Matrix Capelli identities
related to Reflection Equation Algebra

(joint results with D.Gurevich and V.Petrova)

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Report outline:

- Classical objects: Poisson brackets on Fun($gl^*(N)$) and the Weyl-Heisenberg algebra
- Quantum objects: $U(gl(N))$, Reflection Equation Algebra (REA) and quantum partial derivatives
- Classical result by Alfredo Capelli (1887) and some its generalizations
- Matrix generalization of Capelli identity for the quantum double of two REA

Classical objects: Poisson brackets on $\mathrm{Fun}(gl^*(N))$ and the Weyl-Heisenberg algebra

Let x_i^j, $1 \le i, j \le N$ be the complex coordinates of $gl^*(N, C)$ and $\mathrm{Fun}(gl^*(N))$ be the algebra of polynomial functions on $gl^*(N)$.

A. Poisson-Lie structure

$$\{x_i^j, x_k^s\}_{PL} = \delta_k^j\, x_i^s - \delta_i^s\, x_k^j \tag{1}$$

Matrix notation: $\quad X = \|x_i^j\|$,

$$X_k = I^{\otimes k-1} \otimes X \otimes I^{\otimes (p-k)}, \quad \forall\, p \ge k \ge 1$$

is an embedding of $N \times N$ matrix X into the space of $N^p \times N^p$ matrices. In particular for $p = 2$:

$$X_1 = X \otimes I, \qquad X_2 = I \otimes X.$$

Then (1) can be written as matrix equality for $N^2 \times N^2$ matrices:

$$\{X_1, X_2\}_{PL} = P_{12} X_2 - X_2 P_{12}$$

Here $(P_{12})_{i_1 i_2}^{j_1 j_2} = \delta_{i_1}^{j_2} \delta_{i_2}^{j_1}$ is the $N^2 \times N^2$ permutation matrix.

Theorem. The homogeneous polynomials

$$p_k(X) = \mathrm{Tr} X^k, \quad k \ge 0$$

are Poisson central fumctions: $\{p_k(X), f(X)\}_{PL} = 0$ for any $f \in \mathrm{Fun}(gl^*(N))$.

B. Quadratic STS Poisson structure

The quadratic Poisson structure (Semenov-Tian-Schansky):

$$\{X_1, X_2\}_{STS} = r_{21}X_1X_2 - X_1X_2r_{12} + X_2r_{12}X_1 - X_1r_{21}X_2 \qquad (2)$$

Here $N^2 \times N^2$ matrix r_{12} is a classical R-matrix:

$$[r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}] = 0.$$

Theorem. For the $GL(N)$-type R-matrix r_{12} the homogeneous polynomials

$$p_k(X) = \mathrm{Tr}X^k, \quad k \geq 0$$

are Poisson central fumctions: $\{p_k(X), f(X)\}_{STS} = 0$ for any $f \in \mathrm{Fun}(gl^*(N))$.

Example. Drinfeld-Jimbo classical R-matrix for $N = 2$:

$$r_{12} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad X := \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

Then the STS-brackets read:

$$\{a, b\} = -2ab \qquad \{a, c\} = 2ac \qquad \{a, d\} = 0$$
$$\{b, c\} = 2a(d - a) \qquad \{b, d\} = -2ab \qquad \{c, d\} = 2ac$$

Theorem. For a $GL(N)$-type classical R-matrix the brackets $\{,\}_{PL}$ and $\{,\}_{STS}$ are compatible, that is

$$\{,\}_{u,v} = u\{,\}_{PL} + v\{,\}_{STS}$$

is a Poisson bracket for any complex numbers u and v.

Heisenberg-Weyl algebra

The algebra is generated by the coordinates x_i^j and the partial derivatives $\partial_i^j = \partial/\partial x_j^i$:

$$X_1 X_2 = X_2 X_1, \quad D_1 D_2 = D_2 D_1$$

$$D_1 X_2 = X_2 D_1 + P_{12}.$$

The group $GL(N)$ acts on $\mathrm{Fun}(gl^*(N))$ by right multiplications: $\forall\, Q \in GL(N)$:

$$X \mapsto XQ, \quad \Rightarrow \quad D \mapsto Q^{-1} D, \quad D = \|\partial_i^j\|.$$

Theorem. The subalgebra of the right-invariant operators is generated by

$$E_i^j = \sum_{a=1}^{N} x_i^a \partial_a^j$$

and it is isomorphic to $U(gl(N))$:

$$E_1 E_2 - E_2 E_1 = P_{12} E_2 - E_2 P_{12}, \quad E = \|E_i^j\| = XD. \qquad (3)$$

Quantum objects: $U(gl(N))$, Reflection Equation Algebra (REA) and quantum partial derivatives

Quantization of the Poisson structures $\{,\}_{PL}$ and $\{,\}_{PL}$ gives rise to examples of quantum matrix algebras.

- $\{,\}_{PL}$: Fun$(gl^*(N)) \to U(gl(N))$
- $\{,\}_{STS}$: Fun$(gl^*(N)) \to$ REA $\mathcal{M}(R)$.

$$X_1X_2 = X_2X_1 \quad \to \quad M_1R_{12}M_1R_{12} = R_{12}M_1R_{12}M_1, \quad M = \|m_i^j\|.$$

Here R is a $GL(N)$-type R-matrix:

i) $R_{12}R_{23}R_{12} = R_{23}R_{12}R_{23}$

ii) $(R - qI \otimes I)(R + q^{-1}I \otimes I) = 0, \quad q \in C$

iii) $A^{(N+1)}(R) \equiv 0$, rank $A^{(N)} = 1$

$$A^{(1)} = I$$
$$A^{(k)} = \frac{1}{k_q}\, A^{(k-1)} \left(q^{(k-1)}\, I^{\otimes k} - (k-1)_q\, R_{k-1}\right) A^{(k-1)}.$$

iv) R is a skew-invertible matrix $\Rightarrow$ Tr$_R$.

The connection of R and r:

$$q = e^{\epsilon} \quad \Rightarrow \quad PR = I + \epsilon\, r + o(\epsilon)$$

Example. Drinfeld-Jimbo $U_q(sl(N))$ R-matrix. For $N = 2$:

$$r = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \Rightarrow \quad R = \begin{pmatrix} q & 0 & 0 & 0 \\ 0 & \nu & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & q \end{pmatrix} \quad \nu \equiv q - q^{-1}.$$

Quantum deformation of the Heisenberg-Weyl algebra

$$X_1X_2 = X_2X_1 \;\to\; R_{12}M_1R_{12}M_1 = M_1R_{12}M_1R_{12} \quad \mathcal{M}(R)$$

$$D_1D_2 = D_2D_1 \;\to\; R_{12}^{-1}D_1R_{12}^{-1}D_1 = D_1R_{12}^{-1}D_1R_{12}^{-1} \quad \mathcal{D}(R^{-1})$$

$$D_1X_2 = X_2D_1 + P_{12} \;\to\; D_1R_{12}M_1R_{12} = R_{12}M_1R_{12}^{-1}D_1 + R_{12}$$

The action of quantum "derivatives" $D = \|\partial_i^j\|$ are defined by

$$\partial_i^j \triangleright 1_{\mathcal{M}} = 0.$$

Example. For $N = 2$:

$$X := \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \to \quad M := \begin{pmatrix} \hat{a} & \hat{b} \\ \hat{c} & \hat{d} \end{pmatrix}$$

$$\partial_a a = a\,\partial_a + 1 \quad \mapsto \quad \hat{\partial}_a\,\hat{a} = q^{-2}\hat{a}\,\hat{\partial}_a + q^{-1} - \frac{\nu}{q}\,\hat{b}\,\hat{\partial}_b$$

$$\partial_a b = b\,\partial_a \quad \mapsto \quad \hat{\partial}_a\,\hat{b} = \hat{b}\,\hat{\partial}_a - \frac{\nu}{q}\,\hat{a}\,\hat{\partial}_c + \nu^2\,\hat{b}\,\hat{\partial}_d$$

Classical result by Alfredo Capelli (1887) and some its generalizations

A toy example: powers of a differential operator $x\partial_x$.

$$(x\partial_x)^n = x\partial_x\, x\partial_x \ldots x\partial_x \; = x^n\partial_x^n + \ldots$$

but

$$x\partial_x(x\partial_x - 1)(x\partial_x - 2)\ldots(x\partial_x - n + 1) \equiv x^n\partial_x^n.$$

Let us turn to the matrix $E = XD$: $E_i^j = x_i^a\partial_a^j$. The following identitity takes place (A.Capelli):

$$\det{}_r \left\| \begin{array}{lllll} E_1^1 & E_1^2 & E_1^3 & \ldots & E_1^N \\ E_2^1 & E_2^2+1 & E_2^3 & \ldots & E_2^N \\ E_3^1 & E_3^2 & E_3^3+2 & \ldots & E_3^N \\ & & \ldots & & \\ E_N^1 & E_N^2 & E_N^3 & \ldots & E_N^N+N-1 \end{array} \right\| = \det X \det D.$$

An equivalent form (A.Okunkov, 1996)

$$\mathrm{Tr}_{12\ldots N} A^N(P) E_1(E_2+I)(E_2+2I)\ldots(E_N+(N-1)I) = \det X \det D.$$

Here $A^{(N)}(P)$ is an idempotent of the symmeric group S_N corresponding to partition (1^N) (one-column Young diagram) in the matrix representation of S_N in $V^{\otimes N}$, $\dim V = N$.

- Instead of A^N we can put any idempotent of S_k, $k \le N$ (the right hand side is also changed)
- M.Nazarov: superalgebras
- A.Molev et al.: other series of Lie algebras
- Noumi M., Umeda T., Wakayama M. (1994): generralization to quantum group $\mathrm{Fun}_q(sl_N)(RTT$ algebra).

Matrix generalization of Capelli identity for the quantum double of two REA

Introduce the matrix $L = MD$. One can show

$$L_1 R_{12} L_1 R_{12} - R_{12} L_1 R_{12} L_1 = L_1 R_{12} - R_{12} L_1.$$

This is a quantum deformation of $U(gl(N))$ relation (3) on matrix $E = XD$.

The following *matrix identities* take place: for $\forall\, k \geq 1$:

$$\begin{aligned} A^{(k)} L_{\overline{1}} (L_{\overline{2}} + qI) \ \ldots \ (L_{\overline{k}} + q^{k-1}(k-1)_q I\,)\, A^{(k)} \\ = \ q^{k(k-1)} A^{(k)} M_{\overline{1}} \ldots M_{\overline{k}}\, D_{\overline{k}} \ldots D_{\overline{1}} \end{aligned}$$

Here $L_{\overline{1}} = L_1$, $L_{\overline{k+1}} = R_k L_{\overline{k}} R_k^{-1}$.

For $k = N$ and $GL(N)$ type R-matrix we get a quantum version of the Capelli identity:

$$\begin{aligned} \mathrm{Tr}_{R(1\ldots N)} A^{(N)} L_{\overline{1}} (L_{\overline{2}} + qI) \ \ldots \ (L_{\overline{N}} + q^{N-1}(N-1)_q I) \\ = \ q^{-N} \mathrm{det}_R M \, \mathrm{det}_{R^{-1}} D. \end{aligned}$$