Split Casimir operator for simple Lie superalgebras, solutions of Yang-Baxter equations and Vogel parameters

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Introduction

The split Casimir operator $\widehat{C}$ is an element the centre of the tensor product of enveloping algebras of simple Lie algebrsa and superalgebras.

- The operator $\widehat{C}$ can be used to construct projectors onto invariant subspaces of Lie algebras and superalgebra representations in a uniform way.
- The split Casimir operator can be used in the study of the universal Lie algebra. This is a model of all simple Lie algebras (and some Lie superalgebras), in which many quantities that characterize a Lie algebra and its representations can be expressed in a universal way as rational functions of the three Vogel parameters.

In our work, we generalize the results of a recent paper by A.P. Isaev and S. K. Krivonos to the case of Lie superalgebras and derive some additional identities.

Comultiplication and the split Casimir operator

Let $\mathfrak{g} = \mathfrak{g}_{\overline{0}} \oplus \mathfrak{g}_{\overline{1}}$ be a Lie superalgebra over $\mathbb{C}$, and $\mathcal{U}(\mathfrak{g})$ be its enveloping algebra. Then if the Cartan-Killing metric g of $\mathfrak{g}$ is nondegenerate, we can define the quadratic Casimir operator Consider the quadratic Casimir operator $C_2 \in \mathcal{U}(\mathfrak{g})$

$$C_2 = \mathrm{g}^{ij} X_i X_j \ \in \mathcal{U}(\mathfrak{g})\,. \tag{2.1}$$

C_2 is even and $[C_2, X_i] = 0$ for all the generators $X_i \in \mathcal{U}(\mathfrak{g})$, that is, C_2 is invariant under the adjoint action of $\mathfrak{g}$.

We also define the comultiplication $\Delta : \mathcal{U}(\mathfrak{g}) \to \mathcal{U}(\mathfrak{g}) \otimes \mathcal{U}(\mathfrak{g})$ defined on the generators X_i of $\mathcal{U}(\mathfrak{g})$ by

$$\Delta(X_i) = X_i \otimes I + I \otimes X_i \tag{2.2}$$

and extended to any $X_{i_1} \cdot \ldots X_{i_r}$ by being a homomorphism.

Acting on C_2 by Δ yields

$$\Delta(C_2) = C_2 \otimes I + I \otimes C_2 + 2\widehat{C}, \tag{2.3}$$

where

$$\widehat{C} = \overline{\mathrm{g}}^{ij} X_i \otimes X_j \tag{2.4}$$

is called the split Casimir operator of $\mathfrak{g}$ and satisfies

$$[\widehat{C}, \Delta(X_i)] = 0 \qquad \forall X_i. \tag{2.5}$$

Projectors onto invariant subspaces

Let an operator A act on a space V (for example, $A = T_f^{\otimes 2}\,\widehat{C}$ or $\mathrm{ad}^{\otimes 2}\,\widehat{C}$ where T_f and ad are the defining and adjoint representations). Then if A satisfies

$$(A - a_1 I)(A - a_2 I)\dots(A - a_p I) = 0, \tag{2.6}$$

where all the a_i's different, then A is diagonalizable and the projector onto the eigenspace of A with the eigenvalue a_j is given by

$$\mathrm{P}_j \equiv \mathrm{P}_{a_j} = \prod_{\substack{i=1\\ i\neq j}}^{p} \frac{A - a_i I}{a_j - a_i}\ , \tag{2.7}$$

and then $P_i \cdot P_j = \delta_{ij} P_i$.

Split Casimir operator of $osp(M|N)$ in the defining representation

Let ε be a superform on $V_{(M|N)}$, N is even. The $osp(M|N)$ Lie superalgebra is the algebra of operators $A : V_{(M|N)} \to V_{(M|N)}$ that leave the form ε invariant.
Define operators $\mathbf{1}, \mathcal{P}, \mathcal{K} : V_{(M|N)}^{\otimes 2} \to V_{(M|N)}^{\otimes 2}$ with the components

$$\mathbf{1}^{k_1 k_2}{}_{m_1 m_2} = \delta^{k_1}_{m_1} \delta^{k_2}_{m_2} \qquad \mathcal{P}^{k_1 k_2}{}_{m_1 m_2} = (-1)^{[k_1][k_2]} \delta^{k_1}_{m_2} \delta^{k_2}_{m_2} \qquad \mathcal{K}^{k_1 k_2}{}_{m_1 m_2} = \varepsilon^{k_1 k_2} \varepsilon_{m_1 m_2} \; .$$

Here $\mathbf{1}$ is the identity operator, $\mathcal{P}$ is the superpermutation, and

$$\mathcal{P}^2 = \mathbf{1}, \quad \mathcal{K}^2 = \omega \mathcal{K}, \quad \mathcal{P}\mathcal{K} = \mathcal{K}\mathcal{P} = \mathcal{K} \; , \tag{3.8}$$

where $\omega = M - N$. Then $\widehat{C}_f$ can be written as

$$\widehat{C}_f = \frac{1}{2(\omega - 2)} (\mathcal{P} - \mathcal{K}) \; . \tag{3.9}$$

and satisfies

$$\left(\widehat{C}_f - \frac{1}{2(\omega-2)}\right)\left(\widehat{C}_f + \frac{1}{2(\omega-2)}\right)\left(\widehat{C}_f + \frac{\omega - 1}{2(\omega-2)}\right) = 0. \tag{3.10}$$

The invariant projectors are then given by

$$\mathrm{P}_1 = \frac{1}{2}(\mathbf{1} + \mathcal{P}) - \frac{1}{\omega}\mathcal{K} \; , \quad \mathrm{P}_2 = \frac{1}{2}(\mathbf{1} - \mathcal{P}) \; , \quad \mathrm{P}_3 = \frac{1}{\omega}\mathcal{K} \; . \tag{3.11}$$

Solution of YBE in terms of $\widehat{C}_f$

The solution $R^{i_1 i_2}_{\ \ k_1 k_2}(u)$ of the graded Yang-Baxter equation

$$\begin{aligned} & R^{i_1 i_2}_{\ \ j_1 j_2}(u)(-1)^{[j_1][j_2]} R^{j_1 i_3}_{\ \ k_1 j_3}(u+v)(-1)^{[k_1][j_2]} R^{j_2 j_3}_{\ \ k_2 k_3}(v) = \\ & = R^{i_2 i_3}_{\ \ j_2 j_3}(v)(-1)^{[i_1][j_2]} R^{i_1 j_3}_{\ \ j_1 k_3}(u+v)(-1)^{[j_1][j_2]} R^{j_1 j_2}_{\ \ k_1 k_2}(u), \end{aligned} \tag{3.12}$$

which is invariant under the action of $osp(M|N)$ in the defining representation can be written as a rational function of $\widehat{C}_f$:

$$R(u) = \frac{(\omega - 2)\widehat{C}_f + 1/2 + u}{(\omega - 2)\widehat{C}_f + 1/2 - u} \ . \tag{3.13}$$

Split Casimir operator of $osp(M|N)$ in the adjoint representation

To simplify the consideration of $\mathrm{ad}^{\otimes 2}\,\widehat{C} \equiv \widehat{C}_{\mathrm{ad}}$, we introduce analogs of the operators $\mathbf{1}, \mathcal{P}$ and $\mathcal{K}$ used in the case of the defining representation:

$$\mathbf{I}^{A_1 A_2}{}_{B_1 B_2} = \delta^{A_1}_{B_1}\delta^{A_2}_{B_2} \quad \mathbf{P}^{A_1 A_2}{}_{B_1 B_2} = (-1)^{[A_1][A_2]}\delta^{A_1}_{B_2}\delta^{A_2}_{B_2} \quad \mathbf{K}^{A_1 A_2}{}_{B_1 B_2} = \overline{\mathrm{g}}^{A_1 A_2}\mathrm{g}_{B_1 B_2}\ ,$$

where A_i, B_i are vector indices in the adjoint representation, $\mathrm{g}_{B_1 B_2}$ is the Cartan-Killing metric and $\overline{\mathrm{g}}^{A_1 A_2}$ is its inverse.
The operators $\mathbf{I}, \mathbf{P}$ and $\mathbf{K}$ satisfy:

$$\mathbf{P}^2 = \mathbf{I}, \qquad \mathbf{KP} = \mathbf{PK} = \mathbf{K}, \qquad \mathbf{K}^2 = \frac{\omega(\omega-1)}{2}\mathbf{K}, \tag{3.14}$$

where $\frac{\omega(\omega-1)}{2} = \mathrm{sdim}\, osp(M|N)$. We also introduce the symmetric $\widehat{C}_+$ and antisymmetric $\widehat{C}_-$ projections of $\widehat{C}_{\mathrm{ad}}$:

$$\widehat{C}_\pm = \frac{1}{2}(\mathbf{I} \pm \mathbf{P})\widehat{C}_{\mathrm{ad}}, \tag{3.15}$$

which satisfy:

$$\widehat{C}_\pm \widehat{C}_\mp = 0, \qquad \mathbf{P}\widehat{C}_\pm = \pm\widehat{C}_\pm .$$

Split Casimir operator of $osp(M|N)$ in the adjoint representation

Proposition 1

The antisymmetric $\widehat{C}_-$ and symmetric $\widehat{C}_+$ parts of the split Casimir operator of the $osp(M|N)$ Lie superalgebra for $M - N \equiv \omega \neq 0, 1, 2, 4, 8$ satisfies:

$$\widehat{C}_-^2 = -\frac{1}{2}\widehat{C}_- \iff \widehat{C}_-(\widehat{C}_- + \frac{1}{2}) = 0, \tag{3.16}$$

$$\widehat{C}_+\Big(\widehat{C}_+ + \mathbf{I}\Big)\Big(\widehat{C}_+ - \frac{\mathbf{I}}{\omega - 2}\Big)\Big(\widehat{C}_+ + \frac{2\mathbf{I}}{\omega - 2}\Big)\Big(\widehat{C}_+ + \frac{(\omega - 4)\mathbf{I}}{2(\omega - 2)}\Big) = 0. \tag{3.17}$$

The split Casimir operator $\widehat{C}_{\text{ad}} = \widehat{C}_- + \widehat{C}_+$ for $\omega \neq 0, 1, 2, 4, 6, 8$ satisfies:

$$\widehat{C}_{\text{ad}}(\widehat{C}_{\text{ad}} + \frac{1}{2})(\widehat{C}_{\text{ad}} + 1)(\widehat{C}_{\text{ad}} - \frac{1}{\omega - 2})(\widehat{C}_{\text{ad}} + \frac{2}{\omega - 2})(\widehat{C}_{\text{ad}} + \frac{\omega - 4}{2(\omega - 2)}) = 0\,. \tag{3.18}$$

Tensor product of two adjoint representations of $osp(M|N)$

The projectors onto invariant subspaces of $osp(M|N)$ in the adjoint representation are given by

$$\mathrm{P}_1 = \frac{1}{2}(\mathbf{I}-\mathbf{P}) + 2\widehat{C}_- \ , \qquad \mathrm{P}_2 = -2\widehat{C}_- \ , \qquad \mathrm{P}_3 = \frac{2\mathbf{K}}{(\omega-1)\omega} \ ,$$

$$\mathrm{P}_4 = \frac{2}{3}(\omega-2)\widehat{C}_+^2 + \frac{\omega}{3}\widehat{C}_+ + \frac{(\omega-4)(\mathbf{I}+\mathbf{P})}{3(\omega-2)} - \frac{2(\omega-4)\mathbf{K}}{3(\omega-2)(\omega-1)} \ ,$$

$$\mathrm{P}_5 = -\frac{2(\omega-2)^2}{3(\omega-8)}\widehat{C}_+^2 - \frac{(\omega-2)(\omega-6)}{3(\omega-8)}\widehat{C}_+ + \frac{(\omega-4)(\mathbf{I}+\mathbf{P})}{6(\omega-8)} + \frac{2\mathbf{K}}{3(\omega-8)} \ ,$$

$$\mathrm{P}_6 = \frac{4(\omega-2)}{\omega-8}\widehat{C}_+^2 + \frac{4}{\omega-8}\widehat{C}_+ - \frac{4(\mathbf{I}+\mathbf{P})}{(\omega-2)(\omega-8)} - \frac{8(\omega-4)\mathbf{K}}{\omega(\omega-2)(\omega-8)} \ . \tag{3.19}$$

$$\mathrm{str}\,\mathrm{P}_1 = \frac{1}{8}\omega(\omega-1)(\omega+2)(\omega-3) \ , \qquad \mathrm{str}\,\mathrm{P}_4 = \frac{1}{12}\omega(\omega+1)(\omega+2)(\omega-3) \ ,$$

$$\mathrm{str}\,\mathrm{P}_2 = \frac{1}{2}\omega(\omega-1) \ , \qquad \mathrm{str}\,\mathrm{P}_5 = \frac{1}{24}\omega(\omega-1)(\omega-2)(\omega-3) \ ,$$

$$\mathrm{str}\,\mathrm{P}_3 = 1 \ , \qquad \mathrm{str}\,\mathrm{P}_6 = \frac{1}{2}(\omega-1)(\omega+2) \tag{3.20}$$

Definition of $sl(M|N)$ Lie superalgebra

The $s\ell(M|N)$ Lie superalgebra (where $M \neq N$) is defined as the algebra of operators $A : V_{(M|N)} \to V_{(M|N)}$ that satisfy:

$$\mathrm{str}\, A = (-1)^{[a]} A^a{}_a = 0. \qquad (4.21)$$

In terms of the previously introduced operators $\mathbf{1}$ and $\mathcal{P} : V_{(M|N)}^{\otimes 2} \to V_{(M|N)}^{\otimes 2}$ the split Casimir operator in the defining representation reads

$$\widehat{C}_f = \frac{1}{2\omega}\big(\mathcal{P} - \frac{1}{\omega}\mathbf{1}\big) \qquad (4.22)$$

and satisfies the following characteristic equation

$$\big(\widehat{C}_f - \frac{\omega - 1}{2\omega^2}\mathbf{1}\big)\big(\widehat{C}_f + \frac{\omega + 1}{2\omega^2}\mathbf{1}\big) = 0. \qquad (4.23)$$

Here, $\omega = M - N$. The invariant projectors are

$$\mathrm{P}_{\pm} = \pm\Big(\omega\,\widehat{C}_f + \frac{1 \pm \omega}{2\omega}\Big) = \frac{1}{2}(\mathbf{1} \pm \mathcal{P}). \qquad (4.24)$$

Solution of YBE in terms of $\widehat{C}_f$

The solution $R^{i_1 i_2}_{\ \ k_1 k_2}(u)$ of the graded Yang-Baxter equation

$$\begin{aligned} & R^{i_1 i_2}_{\ \ j_1 j_2}(u)(-1)^{[j_1][j_2]} R^{j_1 i_3}_{\ \ k_1 j_3}(u+v)(-1)^{[k_1][j_2]} R^{j_2 j_3}_{\ \ k_2 k_3}(v) = \\ & = R^{i_2 i_3}_{\ \ j_2 j_3}(v)(-1)^{[i_1][j_2]} R^{i_1 j_3}_{\ \ j_1 k_3}(u+v)(-1)^{[j_1][j_2]} R^{j_1 j_2}_{\ \ k_1 k_2}(u), \end{aligned} \tag{4.25}$$

which is invariant under the action of $sl(M|N)$ in the defining representation, can be written in terms of $\widehat{C}_f$:

$$R(u) = \frac{\mathrm{P}_+ + u}{\mathrm{P}_+ - u} = \frac{\omega \widehat{C}_f + \frac{1+\omega}{2\omega} + u}{\omega \widehat{C}_f + \frac{1+\omega}{2\omega} - u}. \tag{4.26}$$

Tensor product of two adjoint representations of $sl(M|N)$

As in the $osp(M|N)$ case, we will use the three operators $\mathbf{I}, \mathbf{P}$ and $\mathbf{K}$ which, this time, satisfy the identities

$$\mathbf{P}^2 = \mathbf{I}, \qquad \mathbf{PK} = \mathbf{K} = \mathbf{KP}, \qquad \mathbf{K}^2 = (\omega^2 - 1)\mathbf{K} \tag{4.27}$$

with $\omega^2 - 1 = \mathrm{sdim}(M|N)$ being the superdimension of $sl(M|N)$. The symmetric and antisymmetric parts of $\widehat{C}_{\mathrm{ad}}$ are

$$\widehat{C}_{\pm} = \frac{1}{2}(\mathbf{I} \pm \mathbf{P})\widehat{C}_{\mathrm{ad}}, \tag{4.28}$$

and satisfy

$$\widehat{C}_{\pm}\widehat{C}_{\mp} = 0, \qquad \mathbf{P}\widehat{C}_{\pm} = \pm\widehat{C}_{\pm}.$$

Tensor product of two adjoint representations of $sl(M|N)$

Proposition 2

The antisymmetric $\widehat{C}_-$ and symmetric $\widehat{C}_+$ parts of the split Casimir operator of the $s\ell(M|N)$ Lie superalgebra for $\omega \neq 0, 1, 2$ satisfy

$$\widehat{C}_-^2 = -\frac{1}{2}\widehat{C}_- \iff \widehat{C}_-(\widehat{C}_- + \frac{1}{2}\mathbf{I}) = 0. \quad (4.29)$$

$$\widehat{C}_+(\widehat{C}_+ + \mathbf{I})(\widehat{C}_+ - \frac{1}{\omega}\mathbf{I})(\widehat{C}_+ + \frac{1}{\omega}\mathbf{I})(\widehat{C}_+ + \frac{1}{2}\mathbf{I}) = 0. \quad (4.30)$$

The split Casimir operator $\widehat{C}_{\rm ad} = \widehat{C}_- + \widehat{C}_+$ for $\omega \neq 0, 1, 2$ satisfies

$$\widehat{C}_{\rm ad}(\widehat{C}_{\rm ad} + \mathbf{I})(\widehat{C}_{\rm ad} - \frac{1}{\omega}\mathbf{I})(\widehat{C}_{\rm ad} + \frac{1}{\omega}\mathbf{I})(\widehat{C}_{\rm ad} + \frac{1}{2}\mathbf{I}) = 0. \quad (4.31)$$

Tensor product of two adjoint representations of $sl(M|N)$

Since $\mathbf{P}_{\pm}^{(\mathrm{ad})}$ are invariant operators, the projectors onto symmetric and antisymmetric invariant spaces can be constructed separately. The explicit expressions are

$$
\begin{aligned}
&\mathrm{P}_1^{(-)} = 2\widehat{C}_- + \mathbf{P}_-^{(\mathrm{ad})}, \qquad\qquad \mathrm{P}_2^{(-)} = -2\widehat{C}_-, \\
&\mathrm{P}_1^{(+)} = \frac{1}{\omega^2-1}\mathbf{K}, \\
&\mathrm{P}_2^{(+)} = -\frac{\omega}{2(\omega+1)(\omega+2)}\mathbf{K} + \frac{\omega^2}{\omega+2}\widehat{C}_+^2 + \frac{\omega}{2}\widehat{C}_+ + \frac{\omega}{2(\omega+2)}\mathbf{P}_+^{(\mathrm{ad})}, \\
&\mathrm{P}_3^{(+)} = \frac{\omega}{2(\omega-1)(\omega-2)}\mathbf{K} - \frac{\omega^2}{\omega-2}\widehat{C}_+^2 - \frac{\omega}{2}\widehat{C}_+ + \frac{\omega}{2(\omega-2)}\mathbf{P}_+^{(\mathrm{ad})}, \\
&\mathrm{P}_4^{(+)} = \frac{4}{\omega^2-4}\left(\omega^2\widehat{C}_+^2 - \mathbf{P}_+^{(\mathrm{ad})} - \mathbf{K}\right).
\end{aligned}
\tag{4.32}
$$

Tensor product of two adjoint representations of $sl(M|N)$

Calculating the traces and supertraces of these projectors, we get the dimensions and superdimensions of the invariant subspaces:

$$\operatorname{str} \mathrm{P}_1^{(-)} = \frac{1}{2}(\omega^2 - 1)(\omega^2 - 4), \qquad \operatorname{str} \mathrm{P}_1^{(+)} = 1,$$
$$\operatorname{str} \mathrm{P}_2^{(-)} = \omega^2 - 1, \qquad \operatorname{str} \mathrm{P}_2^{(+)} = \frac{1}{4}\omega^2(\omega - 1)(\omega + 3),$$
$$\operatorname{str} \mathrm{P}_3^{(+)} = \frac{1}{4}\omega^2(\omega + 1)(\omega - 3),$$
$$\operatorname{str} \mathrm{P}_4^{(+)} = \omega^2 - 1 \tag{4.33}$$

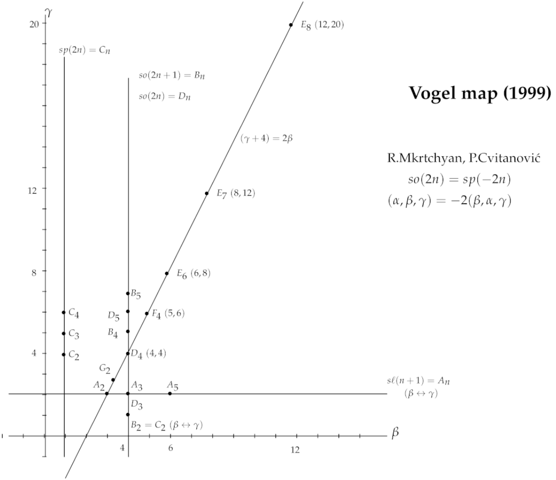
Introductory note on Vogel parameters

The Vogel parameters are defined as three numbers (α, β, γ) modulo a common multiplier and an arbitrary permutation from the symmetric group S3 (or, equivalently, as a point in the space $\mathbb{P}^2/S_3$). Certain values of the Vogel parameters correspond to all simple Lie algebras and some simple Lie superalgebras.

Type	Lie algebra	α	β	γ
A_n	$\mathfrak{sl}_{n+1}$	-2	2	$(n+1)$
B_n	$\mathfrak{so}_{2n+1}$	-2	4	$2n-3$
C_n	$\mathfrak{sp}_{2n}$	-2	1	$n+2$
D_n	$\mathfrak{so}_{2n}$	-2	4	$2n-4$
G_2	$\mathfrak{g}_2$	-2	$10/3$	$8/3$
F_4	$\mathfrak{f}_4$	-2	5	6
E_6	$\mathfrak{e}_6$	-2	6	8
E_7	$\mathfrak{e}_7$	-2	8	12
E_8	$\mathfrak{e}_8$	-2	12	20

Introductory note on Vogel parameters

The Vogel parameters corresponding to simple Lie algebras can be visualized on a Vogel plane:



Universal characteristic identities

For the $osp(M|N)$ and $s\ell(M|N)$ Lie superalgebras (which are denoted by $\mathfrak{g}$ in this section) for $\widehat{C}_+$ in the adjoint representation can be written in the following general form:

$$\widehat{C}_+(\widehat{C}_+ + \mathbf{I})(\widehat{C}_+^3 + \frac{1}{2}\widehat{C}_+^2 - \mu_1 \widehat{C}_+ - 2\mu_2 \mathbf{I}) = 0, \tag{5.34}$$

The parameters μ_1 and μ_2 corresponding to the algebras $osp(M|N)$ and $s\ell(M|N)$ are given in Table 1, where $\omega = M - N$.

Table: The values of μ_1 and μ_2 for the $osp(M|N)$ and $s\ell(M|N)$ Lie superalgebras

	μ_1	μ_2
$osp(M|N)$	$-\frac{\omega-8}{2(\omega-2)^2}$	$\frac{\omega-4}{2(\omega-2)^3}$
$s\ell(M|N)$	$\frac{1}{\omega^2}$	$\frac{1}{4\omega^2}$

Universal characteristic identities

The factorized form of the latter equation is

$$\widehat{C}_+(\widehat{C}_+ + \mathbf{I})(\widehat{C}_+ + \frac{\alpha}{2t}\mathbf{I})(\widehat{C}_+ + \frac{\beta}{2t}\mathbf{I})(\widehat{C}_+ + \frac{\gamma}{2t}\mathbf{I}) = 0. \tag{5.35}$$

The roots

$$a_1 = 0, \qquad a_2 = -1, \qquad a_3 = -\frac{\alpha}{2t}, \qquad a_4 = -\frac{\beta}{2t}, \qquad a_5 = -\frac{\gamma}{2t}, \tag{5.36}$$

of the latter polynomial are normalized a parameter t and satisfy

$$\frac{\alpha}{2t} + \frac{\beta}{2t} + \frac{\gamma}{2t} = \frac{1}{2}. \tag{5.37}$$

Comparing the two forms of the universal characteristic identity we can see that

$$\mu_1 = -\frac{\alpha\beta + \alpha\gamma + \beta\gamma}{4t^2}, \qquad \mu_2 = -\frac{\alpha\beta\gamma}{16t^3}, \tag{5.38}$$

Universal characteristic identities

We choose $t = h^\vee$ where $h^\vee$ is the dual Coxeter number of $\mathfrak{g}$. The parameters α, β and γ are called the Vogel parameters. Their values for the algebras $osp(M|N)$ and $s\ell(M|N)$ are given in Table 2.

Table: The Vogel parameters for the $osp(M|N)$ and $s\ell(M|N)$ Lie superalgebras

	$s\ell(M\|N)$	$osp(2m+1\|N),\ \omega > 1$ $osp(2m\|N),\ \omega > 0$	$osp(2m+1\|N),\ \omega \leq 1$ $osp(2m\|N),\ \omega \leq 0$
α	-2	-2	1
β	2	4	-2
γ	ω	$\omega - 4$	$-\frac{1}{2}(\omega - 4)$
t	ω	$\omega - 2$	$-\frac{1}{2}(\omega - 2)$

Universal characteristic identities

The universal characteristic identity allows us to write down a universal form of the projectors $\mathsf{P}^{(+)}_{(a_i)}$ onto the invariant subspaces $V_{(a_i)}$ of the symmetric space $\mathbf{P}_+(V_{\mathrm{ad}}^{\otimes 2})$:

$$\mathsf{P}^{(+)}_{(-\frac{\alpha}{2t})} = \mathsf{P}^{(+)}(\alpha|\beta,\gamma), \qquad \mathsf{P}^{(+)}_{(-\frac{\beta}{2t})} = \mathsf{P}^{(+)}(\beta|\alpha,\gamma), \qquad \mathsf{P}^{(+)}_{(-\frac{\gamma}{2t})} = \mathsf{P}^{(+)}(\gamma|\alpha,\beta)$$
$$\mathsf{P}^{(+)}_{(-1)} = \frac{1}{\mathrm{sdim}\,\mathfrak{g}}\mathbf{K}, \tag{5.39}$$

where we denoted

$$\mathsf{P}^{(+)}(\alpha|\beta,\gamma) = \frac{4t^2}{(\beta-\alpha)(\gamma-\alpha)}\Big(\widehat{C}_+^2 + (\frac{1}{2} - \frac{\alpha}{2t})\,\widehat{C}_+ + \frac{\beta\gamma}{8t^2}\big(\mathbf{I} + \mathbf{P}^{(\mathrm{ad})} - \frac{2\alpha}{\alpha - 2t}\mathbf{K}\big)\Big). \tag{5.40}$$

Universal characteristic identities

The supertrace of $\mathrm{P}^{(+)}(\alpha|\beta,\gamma)$ is

$$\mathrm{str}\,\mathrm{P}^{(+)}(\alpha|\beta,\gamma) = -\frac{(3\alpha-2t)(\beta-2t)(\gamma-2t)(\beta+t)(\gamma+t)t}{\alpha^2(\alpha-\beta)(\alpha-\gamma)\beta\gamma}. \tag{5.41}$$

Thus, we get the superdimensions of the invariant subspaces $V_{(-1)}$, $V_{(-\frac{\alpha}{2t})}$, $V_{(-\frac{\beta}{2t})}$ and $V_{(-\frac{\gamma}{2t})}$ extracted by the projectors (5.39):

$$\begin{aligned}
\mathrm{sdim}\,V_{(-1)} &= 1,\\
\mathrm{sdim}\,V_{(-\frac{\alpha}{2t})} &= -\frac{(3\alpha-2t)(\beta-2t)(\gamma-2t)(\beta+t)(\gamma+t)t}{\alpha^2(\alpha-\beta)(\alpha-\gamma)\beta\gamma},\\
\mathrm{sdim}\,V_{(-\frac{\beta}{2t})} &= -\frac{(3\beta-2t)(\alpha-2t)(\gamma-2t)(\alpha+t)(\gamma+t)t}{\beta^2(\beta-\alpha)(\beta-\gamma)\alpha\gamma},\\
\mathrm{sdim}\,V_{(-\frac{\gamma}{2t})} &= -\frac{(3\gamma-2t)(\beta-2t)(\alpha-2t)(\beta+t)(\alpha+t)t}{\gamma^2(\gamma-\beta)(\gamma-\alpha)\beta\alpha}.
\end{aligned} \tag{5.42}$$

Eigenvalues of higher Casimir operators in the adjoint representation

Since $\widehat{C}_{\text{ad}}$ is an ad-invariant operator, then so is an its arbitrary power $\widehat{C}^k_{\text{ad}}$. Taking the supertrace of its second component yields another ad-invariant operator

$$\text{ad}(C_k) \equiv \text{str}_2(\widehat{C}^k_{\text{ad}}) = \text{g}^{a_1 \cdots a_n} \, \text{ad}(X_{a_1}) \cdots \text{ad}(X_{a_n}), \tag{5.43}$$

where

$$\text{g}^{a_1 \cdots a_2} = (-1)^{\sum_{i>j}^n [a_i][a_j]} \text{g}^{a_1 b_1} \cdots \text{g}^{a_n b_n} \, \text{str}\left(\text{ad}(X_{b_1} \cdots \text{ad}(X_{b_n}) \right). \tag{5.44}$$

Let us c_k be the eigenvalue of $\text{ad}(C_k)$. The for its generating function $c(z)$ we have

$$c(z) \cdot I \equiv \sum_{p=0}^{\infty} c_p z^p = \text{str}_2 \left(\sum_{p=0}^{\infty} \widehat{C}^p_{\text{ad}} z^p \right) = \text{str}_2 \left(\sum_{p=0}^{\infty} \widehat{C}^p_{+} z^p + \sum_{p=0}^{\infty} \widehat{C}^p_{-} z^p \right), \tag{5.45}$$

where we assume $\widehat{C}^0_{\pm} = \frac{1}{2}(\mathbf{I} \pm \mathbf{P})$, and $\widehat{C}^0_{\text{ad}} = \mathbf{I}$. Using expressions for $\widehat{C}_+$ and $\widehat{C}_-$ in terms of invariant projectors and calculating their supertraces str_2 yields

$$\begin{aligned} c(z) =& \frac{(\alpha - 2t)(\beta - 2t)(\gamma - 2t)}{\alpha\beta\gamma} + \\ & z^2 \frac{96t^3 + 168t^3 z + 6(14t^3 + tt_2 - t_3)z^2 + (13t + 3tt_2 - 4t_3)z^3}{6(2t + \alpha z)(2t + \beta z)(2t + \gamma z)(2 + z)(1 + z)}. \end{aligned} \tag{5.46}$$

Conclusions

- We have found explicit universal formulas for the projectors onto the invariant subspaces of the representation $\mathrm{ad}^{\otimes 2}$ of the $osp(M|N)$ and $sl(M|N)$ Lie superalgebras.
- The split Casimir operator has been used for deriving a universal formula for eigenvalues of higher Casimir operators in the adjoint representation.

To do:

- Find universal formulas for projectors onto invariant subspaces of the $\mathrm{ad}^{\otimes 3}$ and $\mathrm{ad}^{\otimes 4}$ representations of simple Lie algebras and superalgebras.
- Find a universal solution of the Yang-Baxter equation in the adjoint representation.

Thank you
for your attention