_3+\nu_4+\nu}(y+\alpha_{13st})S_{\nu_3+\nu}(y+2Q)S_{-\nu_2+\nu}(y+\alpha_{24st})\frac{dy}{2\mathrm{i}},
\end{aligned}$$

where $D = \nu_2\nu_3\nu_4 + \nu_2\nu_3 + \nu_2\nu_4 + \nu_3\nu_4 + \nu_2 + \nu_4$.

$$\Omega(\underline{\alpha},\underline{\nu}) = (-1)^{\nu_3(\nu_2+\nu_4)} S_{\nu_3}(Q+\alpha_s-\alpha_3-\alpha_4) S_{\nu_4}(Q+\alpha_s-\alpha_1-\alpha_2)$$
$$\times S_{\nu_2}(Q-\alpha_t+\alpha_2-\alpha_3) S_{\nu_1}(Q-\alpha_t+\alpha_4-\alpha_1) S_{\nu_2}(-Q+\alpha_t+\alpha_2+\alpha_3$$
$$\times S_{\nu_3}(Q-\alpha_s+\alpha_3-\alpha_4) S_{\nu_1}(-Q+\alpha_t+\alpha_4+\alpha_1) S_{\nu_4}(Q-\alpha_s+\alpha_1-\alpha_2)$$

The integral part of $\left\{ \begin{array}{ccc} \alpha_1^{a_1} & \alpha_3^{a_3} & \alpha_s^{a_s} \\ \alpha_2^{a_2} & \alpha_4^{a_4} & \alpha_t^{a_t} \end{array} \right\}_{\nu_1\nu_2}^{\nu_3\nu_4}$ obviously coincides with $J(\underline{\alpha},\underline{\nu})$ for $a_i = 0$.

The integral part of $\left\{ \begin{array}{ccc} \alpha_1^{a_1} & \alpha_3^{a_3} & \alpha_s^{a_s} \\ \alpha_2^{a_2} & \alpha_4^{a_4} & \alpha_t^{a_t} \end{array} \right\}_{\nu_1\nu_2}^{\nu_3\nu_4}$ for general a_i is given by the supersymmetric hypergeometric function

$$\begin{aligned} &J(\underline{\alpha}, \underline{\nu}, \underline{a}) = \\ &\int_{-\mathrm{i}\infty}^{\mathrm{i}\infty} \sum_{\nu=0}^{1} \Lambda(-y-\alpha_{23t}, \nu_2-\nu_3-\nu-a_s-a_t)\Lambda(-y-\alpha_{34s}, -\nu) \\ &\times\Lambda(-y-\alpha_{14t}, \nu_2-\nu_4-\nu)\Lambda(-y-\alpha_{12s}, -\nu_3-\nu_4-\nu) \\ &\times\Lambda(y+\alpha_{1234}, \nu_4+\nu+1)\Lambda(y+\alpha_{24st}, -\nu_2+\nu+1-a_2) \\ &\times\Lambda(y+\alpha_{13st}, -\nu_2+\nu_3+\nu_4+\nu-1+a_2+a_t) \\ &\times\Lambda(y+2Q, \nu_3+\nu-1+a_s)\frac{dy}{2\mathrm{i}}\,. \end{aligned} \tag{0.64}$$

Using parafermionic Teschner-Vartanov in the opposite direction

$$J(\underline{\alpha},\underline{\nu},\underline{a}) = (-1)^{\nu_2+\nu_4+a_2-a_s}\Omega^{-1}(\underline{\alpha},\underline{\nu},\underline{a}) I_{\alpha_s^{a_s},\alpha_t^{a_t}} \left[\begin{array}{cc} \alpha_3^{a_3} & \alpha_2^{a_2} \\ \alpha_4^{a_4} & \alpha_1^{a_1} \end{array}\right]_{\nu_1\nu_2}^{\nu_3\nu_4},$$

where

$$
I_{\alpha_s^{a_s},\alpha_t^{a_t}}\left[\begin{array}{cc}\alpha_3^{a_3} & \alpha_2^{a_2}\\ \alpha_4^{a_4} & \alpha_1^{a_1}\end{array}\right]_{\nu_1\nu_2}^{\nu_3\nu_4} =
$$

$$
\int_{-i\infty}^{i\infty}\sum_{\nu=0}^{1}\Lambda(y+\gamma_1^{\circ},\nu_3+a_2+a_s+a_t+\nu)\Lambda(y+\gamma_2^{\circ},\nu+\nu_4+a_2)
$$

$$
\times\Lambda(y+\gamma_3^{\circ},\nu+\nu_3+a_s)\Lambda(y+\gamma_4^{\circ},\nu-\nu_4+a_s)
$$

$$
\times\Lambda(-y+\beta_1^{\circ},-\nu-1-a_s)\Lambda(-y+\beta_4^{\circ},-\nu-1)
$$

$$
\times\Lambda(-y+\beta_2^{\circ},-\nu+1+\nu_2-\nu_3-a_2-a_s-a_t)
$$

$$
\times\Lambda(-y+\beta_3^{\circ},-\nu+1-\nu_2-\nu_3-a_2-a_s)\frac{dy}{2i}
$$

we get integral part of the SUSY 6j-symbols in the Ponsot-Teschner parametrization:

$$I_{\alpha_s^{a_s},\alpha_t^{a_t}}\left[\begin{array}{cc}\alpha_3^{a_3} & \alpha_2^{a_2}\\ \alpha_4^{a_4} & \alpha_1^{a_1}\end{array}\right]_{\nu_1\nu_2}^{\nu_3\nu_4}=$$

$$(-1)^F\int_{-i\infty}^{i\infty}\sum_{\nu=0}^{1}(-1)^E S_{1+\nu+\nu_3+a_4}(y+\gamma_1^\circ)S_{1+\nu+\nu_4+a_2}(y+\gamma_2^\circ)$$

$$\times S_{1+\nu+\nu_3+a_s}(y+\gamma_3^\circ)S_{1+\nu+\nu_4+a_s}(y+\gamma_4^\circ)S_{\nu-a_s}(-y+\beta_1^\circ)$$

$$\times S_{\nu+\nu_2+\nu_3-a_4}(-y+\beta_2^\circ)S_{\nu+\nu_2+\nu_3-a_1}(-y+\beta_3^\circ)S_\nu(-y+\beta_4^\circ),\frac{dy}{2i},$$

$$E = \nu(\nu_2 + \nu_4) + \nu(\nu_3 a_2 + \nu_2 a_t + \nu_4 a_2 + \nu_4 a_s + a_2 a_s + a_s + a_t) \text{ and}$$

$$F = \nu_3\nu_2 + \nu_4 + \nu_2\nu_3 a_t + \nu_2 a_2 a_t + \nu_2 a_s a_t + \nu_3 a_2 a_s + \nu_3 a_t + \nu_2 a_s + \nu_2 a_2 + \nu_4 a_s + a_2 a_t + a_s a_t + a_s \ ,$$