Super Yangians, Affine Super Yangians and Quantum Loop Superalgebras

Vladimir Stukopin (MIPT, SMI of VSC RAS, MCCME)

SQS – 2022

Introduction

We describe associative superalgebra and Hopf superalgebra structures on super Yangians, quantum loop superalgebra and affine super Yangians. We use an isomorphism (of associative superalgeras) between the completions of the Yangian of a special linear Lie superalgebra and the quantization of a loop superalgebra of a special linear Lie superalgebra. This last result is natural generalization of result by Sachin Gautam and Valerio Toledano Laredo on isomorphism between completions of Yangian of general linear algebra and quantum loop algebra of $L\mathfrak{gl}_{\mathfrak{n}}$ ([4], see also the paper [6] for futher development of this theory). Let's note that from existence of such isomorphism Φ between completions of super Yangian and quantum loop superalgebra does not imply equivalence representation categories because this isomorphism Φ is given by formal power series which do not converge on these representations, and do not allow for the numerical specialisation of the deformation parameters. We also consider separetly the problem on categories representation equivalence. We use idea by S. Gautam and V. Toledano-Laredo that the matrix coefficient of image of Cartan field under representation $\rho(\psi(z)) = \dfrac{qz - q^{-1}a}{z-a}$ is the monodromy of the additive difference equation $f(u+1) = Af(u)$

where$A(u) = \dfrac{u + \hbar - a}{u - a}$ is image Cartan field of Yangian (in the case of even roots) under representation. The main our results related to the problem of description Hopf superalgebra structures on super Yangian and quantum loop superalgebra and also on quantized enveloping superalgebras of some basic Lie superalgebras. The main tool of such description is a quantum Weyl groupoid. We define the structures of quantum Weyl groupoid and quantum affine Weyl groupoid in terms of action of elements of these groupoids and describe Hopf superalgebra structures on above mentioned quantum superalgebras.

The report presents a work that is a natural continuation of [3], [10]. At the end it is briefly presented some results related to the super Yangian of affine Kac-Moody algebras, namely related to the afiine super Yangian $Y_{\varepsilon_1,\varepsilon_2}(\tilde{\mathfrak{sl}}(m,n))$, where $\tilde{\mathfrak{sl}}(m,n)$ is defined by arbitrary simple root system Π. We show that such super Yangian are isomorphic for different simple root systems. We alsdo consider affine Weil groupoid and consider description Hopf superalgebra strucures.

Quantization

I recall the notion of quantization of Poisson manifolds.

Definition

Given a Poisson algebra $(A, \{\cdot, \cdot\})$, a deformation quantization is an associative unital product $*$ on the algebra of formal power series in $\hbar$, $A[[\hbar]]$, subject to the following two axioms

1. $f * g = fg + O(\hbar)$
2. $[f, g] = f * g - g * f + O(\hbar)$
3. $f * g = fg + \sum_{n=0}^{\infty} \hbar^k B_k(f \otimes g)$,

where B_k are linear bidifferential operators of degree at most k.

The Drinfeld quantization of Poisson groups is, in essence, a special case of deformation quantization. It can be defined in an equivalent way as a deformation of the structure of a Lie bialgebra.
The Lie bialgebra is the tangent space to the Poisson group.
In short, a Lie bialgebra is a Lie algebra $\mathfrak{g}$ such that the dual vector space $\mathfrak{g}^*$ is also a Lie algebra. Moreover, the bracket on $\mathfrak{g}$ induced by the operation of the bracket on $\mathfrak{g}^*$ it is one cocycle with coefficients in the module $\mathfrak{g}\otimes\mathfrak{g}$,where module structure is defined by
$a\cdot(b\otimes c)=[a\otimes 1+1\otimes a, b\otimes c]=[a,b]\otimes c+b\otimes[a,c]$.
The main object of my report are Yangian and quantum loop algebra which are quantization of the following bialgebras, correspondingly: current Lie algebra $\mathfrak{g}[\mathfrak{t}]$ and loop Lie algebra $\mathfrak{g}[t,t^{-1}]$ with cobrackets which can be defined using some standard r-matrices or by equivalent way using language of Manin triples.
Let $A_\hbar$ be a QUE superalgebra: $A_\hbar/\hbar A_\hbar\cong U(\mathfrak{g})$. Then the Lie superalgebra $\mathfrak{g}$ has a natural structure of a Lie superbialgebra defined by $\delta(x)=\hbar^{-1}(\Delta(\tilde{x})-\Delta^{op}(\tilde{x}))\mod \hbar$,where $x\in\mathfrak{g}$, $\tilde{x}\in A_\hbar$ is a preimage of x, Δ is a comultiplication in A and $\Delta^{op}:=\tau_{U(\mathfrak{g}),U(\mathfrak{g})}\circ\Delta$, where $\tau_{U(\mathfrak{g}),U(\mathfrak{g})}$ be a (super)permutation of tensor multipliers.

More precisely, we will deal with the superanalogues of the Lie bialgebras introduced above, that is, $\mathfrak{g}$ will be some Lie superalgebra. More precisely, we restrict ourselves to considering the particular case when $\mathfrak{g} = sl(m+1, n+1) = A(m, n)$ is a special linear superalgebra.

Completion

Suppose that E is an abelian group (vector space, ring and so on) with descending filtration $E = F^0E \supset F^1E \supset F^2E \supset \cdots$ of subgroups. One then defines the completion with respect to the filtration as the inverse limit:

$$\hat{E} := \lim_{\leftarrow}(E/F^nE)$$

This again an algebraic group. If E has additional algebraic structure compatible with filtration then its completion is an again an object with the same structure that is completenin the topology determined by the filtration. One of the example is a so called Krull topology which defined by the ideal I of ring R and filtration $F^nR := I^n$. If A be an algebra over ring of formal power series $k[[\hbar]]$the ideal $I = F^1A$, where filtration is defined by degrees of $\hbar$. In the last case we obtain completion in the $\hbar$-adic toplogy which used further.
Basic problems

1. Isomorphism $\Phi : \hat{U}_q(L\mathfrak{g}) \to \hat{Y}_\hbar(\mathfrak{g})$ between completions of (super)Yangian and quantum loop superalgebra. It has been constructed for the Lie superalgebras of $A(m,n)$ (for other types is interesting question which in progress).

2. Equivalence between some counterparts of category $\mathfrak{O}$ of representations of super Yangian and quantum loop superalgebra.

3. Description Hopf superalgebra structures for superYangians and quantum loop superalgebras and relations between these structures. Construction of quantum affine Weyl groupoid

4. Geometric representation theory, study of Nakajima supervarities.

5. Affine super Yangians or Yangians of Kac-Moody superalgebras.

Current Lie superalgebra $\mathfrak{g}[\mathfrak{t}]$ quantization $\to Y_\hbar(\mathfrak{g})$, loop Lie superalgebra $\mathfrak{g}[\mathfrak{t},\mathfrak{t}^{-1}]$ quantization $U_q(L\mathfrak{g})$. Let's consider homomorphism defined on generators by the following formula

$$\exp^* : \mathfrak{g}[\mathfrak{t},\mathfrak{t}^{-1}] \to \hat{\mathfrak{g}}[s], \quad X \otimes t^n \mapsto X \otimes e^{ns}, n \in \mathbb{Z}.$$

We would to quantize the mapping $\exp^* : \mathfrak{g}[\mathfrak{t},\mathfrak{t}^{-1}] \to \hat{\mathfrak{g}}[s]$ and obtain $\Phi : \hat{U}_q(L\mathfrak{g}) \to \hat{Y}_\hbar(\mathfrak{g})$.

Special linear superalgebra

Definition

The Lie superalgebra $\mathfrak{g} = A(m,n)$ is generated by the generators: $h_i, x_i^{\pm}$, $i \in I$. The generators $x_{m+1}^{\pm}$ are odd, while the remaining generators are even, that is, the parity function p takes on the following values: $p(h_i) = 0, i \in I, p(x_j^{\pm}) = 0, j \neq m+1, p(x_{m+1}^{\pm}) = 1$. These generators satisfy the following defining relations:

$[h_i, h_j] = 0, \quad [h_i, x_j^{\pm}] = \pm a_{ij} x_j^{\pm}$,

$[x_i^+, x_j^-] = \delta_{ij} h_i$,

$[[x_m^{\pm}, x_{m+1}^{\pm}], [x_{m+1}^{\pm}, x_{m+2}^{\pm}]] = 0$ if α_{m+1} (This definition is suitable for arbitrary Dynkin diagram if α_{m+1} be an arbitrary odd simple root.)

$ad^{1-\tilde{a}_{ij}}(x_i^{\pm}) x_j^{\pm} = 0$.

$[x_i^{\pm}, [x_i^{\pm}, x_j^{\pm}]] = 0$ if $|i-j| = 1$, $[x_i^{\pm}, x_j^{\pm}] = 0$ elsewhere $(i \neq j)$.

What is the analogue of the Weyl group for Lie superalgebras?

Weyl groupoid

We will use notion of Cartan data.

Definition

Let $\mathscr{C} = \mathscr{C}(A, D, E, (\Pi^d)_{d\in D}, (\rho_\alpha)_{\alpha\in\tau^d, d\in D}, (C^d)_{d\in D})$ be a Cartan scheme, $A = \{a_d\}_{d\in D}$ be a set of Dynkin diagrams, Π^d is a basis of E, . For all $d \in D$ and $\alpha, \beta \in \Pi^d$ define the function $\sigma_\alpha^d \in GL_N(E)$ by

$$\sigma_\alpha^d(\beta) = \beta - c_{\alpha,\beta}^d \alpha.$$

We will also use equivalent data $d = (E, \Pi, \Pi^*, p)$, where $p : E \to E$ be a parity function $(\alpha_i, \check{\alpha}_j) = c_{ij}^d$, $\alpha_i \in \Pi, \check{\alpha}_j \in \Pi^*$. The Weyl groupoid of $\mathscr{C}$ is the supercategory $\mathscr{W}(\mathscr{C})$ such that $\mathrm{Obj}(\mathscr{W}(\mathscr{C})) = A$ and the morphisms are compositions of maps σ_α^d with $d \in D$ and $\alpha \in \tau^d$, where σ_α^d is considered as an element of super vector space $\mathrm{Hom}_{\mathscr{W}(\mathscr{C})}(a_d, \rho_\alpha(a_d))$. The cardinality of D is the rank of $\mathscr{W}(\mathscr{C})$.

Quantum Loop Superalgebra

Let denote by $L\mathfrak{g}$ the Lie algebra of (Laurent polynomial) loops with values in the simple Lie algebra (basic Lie superalgebra) $\mathfrak{g}$. We note that this is simply an affine (super) Kac-Moody algebra without a central element (and a grading element d that defines differentiation d) or, more precisely, factoralgebra of (derived) affine (super)algebra by center.
Let $I = \{1, 2, \ldots, m+1, \ldots m+n+1\}$. We will identify these set with the set of simple roots $\Gamma = \{\alpha_1, \alpha_2, \ldots, \alpha_m, \alpha_{m+1}, \ldots \alpha_{m+n+1}\}$ of basic Lie superalgebra. We'll suppose that α_{m+1} be an odd simple root, that is, we will deal with the isolated system of simple roots of the basic Lie superalgebra $A(m, n)$.

Definition of the Quantum Loop Superalgebra

Definition

Let $U_\hbar(L\mathfrak{g})$ be an associative superalgebra over $C[[\hbar]]$ generated by $\{E_{i,k}, F_{i,k}, H_{i,k}\}_{i\in I, k\in\mathbb{Z}}$, such that:

Q1) For all $i,j \in I$ and $r,s \in \mathbb{Z}$ $[H_{i,r}, H_{j,s}] = 0$.

Q2) For all $i,j \in I$ and $k \in \mathbb{Z}$ $[H_{i,0}, E_{j,k}] = a_{i,j} E_{j,k}, \quad [H_{i,0}, F_{j,k}] = -a_{i,j} F_{j,k}$.

Q3) For all $i,j \in I$ and $r,s \in \mathbb{Z}\backslash\{0\}$

$[H_{i,r}, E_{j,k}] = \frac{[ra_{i,j}]_{q_i}}{r} E_{j,r+k}, \quad [H_{i,r}, F_{j,k}] = -\frac{[ra_{i,j}]_{q_i}}{r} F_{j,r+k}$.

Q4) For all $i,j \in I$ and $k,l \in \mathbb{Z}$

$E_{i,k+1}E_{j,l} - q_i^{a_{ij}} E_{j,l}E_{i,k+1} = q_i^{a_{ij}} E_{i,k}E_{j,l+1} - E_{j,l+1}E_{i,k}$,

$F_{i,k+1}F_{j,l} - q_i^{-a_{ij}} F_{j,l}F_{i,k+1} = q_i^{-a_{ij}} F_{i,k}F_{j,l+1} - F_{j,l+1}F_{i,k}$.

Q5) For all $i,j \in I$ and $k,l \in \mathbb{Z}$ $[E_{i,j}, F_{k,l}] = \delta_{i,j} \frac{\psi_{i,k+l} - \varphi_{i,k+l}}{q_i - q_i^{-1}}$.

Q6) Let $i \neq j \in I$ and let $M = 1 - a_{ij}$. For every $k_1, \ldots, k_M \in \mathbb{Z}$ and $l \in \mathbb{Z}$

$\sum_{\pi\in\mathfrak{S}_M} \sum_{s=0}^{M} (-1)^s \begin{bmatrix} M \\ s \end{bmatrix}_{q_i} E_{i,k_{\pi(1)}} \cdot \ldots \cdot E_{i,k_{\pi(s)}} \cdot E_{j,l} \cdot E_{i,k_{\pi(s+1)}} \cdot \ldots \cdot E_{i,k_{\pi(M)}} = 0$,

$\sum_{\pi\in\mathfrak{S}_M} \sum_{s=0}^{M} (-1)^s \begin{bmatrix} M \\ s \end{bmatrix}_{q_i} F_{i,k_{\pi(1)}} \cdot \ldots \cdot F_{i,k_{\pi(s)}} \cdot F_{j,l} \cdot F_{i,k_{\pi(s+1)}} \cdot \ldots \cdot F_{i,k_{\pi(M)}} = 0$,

Q7) $[[E_{m,k}, E_{m+1,0}]_q, [E_{m+1,0}, E_{m+2,r}]_q]_q = 0$,

$[[F_{m,k}, F_{m+1,0}]_q, [F_{m+1,0}, F_{m+2,r}]_q]_q = 0$,

where the elements $\psi_{i,r}, \varphi_{i,r}$ are defined by the following formulas:
$\psi_i(z) = \sum_{r \geq 0} \psi_{i,r} z^{-r} = \exp(\frac{\hbar d_i}{2} H_{i,0}) \exp((q_i - q_i^{-1}) \sum_{s \geq 1} H_{i,s} z^{-s}),$
$\varphi_i(z) = \sum_{r \geq 0} \varphi_{i,r} z^{-r} = \exp(-\frac{\hbar d_i}{2} H_{i,0}) \exp(-(q_i - q_i^{-1}) \sum_{s \geq 1} H_{i,-s} z^{s}),$
where $\psi_{i,-k} = \varphi_{i,k} = 0$ for $k \geq 1$. Here, $p(H_{i,r}) = 0$ for $i \in I, r \in \mathbb{Z}_+$, and $p(E_{i,r}) = p(F_{i,r}) = 0$ for $i \in I \backslash \{m+1\}, r \in \mathbb{Z}$, and $p(E_{m+1,r}) = p(F_{m+1,r}) = 0$ for $r \in \mathbb{Z}$.
Let us note, that this definition is suitable for the an arbitrary Dynkin diagram, but in this case $E_{m+1} = E_{\alpha_{m+1}}, F_{m+1} = F_{\alpha_{m=1}}$, α_{m+1}be an arbitrary simple odd root.

Yangian of special linear Lie superalgebra

Definition

Let, $\mathfrak{g} = A(m,n)$, $Y_\hbar(\mathfrak{g})$ be an associative (super)algebra generated by generators $\{x_{i,r}^{\pm}, h_{i,r}\}_{i\in I, r\in\mathbb{Z}_+}$, which satisfied the following defining relations:

Y1) For all $i,j \in I$ and $r,s \in \mathbb{Z}_+$ $[h_{i,r}, h_{j,l}] = 0$.

Y2) For all $i,j \in I$ and $s \in \mathbb{Z}_+$ $[h_{i,0}, x_{j,s}^{\pm}] = \pm d_i a_{ij} x_{j,s}^{\pm}$.

Y3) For all $i,j \in I$ and $r,s \in \mathbb{Z}_+$

$[h_{i,r+1}, x_{j,s}^{\pm}] - [h_{i,r}, x_{j,s+1}^{\pm}] = \pm\frac{d_i a_{ij}\hbar}{2}(h_{i,r}x_{j,s}^{\pm} + x_{j,s}^{\pm}h_{i,r})$.

Y4) For every $i,j \in I$ and $r,s \in \mathbb{Z}_+$

$[x_{i,r+1}^{\pm}, x_{j,s}^{\pm}] - [x_{i,r}^{\pm}, x_{j,s+1}^{\pm}] = \pm\frac{d_i a_{ij}\hbar}{2}(x_{i,r}^{\pm}x_{j,s}^{\pm} + x_{j,s}^{\pm}x_{i,r}^{\pm})$.

Y5)For every $i,j \in I$ and $r,s \in \mathbb{Z}_+$ $[x_{i,r}^{+}, x_{j,l}^{-}] = \delta_{i,j}h_{i,r+s}$.

Y6) Let $i \neq j \in I$ and let $M = 1 - d_i a_{ij}$. Then for all $k_1, \ldots, k_M \in \mathbb{Z}$ and $l \in \mathbb{Z}_+$ $\sum_{\pi\in\mathfrak{S}_M}[x_{i,r_{\pi(1)}}^{\pm}, [x_{i,r_{\pi(2)}}^{\pm}, \ldots, [x_{i,r_{\pi(M)}}^{\pm}, x_{j,s}^{\pm}]\ldots]] = 0$.

Y7) $[[x_{m,k}^{\pm}, x_{m+1,0}^{\pm}], [x_{m+1,0}^{\pm}, x_{m+2,t}^{\pm}]] = 0$, $k,t \in \mathbb{Z}$.

The same way we can define super Yangian for arbitrary simple root system.

Definition

Let's note that $Y_\hbar(\mathfrak{g})$ be a $\mathbb{Z}_+$-graded superalgebra with grading defining by the following conditions on generators: $deg(h_{i,r}) = deg(x_{i,r}^{\pm}) = r$ и $deg(\hbar) = 1$. Moreover, $p(h_{i,r}) = 0$ for $i \in I, r \in \mathbb{Z}_+$, and $p(x_{i,r}^{\pm}) = 0$ for $i \in I \backslash \{m+1\}$, and $p(x_{m+1,r}^{\pm}) = 1$ for $r \in \mathbb{Z}_+$. As above, this definition is suitable for the an arbitrary Dynkin diagram, but in this case $x_{m+1}^{\pm} = x_{\alpha_{m+1}}^{\pm}$ and α_{m+1}be an arbitrary simple odd root.

We consider two types of the reflections $s_i : E = E_0 \oplus E_1 \to E = E_0 \oplus E_1$, $\tilde{s}_i : E \to E$, where in the second case we consider E as a vector space without graiding. In the first case $s_i(\Pi) = \Pi_1$, but $\tilde{s}_i(\Pi) \subset R$, where R is the root slattice generated by Π.

Theorem

Let Π, Π_1 be a two different system of simple roots and $s : E \to E$ be an element of Weyl groupoid such that $s(\Pi) = \Pi_1$, then exist isomorphism

$$T_s : Y(\mathfrak{g}(E, \Pi, p)) \to Y(\mathfrak{g}(E, \Pi_1, p)) \tag{1}$$

which induced by sand defined by the following formulas:

$$T_i(x^{\pm}_{\alpha_j,k}) = x^{\pm}_{\tilde{s}_i^{-1}(\beta_j),k}, \quad T_i(h_{\alpha_j,k}) = h_{\tilde{s}_{ii}^{-1}(\beta_j),k}, \quad \beta_j = s_i(\alpha_j).$$

Representation theory of super Yangian and quantum loop superalgebra

We formulate here the main results on classification finite dimensional irreducible representations of Yangians and qiantum loop algebras for special linear superalgebra.

Theorem

([9]) 1) Every irreducible finite-dimensional $Y_\hbar(A(m,n))$-module V is a module with highest weight d :$V = V(d)$, i. e.,
$h_i(u)v_0 = \left(1+\hbar\sum_{k=0}^{\infty} h_{i,k}\cdot u^{-k-1}\right) v_0 = \left(1+\hbar\sum_{k=0}^{\infty} d_{i,k}\cdot u^{-k-1}\right) v_0$ *, where v_0 is a highest vector, and $i = \{1,2,\ldots,m+n+1\}$.*
2) The module $V(d)$ is finite-dimensional if and only if there exist polynomials P_i^d, $i\in\{1,2,\ldots,m,m+2,\ldots m+n+1\} = I\backslash\{m+1\}$, as well as polynomials P_{m+1}^d, Q_{m+1}^d, which satisfy the following conditions:
a) all these polynomials with leading coefficients equal to 1 (or monic);
b) $\dfrac{P_i^d(u+d_ia_{ii}\hbar/2)}{P_i^d(u)} = 1+\hbar\sum_{k=0}^{\infty} d_{i,k}\cdot u^{-k-1}, \quad i\in I\backslash\{m+1\},$
$\dfrac{P_{m+1}^d(u)}{Q_{m+1}^d(u)} = 1+\hbar\sum_{k=0}^{\infty} d_{m+1,k}\cdot u^{-k-1}$. *Here d_ia_{ii} is the matrix element of the symmetrized Cartan matrix of the Lie superalgebra $A(m,n)$.*

Theorem

([5]) 1. Every irreducible finite-dimensional $U_\hbar(LA(m,n))$ *-module* V *is a module with highest weight* δ *:*$V = V(\delta)$*, i.e.*
$\psi_i(z)v_0 = \left(\sum_{k=0}^{\infty} \delta_{i,k}^{+} \cdot z^{-k}\right) v_+, \quad \varphi_i(z)v_0 = \left(\sum_{k=0}^{\infty} \delta_{-i,k}^{-} \cdot z^{k}\right) v_+$, *where* v_0 *is a highest vector and* $i = \{1, 2, \ldots, m+n+1\}$.
2. The module $V(\delta)$ *is finite-dimensional if and only if there exist polynomials* P_i^δ, $i \in \{1, 2, \ldots, m, m+2, \ldots m+n+1\} = I \backslash \{m+1\}$, *as well as polynomials* $P_{m+1}^\delta, Q_{m+1}^\delta$, *which satisfy the following conditions:*
a) all these polynomials with leading coefficients equal to 1 and non-zero free terms;
b) $q^{-d_i a_{ii}/2} \dfrac{P_i^\delta(q^{d_i a_{ii}} z)}{P_i^\delta(z)} = \sum_{k=0}^{\infty} \delta_{i,k}^{+} \cdot z^{-k} = \sum_{k=0}^{\infty} \delta_{i,-k}^{-} \cdot z^{k}, \quad i \in I \backslash \{m+1\}$,
$\dfrac{P_{m+1}^\delta(z)}{Q_{m+1}^\delta(z)} = \sum_{k=0}^{\infty} \delta_{m+1,k}^{+} \cdot z^{-k} = \sum_{k=0}^{\infty} \delta_{m+1,-k}^{-} \cdot z^{k}$.

Mapping Φ

Let $\{E_{i,r}, F_{i,r}, H_{i,r}\}_{i\in I, r\in\mathbb{Z}}$ be current generators of Quantum Loop Superalgebra $U_\hbar(L\mathfrak{g})$, and $\{e_{i,k}, f_{i,k}, h_{i,k}\}_{i\in I, k\in\mathbb{Z}_+}$ be generators of Yangian $Y_\hbar(\mathfrak{g})$. Let's define the map ([10])

$$\Phi : U_\hbar((L\mathfrak{g}) \to \widehat{Y_\hbar(\mathfrak{g})} \tag{2}$$

on generators by the following formulas:

$$\Phi(H_{i,r}) = \frac{\hbar}{q_i - q_i^{-1}} \sum_{k\geq 0} t_{i,k} \frac{r^k}{k!}, \tag{3}$$

$$\Phi(E_{i,r}) = e^{r\sigma_i^+} \sum_{m\geq 0} g_{i,m}^+ e_{i,m}, \Phi(F_{i,r}) = e^{r\sigma_i^-} \sum_{m\geq 0} g_{i,m}^- f_{i,m}. \tag{4}$$

Here we use the notations: $q = e^{\hbar/2}$, $q_i = q^{d_i}$, d_i be elements of symmetrizable matrix $D = diag[d_1, \ldots, , d_{m+n+1}]$ of Cartan matrix $A = (a_{i,j})$ of Lie Superalgebra $\mathfrak{g} = A(m,n)(d_i = 1, i \in \{1, \ldots, m\}$, $d_i = -1, i \in \{m+1, \ldots, m+n+1\}$.

We'll use the logarithmic generators $\{t_{i,r}\}_{i\in I, r\in\mathbb{N}}$ of commutative subsuperalgebra $Y_\hbar(\mathfrak{h}) \subset Y_\hbar(\mathfrak{g})$ which generated by generators $\{h_{i,r}\}_{i\in I, r\in\mathbb{Z}_{\nvdash}}$. These logarithmic generators of quantum universal enveloping superalgebra defined by the followinq equality for generating functions of generators: $\hbar\sum_{r\geq 0} t_{i,r}u^{-r-1} = \log(1+\sum_{r\geq 0} h_{i,r}u^{-r-1})$.

The elements $\{g_{i,m}^{\pm}\}_{i\in I, m\in Z_0}$ belongs to the completion $\widehat{Y^0}$ of superalgebra $Y^0 = Y_\hbar(\mathfrak{h})$ and defined as follows. Let's consider the following formal power series : $G(v) = \log\left(\frac{v}{e^{v/2}-e^{-v/2}}\right) \in Q[[v]]$ and define $\gamma_i \in \hat{Y^0}[v]$ by formula:

$$\gamma_i(v) = \hbar \sum_{r\geq 0} \frac{t_{i,r}}{r!}\left(-\frac{d}{dv}\right)^{r+1} G(v).$$

Then,

$\sum_{m\geq 0} g_{i,m}^{\pm} v^m = \left(\frac{\hbar}{q_i - q_i^{-1}}\right)^{1/2} \exp\left(\frac{\gamma_i(v)}{2}\right).$

Finally, $\sigma_i^{\pm}$ are homomorphisms of subsuperalgebras $\sigma_i^{\pm} : Y_\hbar(\mathfrak{b}_{\pm})(\subset Y_\hbar(\mathfrak{g})) \to Y_\hbar(\mathfrak{b}_{\pm})$, which defined onto generators $\{h_{i,r}, e_{i,r} = x_{i,r}^{+}, f_{i,r} = x_{i,r}^{-}\}$ as follows. They leave as fixed generators $h_{i,k}$, and on other generators they act as shifts:
$\sigma_i^{+} : e_{j,r} \to e_{j,r+\delta_{ij}}, \sigma_i^{-} : f_{j,r} \to f_{j,r+\delta_{ij}}$. These homomorphisms are little modifications introduced by Drinfel'd homomorphisms $T_{\pm}$. These mappings : $\sigma_i^{\pm} : x_{j,r}^{\pm} \mapsto x_{j,r+\delta_{i,j}}^{\pm}, \quad h_{j,r} \mapsto h_{j,r}$ can be continued to homomorphism of associative superalgebras $\hat{Y}^{\pm} \to \hat{Y}^{\pm}$.

Main result

Theorem

1) The mapping $\Phi : U_\hbar(L\mathfrak{g}) \to \widehat{Y_\hbar(\mathfrak{g})}$, is uniquely defined by formulas (2), (3) and it is homomorphism of associative superalgebras.
2) The mapping Φ uniquely continued to homomorphism of topological completion : $\hat{\Phi} : \widehat{U_\hbar(L\mathfrak{g})} \to \widehat{Y_\hbar(\mathfrak{g})}$ of these superalgebras. Moreover, mapping $\hat{\Phi}$ be an isomorphism of topological superalgebras.

Proof of the theorem

I briefly describe the scheme of proving of the main result. The proof based on the classification of irreducible finite dimensional representations of Yangian of special linearsuperalgebra and ones for the quantum loop superalgebra. We also use the reduction to the partial cases $\mathfrak{sl}(1,1)$ and $\mathfrak{sl}(2)$. I recall definitions of Yangian and the quantum loop algebra in the firrst case of Lie superalgebra $\mathfrak{sl}(1,1)$.

Definition

Yangian $Y(\mathfrak{sl}(1,1))$ is generated by the generators $h_n, e_n, f_n, n \in \mathbf{Z}_0$, which satisfy the following defining relations:

$[h_k, h_l] = 0, \quad [h_k, e_l] = [h_k, f_l] = 0,$

$[e_k, e_l] = [f_k, f_l] = 0,$

$[e_k, f_l] = h_{k+l}.$

The quantum loop algebra $U_\hbar(L\mathfrak{sl}(1,1))$ is generated by the generators $\{E_n, F_n, H_n\}_{n\in\mathbb{Z}}$, which satisfy the following system of defining relations:

$[H_r, H_s] = 0,$

$[H_r, E_s] = [H_r, F_s] = 0, \qquad [E_r, E_s] = [F_r, F_s] = 0,$

$$[E_r, F_s] = \frac{\psi_{r+s} - \varphi_{r+s}}{e^{\hbar/2} - e^{-\hbar/2}},$$

for all $r, s \in \mathbb{Z}$. Here as above elements ψ_r, φ_r are defined by the following formulas: $\psi_i(z) = \sum_{r\geq 0} \psi_r z^{-r} = \exp(\frac{\hbar}{2} H_0) \exp((e^{\hbar/2} - e^{-\hbar/2}) \sum_{s\geq 1} H_s z^{-s}),$
$\varphi_i(z) = \sum_{r\geq 0} \varphi_r z^r = \exp(-\frac{\hbar}{2} H_0) \exp(-(e^{\hbar/2} - e^{-\hbar/2}) \sum_{s\geq 1} H_{-s} z^s).$

Lemma

The following equality holds

$$\Phi\left(\frac{\psi_k - \varphi_k}{e^{\hbar/2} - e^{-\hbar/2}}\right) = \frac{\hbar}{e^{\hbar/2} - e^{-\hbar/2}} e^{kv} \exp(\gamma(v))|_{v^n = h_n}.$$

Proof.

We first calculate the left-hand side of the equality using description of action of elements of quantum superalgebras in the above mentioned universal modules. $\square$

Easy to see that from lemma follows equality :
$[\Phi(E_r), \Phi(F_l)] = \dfrac{\Phi(\psi_{r+s}) - \Phi(\varphi_{r+s})}{e^{\hbar/2} - e^{-\hbar/2}}$, since, as was proved earlier, by the injectivity of the mapping D^U, which follows from the first and second lemmas, left-hand side of proved equality is equal to $e^{kv}\exp(\gamma v)|_{v^n=h_n}$.
From the second lemma and the injectivity of the map D^Y, which also follows from the injectivity of homomorphisms D^Y and D^U, it follows that the right-hand side of the proved equality also is equal to $e^{kv}\exp(\gamma v)|_{v^n=h_n}$. Theorem is proved.
We reduced general case to the partial. The proof more technical.

Categories of modules

Let Π, Ω be some subsets of the complex plane are invariant with respect to the group of additive shifts on $\hbar$, respectively, multiplicative shifts by q. We introduce the category $\mathfrak{O}^{\Pi}(Y_{\hbar}(\mathfrak{g}))$ as a full subcategory of category $\mathfrak{O}(Y_{\hbar}(\mathfrak{g}))$ consisting of the representations V such that, for every $(\lambda, \{P_i^d\}, Q_{m+1}^d) \in \Pi_+^Y$ for which $[V : L(\lambda, \{P_i^d\}, Q_{m+1}^d)] \neq 0$, the roots of $P_i^d, i \in I, Q_{m+1}^d$ lie in Π.
Let similarly $\Omega \in \mathbb{C}^*$ be a subset stable under multiplication by $q^{\pm}$. We define $\mathfrak{O}^{\Omega}(U_q(L\mathfrak{g}))$ to be a the full subcategory of $\mathfrak{O}(U_q(L\mathfrak{g}))$ consisting those U such that for every $(\lambda, \{P_i^{\delta}\}, Q_{m+1}^{\delta}) \in \Pi_+^U$ for which $[U : L(\lambda, \{P_i^{\delta}\}, Q_{m+1}^{\delta})] \neq 0$, the roots of $P_i^{\delta}, i \in I, Q_{m+1}^{\delta}$ lie in Ω.
We have the folloing important result

Theorem

([3])(1) Let $V \subset \mathfrak{O}_{int}(Y_\hbar(\mathfrak{g}))$ *. Then the following conditions are equivalent.*
$V \subset \mathfrak{O}^{\Pi}_{int}(Y_\hbar(\mathfrak{g}))$ *.*
(ii) $\sigma(V) \subset \Pi$ *.*
(iii) Poles of $h_i(u)^{\pm 1}$ *are contained in* Π *for* $i \in I$*.*
(iv) Eigenvalues of $h_i(u)$ *are rational functions of parameter* u*, which have poles and zeroes in* Π*.*
(2) Let $U \subset \mathfrak{O}_{int}(U_q(L\mathfrak{g}))$*. Then the following conditions are equivalent.*
(i) $U \subset \mathfrak{O}^{\Omega}_{int}(U_q(L\mathfrak{g}))$*.*
(ii) $\sigma(U) \subset \Omega$*.*
(iii) Poles of $\psi_i^{\pm 1}(z)$ *are contained in* Ω*.*
(iv) Eigenvalues of $\psi_i^{\pm 1}(z)$ *are rational functions of parameter* z*, which have poles and zeroes in* Ω*.*

Now we formulate the second main result

Theorem

([3]) i) $\mathfrak{O}^{\Pi}(Y_{\hbar}(\mathfrak{g}))$ and $\mathfrak{O}^{\Omega}(U_q(L\mathfrak{g}))$ are Serre subcategories of $\mathfrak{O}(Y_{\hbar}(\mathfrak{g}))$ and $\mathfrak{O}(U_q(L\mathfrak{g}))$, respectively. Other words, they are closed under taking direct sum, subobjects, quotients and extensions.
(ii) $\mathfrak{O}^{\Pi}(Y_{\hbar}(\mathfrak{g}))$ and $\mathfrak{O}^{\Omega}(U_q(L\mathfrak{g}))$ are closed under tensor product.
(iii) Categories $\mathfrak{O}^{\Pi}(Y_{\hbar}(\mathfrak{g}))$ and $\mathfrak{O}^{\Omega}(U_q(L\mathfrak{g}))$ are equivalent.
(iv) There is exist exact faithful monoidal functor

$$\Phi : \mathfrak{O}^{\Pi}(Y_{\hbar}(\mathfrak{g})) \to \mathfrak{O}^{\Omega}(U_q(L\mathfrak{g})) \tag{5}$$

which is category equivalence.

Drinfeld comultiplication

Now we consider so called Drinfeld comultiplication. We define Drinfeld comultipplication on the imagesof genrators in tensor product of representations $V \otimes W$, where $V, W \in Rep_{fd}(Y(\mathfrak{g}))$. Then Drinfeleld comultiplicaton on super Yangian $Y_\hbar(\mathfrak{g})$,$\mathfrak{g} = A(m,n)$, is defined by the following formulas

$\Delta_s(h_i(u)) = h_i(u-s) \otimes h_i(u), \quad \Delta_s(x_i^+(u)) =$
$x_i^+(u-s) \otimes 1 + \oint_{C_2} \dfrac{1}{u-v} h_i(v-s) \otimes x_i^+(v)dv, \quad \Delta_s(x_i^-(u)) =$
$\oint_{C_1} \dfrac{1}{u-v} x_i^-(v-s) h_i(v)dv + 1 \otimes x_i^-(u).$

We also can defined the Drinfeld comultiplication on $U_q(L\mathfrak{g})$ by the formulas

$\Delta_u(\Psi_i(z)) = \Psi_i(u^{-1}z) \otimes \Psi_i(z), \quad \Delta_u(X_i^+(z)) =$
$X_i^+(u^{-1}z) \otimes 1 + \oint_{C_2} \dfrac{zw^{-1}}{z-w} \Psi_i(u^{-1}w) \otimes X_i^+(w)dw, \quad \Delta_u(X_i^-(z)) =$
$1 \otimes X_i^-(u^{-1}z) + \oint_{C_1} \dfrac{zw^{-1}}{z-w} X_i^-(w) \otimes \Psi_i(u^{-1}w)dw.$
$X_i^\pm(z) = \sum_{k\in\mathbb{Z}} X_{i,k}^\pm z^k, \quad \Psi_i(z) = \psi_i^+(z) - \psi_i^-(z), \quad X_{i,k}^+ = E_{i,k}, X_{i,k}^- = F_{i,k}.$
new

Theorem

(1) Above defined exact faithful monoidal functor $\Phi : \mathfrak{O}^{\Pi}(Y_{\hbar}(\mathfrak{g})) \to \mathfrak{O}^{\Omega}(U_q(L\mathfrak{g}))$ induced isomorphism of Hopf superalgebra isomorphism

$$\hat{\Phi} : \widehat{U_{\hbar}(L\mathfrak{g})} \to \widehat{Y_{\hbar}(\mathfrak{g})}$$

of completions of super Yangian and quantum loop superalgebra with above defined Drinfeld comultiplications.

Quantum affine Weyl groupoid and classification of Hopf superalgebra structures

1. What happens to the structure of the Hopf superalgebra when the isomorphism defined above is applied?
2. How to describe all possible structures of the Hopf superalgebra on the Yangian and the quantum loop superalgebra if we consider them only with a fixed structure of the associative superalgebra?

The answers to these questions can be given in terms of the so-called Weyl quantum affine groupoid, and this is an important part of the still unfinished work.
What does mean Weyl groupoid?
We define the Weyl groupoid as a some supercategory. Based on this abstract definition, we give an explicit realization of the Weyl quantum groupoid in terms of isomorphisms of quantum superalgebras generated by isomorphisms, which are induced by reflections (relatively both even and odd roots). We interpret the elements of the quantum Weyl groupoid as adjoint maps given by the elements of the some extension of quantum superalgebra $U_q(sl(m|n))$. Thus, we associate the elements of the quantum Weyl groupoid with the group-like elements of the quantum superalgebra. Among other things, we note that these elements satisfy some analogue of the braid group relations and define braid groupoid structure. It should be

noted that the classical Weyl groupoid, acting by reflections with respect to odd roots, permutes non-conjugate Borel subalgebras. Thus, it permutes the corresponding non-isomorphic Manin triples defined by these Borel subalgebras. This gives a natural description and, moreover, a classification of the structures of Lie bisuperalgebras on superalgebra $sl(m|n)$. We also consider quantum affine Weyl groupoid and present its elements as elements of extension of completions of super Yangian and quantum affine superalgebra (and quantum loop superalgebra). Quantum Weyl groupoid for super Yangian.

$$T_i = \exp adx_{i,0}^{+} \exp adx_{i,0}^{-} \exp adx_{i,0}^{+}.$$

Theorem

$$T_i(x^+_{j,1}) = \begin{cases} -x^-_{i,1} + \frac{\hbar}{2}\{h_{i,0}, x^-_{i,0}\}, & \textit{if} \quad i = j, \\ [x^+_{i,0}, x^+_{j,1}], & \textit{if} \quad a_{ij} = \pm 1, \\ x^+_{j,1}, & \textit{if} \quad a_{ij} = 0 \end{cases},$$

$$T_i(\tilde{h}_{j,1}) = \begin{cases} -\tilde{h}_{i,1} - \frac{\hbar}{2}\{x^+_{i,0}, x^-_{i,0}\}, & \textit{if} \quad i = j, \\ \tilde{h}_{i,1} + \tilde{h}_{j,1} + \frac{\hbar}{2}\{x^+_{i,0}, x^-_{i,0}\}, & \textit{if} \quad a_{ij} = \pm 1, \\ \tilde{h}_{j,1}, & \textit{if} \quad a_{ij} = 0 \end{cases}.$$

$$T_i(x^-_{j,1}) = \begin{cases} -x^-_{i,1} + \frac{\hbar}{2}\{h_{i,0}, x^+_{i,0}\}, & \textit{if} \quad i = j, \\ -[x^-_{i,0}, x^-_{j,1}], & \textit{if} \quad a_{ij} = \pm 1, \\ x^-_{j,1}, & \textit{if} \quad a_{ij} = 0 \end{cases},$$

Theorem

The action of generators T_i is compatible with comultiplication and Drinfel'd comultiplication

$$\Delta(T_i) = (T_i \otimes T_i) \circ \Delta, \quad \Delta^D_s(T_i) = (T_i \otimes T_i) \circ \Delta^D_s.$$

Quantum Weyl group action on quantum loop superalgebra. We introduce the element T_i^L which is natural analogue of Lusztig automorphisms.

$$T_i^L(x) = U_i x U_i^{-1},$$

$$U_i = \exp_{q_i^{-1}}(-q_i^{-1} F_{i,0} q_i^{H_{i,0}}) \exp_{q_i^{-1}}(E_{i,0}) \exp_{q_i^{-1}}(-q_i^{-1} F_{i,0} q_i^{H_{i,0}}) q_i^{H_{i,0}(H_{i,0}+1)/2}.$$

$$\exp_p(x) = \sum_{n \geq 0} \frac{1}{[n]_p} p^{m(m-1)/2} x^n$$

Theorem

The action of generators T_i^L is compatible with comultiplication and Drinfel'd comultiplication

$$\Delta_z(T_i^L) = (T_i^L \otimes T_i^L) \circ \Delta_z, \quad \Delta_z^D(T_i^L) = (T_i^L \otimes T_i^L) \circ \Delta_z^D$$

Quiver super varieties and geometric realization of super Yangians and quantum loop superalgebras

Now, we describe super analogues of Nakajima quiver varieties. We roughly describe the idea of definition. The quiver variety corresponding to a weight space of the representation of $\mathfrak{sl}(m|n)$ is a cotangent bundle of an N-step partial (super)flag variety. In particular, for N $=2$, it is the cotangent bundle of a (super)Grassmannian. Let us quiver Q which is a linear graph with two univalent vertices, N_Q bi-valent vertices and $N_Q$edges. A vertex (i) is assigned a non-negative integers $n_i=(n_i^0,n_i^1)$ in representing the n_i- (super)dimensional vector space V_i and the group $G_i=GL(V_i)$. We omit now explicite defintion of this action. We assume that $n_0=n_{N_Q}=(0.0)$. An oriented edge connecting the vertices (i$-$1) and (i) is assigned a symplectic variety (X_i^s,ω) with the Hamiltonian action of $G_{i-1}\times G_i$ and the corresponding moment maps μ_{i-1}^R and μ_i^L. The quiver variety X_Q is a result of the Hamiltonian reduction of the product of edge varieties with respect to all vertex groups $X_Q:=X_e|_{\mu_i=0}//G_v$, where $X_e=\prod_{i=1}^{N_Q}X_i^s$, $G_v=\prod_{i=0}^{N_Q}GL(V_i)$, $GL(V_i)=GL(V_i^0)\oplus GL(V_i^1)$ $\mu_i=\mu_i^L+\mu_i^R$is the total moment map of vertex (i). We will consider two types of symplectic manifolds X_i^s arroy type and bow (twisted) type. We will use also Legendre transformation to change parity and obtain supervariety.

We have a family of quivers and action of superWeyl group on corresponding family of quiver variteies. The odd elements of Weyl group permutate the elements of these family.

Affine super Yangian

Now, we define affine super Yangian $Y(\tilde{\mathfrak{g}}(E,\Pi,p)$ of affine superalgebra $\tilde{\mathfrak{g}}(E,\Pi,p) = \tilde{sl}(m,n)$ for arbitrary system of simple roots Π. Let $\Pi = \alpha_0, \alpha_1, \ldots, \alpha_{m+n-1}$ be a simple rooot sistem.

Definition

The affine super Yangian $Y_{\varepsilon_1,\varepsilon_2}(\tilde{sl}(E,\Pi,p))$ is an associative superalgebra over $\mathbb{C}$ generated by $x_{i,r}^{\pm}, h_{i,r}$, $0 \leq i \leq m+n-1$, $\quad r = 0,1$, $\tilde{h}_{i,1} = h_{i,1} - \frac{1}{2}h_{i,0}^2$, subject to the defining relations

$$[h_{i,0}, h_{j,0}] = 0, [h_{i,1}, h_{j,0}] = 0, [x_{i,0}^{+}, x_{j,1}^{-}] = [x_{i,1}^{+}, x_{j,0}^{-}] = [x_{i,0}^{+}, x_{j,1}^{-}] = \delta_{i,j} h_{i,1}, \quad (6)$$

$$[h_{i,0}, x_{j,r}^{\pm}] = \pm a_{ij} x_{j,r}^{\pm}, \quad [\tilde{h}_{i,1}, x_{j,0}^{\pm}] = \pm a_{ij}(x_{j,1}^{\pm} - (a_{ij} - \delta_{ij} a_{ii})\frac{\varepsilon_1 - \varepsilon_2}{2} x_{j,0}^{\pm}), \quad (7)$$

$$[x_{i,1}^{\pm}, x_{j,0}^{\pm}] - [x_{i,0}^{\pm}, x_{j,1}^{\pm}] = \pm a_{ij} \frac{\varepsilon_1 + \varepsilon_2}{2} \{x_{i,0}^{\pm}, x_{j,0}^{\pm}\} - (a_{ij} - a_{ii}\delta_{ij}) \frac{\varepsilon_1 - \varepsilon_2}{2} [x_{i,0}^{\pm}, x_{j,0}^{\pm}], \quad (8)$$

$$ad(x_{i,0}^{\pm})^{1+|a_{ij}|}(x_{j,0}^{\pm}) = 0, [x_{l,0}^{\pm}, x_{l,0}^{\pm}] = 0, \quad [[x_{l-1,0}^{\pm}, x_{l,0}^{\pm}], [x_{l,0}^{\pm}, x_{l+1,0}^{\pm}]] = 0, l \in \Pi_1. \quad (9)$$

Main theorem

Theorem

The affine super Yangian $Y_{\varepsilon_1,\varepsilon_2}(\tilde{sl}(E,\Pi,p))$ is isomorphic an associative superalgebra over $\mathbb{C}$ generated by $x^{\pm}_{i,r}, h_{i,r}$, $0 \le i \le m+n-1, \quad r = 0,1$, $\tilde{h}_{i,1} = h_{i,1} - \frac{1}{2}h^2_{i,0}$, subject to the defining relations

$$[h_{i,r}, h_{j,s}] = 0, [x^{+}_{i,r}, x^{-}_{j,s}] = \delta_{i,j} h_{i,r+s}, [h_{i,0}, x^{\pm}_{j,r}] = \pm a_{ij} x^{\pm}_{j,r}$$

$$[h_{i,r+1}, x^{\pm}_{j,s}] - [h_{i,r}, x^{\pm}_{j,s+1}] = \pm a_{ij}(\{h_{i,r} x^{\pm}_{j,s}\} - b_{ij}\frac{\varepsilon_1 - \varepsilon_2}{2}[h_{i,r}, x^{\pm}_{j,s}]),$$

$$[x^{\pm}_{i,r+1}, x^{\pm}_{j,s}] - [x^{\pm}_{i,r}, x^{\pm}_{j,s+1}] = \pm a_{ij}\frac{\varepsilon_1 + \varepsilon_2}{2}\{x^{\pm}_{i,0}, x^{\pm}_{j,0}\} - b_{ij}\frac{\varepsilon_1 - \varepsilon_2}{2}[x^{\pm}_{i,0}, x^{\pm}_{j,0}],$$

$$\sum_{w \in \mathfrak{S}_{1+|a_{ij}|}} [x^{\pm}_{i,r_{w(1)}}, [x^{\pm}_{i,r_{w(2)}}, \dots, [x^{\pm}_{i,r_{w(1+|a_{ij}|)}}, x^{\pm}_{j,s}] \dots]] = 0,$$

$$[x^{\pm}_{l,s}, x^{\pm}_{l,r}] = 0, \quad [[x^{\pm}_{l-1,r}, x^{\pm}_{l,0}], [x^{\pm}_{l,0}, x^{\pm}_{l+1,0}]] = 0, l \in \Pi_1.$$

where $b_{ij} = (a_{ij} - a_{ii}\delta_{ij})$.

Theorem

Super Yangians $Y_{\varepsilon_1,\varepsilon_2}(\tilde{sl}(E,\Pi,p))$ *and* $Y_{\varepsilon_1,\varepsilon_2}(\tilde{sl}(E,\Pi_1,p_1))$ *are isomorphic for any simple root systems* Π *and* Π_1*.and isomorphism*

$$T_s : Y_{\varepsilon_1,\varepsilon_2}(\tilde{sl}(E,\Pi,p)) \to Y_{\varepsilon_1,\varepsilon_2}(\tilde{sl}(E,\Pi_1,p_1))$$

is indused by element $s \in \hat{W}$ *of affine Weyl groupid* $\hat{W}$*.*

References

- V. Stukopin : Representations of the Yangian of a Lie superalgebra of type A(m,n). – Izvestija Mathematics. -- 77(2013), No 5, 1021-- 1043 pp.
  - Stukopin V. Isomorphism of the Yangian of the special Lie superalgebra and the quantum loop superalgebra. – Theoretical and mathematical physics,v.198(2019), no 1, p. 129 – 144.
  - Stukopin V. Relation between categories of representations of the super-Yangian of a special linear Lie superalgebra and quntum loop superalgebra. – Theoretical and mathematical physics, v.204(2020), no 3, p. 1227 – 1243.
  - S. Gautam, V. Toledano Laredo Yangians and Quantum Loop Algebras.– Selecta Math., 19(2013), 271–336.
  - Huafeng Zhang Representations of Quantum Affine Superalgebras. – arXiv:math/1309. 5250 [math.QA].
  - S. Gautam, V. Toledano Laredo Meromorphic tensor equivalence for Yangians and quantum loop algebras. – Publ. Math. Inst. Hautes ´ Etudes Sci., 125(2017), 267–337.

S. Gautam, V. Toledano Laredo Yangians, quantum loop algebras and abelian difference equations. –J. Amer. Math. Soc., 29(2016), 775–824.

V. Stukopin Isomorphism between super Yangian and quantum loop superalgebra. – arXiv:math/1804. 06678 [math.QA].

A. Mazurenko, V.A. Stukopin, $\mathfrak{R}$-matrix for quantum superalgebra $\mathfrak{sl}(2|1)$ at roots of unity and its application to centralizer algebras, arXiv:1909.11613 [mathQA].

A. Mazurenko, V.A. Stukopin,Classification of Hopf superalgebras associated with quantum special linear superalgebra at roots of unity using Weyl groupoid, arXiv:2006.06610 math[QA].

A. Mazurenko, V.A. Stukopin,Relationship between super Yangians and quantum loop su[peralgebra via Weyl groupoid, in progress.

Thank you for attention