Relationship between AdS_5 and dS_5 $(N=1)$ - supersymmetries via cliffonic dressing

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Abstract

The concept of cliffonic dressing for two different associative (or Lie) (super)algebras of the same demension is introduced. It is shown that the supersymmetries of anti-de Sitter and de Sitter, which have different gradings (Z_2 and $Z_2 \times Z_2$), are related to each other using cliffonic dressing.

"Everything not forbidden is compulsory"
M. Gell-Mann[1].

[1] The Gell-Mann's totalitarian principle (M. Gell-Mann, Il Nuovo Cimento (1955-1965) 4, 848 (1956)) (see Xiangyi Meng, Phys. Rev. **D104** 016016 (2021), arXiv:2012.05379v2 [hep-th])

Contents

By definition the Clifford algebra $Cl_n^{\circledast}(p,q)$ of the rank $n=p+q$, with the signature (p,q), is a real or complex unital (with a unite) associative algebra generated by the generators c_i, $(i=1,2,\ldots,n)$, which will be called the *cliffons*, with the defining relations

$$\{c_i,c_j\}:=c_ic_j+c_jc_i=\eta_{ij}e,\quad i,j=1,\ldots n, \tag{1}$$

where e is identity element, $\eta=||\eta_{ij}||=\mathrm{diag}(1,\ldots,1,-1,\ldots,-1)$ is the diagonal $(n\times n)$-matrix with its first p entries equal to 1 and the last q entries equal to -1 on the diagonal. The star $\circledast$ is the involutive antiautomorphysm (or automorphism) which acts on the generating cliffons as follows: $(c_i^{\circledast})^{\circledast}=c_i$, $(c_ic_j\cdots c_k)^{\circledast}=c_k^{\circledast}\cdots c_j^{\circledast}c_i^{\circledast}$ (or $(c_ic_j\cdots c_k)^{\circledast}=c_i^{\circledast}c_j^{\circledast}\cdots c_k^{\circledast}$) for $i,j,\ldots,k=1,2,\ldots,n$. The Clifford (cliffonic[2]) algebras $Cl_n^{\circledast}(p,q)$ with the different stars[3] are different and they have the different applications. We consider here a special $Cl_n(p,q)$-application called *cliffonic dressing*. Let $\mathfrak{g}$ and $\mathfrak{g}'$ be two different non-isomorphic (real or complex) Lie (super)algebras (or associative (super)algebras) of the same dimension. Cliffonic dressing is an exact (non-degenerate) special homomorphism $\mathfrak{g}\mapsto Cl_n^{\circledast}(p,q)\otimes\mathfrak{g}'$ for a some clifonic algebra $Cl_n^{\circledast}(p,q)$. In other words, we get the realization of algebra $\mathfrak{g}$ by "dressing" another equidimensional algebra $\mathfrak{g}'$ with the help of $Cl_n^{\circledast}(p,q)$ cliffons.

[2] Below we will also use the name "cliffonic algebra" by analogy with bosonic and fermionic associative algebras.

[3] We will also use a cliffonic algebra $Cl_n(p,q)$ without the star, i.e. $\circledast=\emptyset$.

A. *Cliffonic dressing for the construction of real forms.*

Let's consider a cliffonic algebra $Cl_{\bar{n}}^{*} := Cl_n^{*}(0,n)$ with a negative signature $\eta = ||\eta_{ij}|| = \mathrm{diag}(-1,-1,\ldots,-1)$, and with the following real cliffons: $(c_i c_j\cdots c_k)^{*} = c_i^{*}c_j^{*}\cdots c_k^{*} = c_i c_j\cdots c_k$ for $i,j,\ldots,k = 1,2,\ldots,n$.
In the case of a one-cliffonic algebra $Cl_{\bar{1}}^{*}$, we put $J := c_1$ and we have $J=-1$ and $J^{*}=J$. The element J with such properties is named the operator of a fundamental symmery in the theory of Krein spaces (see [YuVernov]).
It is easy to see that the cliffonic algebras $Cl_n^{\circledast}(n,0)$ and $Cl'^{\circledast}_n(p,q)$ with different signatures and the same star $\circledast$ are connected to each other by means of $Cl_{\bar{1}}^{*}$-dressing, namely,

$$\begin{aligned} c_i' &= ec_i, \quad i=1,\ldots,p,\\ c_i' &= Jc_i, \quad i=p+1,\ldots,n, \end{aligned} \tag{2}$$

or the inverse $Cl_{\bar{1}}^{*}$-dressing

$$\begin{aligned} c_i &= ec_i', \quad i=1,\ldots,p,\\ c_i &= Jc_i', \quad i=p+1,\ldots,n, \end{aligned} \tag{3}$$

where c_i $(i=1,\ldots,n)$ are the cliffons of $Cl_n^{\circledast}(n,0)$ and c_i' $(i=1,\ldots,n)$ are the cliffons of $Cl'^{\circledast}_n(p,q)$ and everywhere we use short notation $ec_i := e\otimes c_i$, $Jc_i := J\otimes c_i$. In the following sections we will use the cliffonic dressing to construct real de Sitter and anti-de Sitter superalgebras.

B. Cliffonic cross-dressing to change Lie (super)brackets

In this special case we will use the cliffonic algebra $Cl_n^* := Cl_n^*(n,0)$ with a positive signature $\eta = ||\eta_{ij}|| = \mathrm{diag}(1,1,\ldots,1)$, and with the following real cliffons: $(c_ic_j\cdots c_k)^* = c_i^*c_j^*\cdots c_k^* = c_ic_j\cdots c_k$ for $i,j,\ldots,k = 1,2,\ldots,n$. We will apply the Cl_n^*-dressing to the well-known bosonic and fermionic associative algebras. Recall that the bosonic algebra B_Λ $(\Lambda\in\mathbb{N})$ generated by the creation b_i^+ and annihilation b_i $(i=1,2,\ldots,\Lambda)$ operators with the following defining relations:

$$\begin{aligned}
&[b_i,b_j] = [b_i^+,b_j^+] = 0,\\
&[b_i,b_j^+] = \delta_{ij}I
\end{aligned}\tag{4}$$

for $i,j=1,2,\ldots,\Lambda$. Analogously, the fermionic algebra F_Λ generated by the creation a_i^+ and annihilation a_i $(i=1,2,\ldots,\Lambda)$ operators with the following defining relations:

$$\begin{aligned}
&\{a_i,a_j\} = \{a_i^+,a_j^+\} = 0,\\
&\{a_i,a_j^+\} = \delta_{ij}I
\end{aligned}\tag{5}$$

for $i,j=1,2,\ldots,\Lambda$.

Now we applay the Cl_n^*-cliffonic dressing to the bosonic and fermion algebras. Let $[\lambda_1,\lambda_2,\ldots,\lambda_n]$, where $\lambda_1\geq\lambda_2\geq\cdots\geq\lambda_n\geq 1$, be the partition of a closed integer interval $[1,\Lambda]$ where $\Lambda=\lambda_1+\lambda_2+\cdots+\lambda_n$. We set for the the bosonic algebra B_Λ:

$$\begin{array}{llll}
\tilde{b}_{\lambda_1 j}=c_1b_j, & \tilde{b}^+_{\lambda_1 j}=c_1b^+_j, & j=1,2,\ldots,\lambda_1, \\
\tilde{b}_{\lambda_2 j}=c_2b_j, & \tilde{b}^+_{\lambda_2 j}=c_2b^+_j, & j=\lambda_1+1,\ldots,\lambda_1+\lambda_2, \\
\ldots\quad\ldots & \ldots\quad\ldots & \ldots\quad\ldots\quad\ldots \\
\tilde{b}_{\lambda_n j}=c_nb_j, & \tilde{b}^+_{\lambda_n j}=c_nb^+_j, & j=\Lambda_{n-1}+1,\ldots,\Lambda_{n-1}+\lambda_n,
\end{array}\tag{6}$$

and also $\tilde{I}=eI$, where $\Lambda_{n-1}:=\sum_{s=1}^{n-1}\lambda_s$. It is easy to check that the new generators (6) satisfy the following relations (instead of (4)) for the given partition $[\lambda_1,\lambda_2,\ldots,\lambda_n]$:

$$\{\tilde{b}_{\lambda_r i},\tilde{b}_{\lambda_{r'} j}\}=\{\tilde{b}^+_{\lambda_r i},\tilde{b}^+_{\lambda_{r'} j}\}=\{\tilde{b}_{\lambda_r i},\tilde{b}^+_{\lambda_{r'} j}\}=0,\tag{7}$$

for $r\neq r'$, and

$$\begin{array}{l}
[\tilde{b}_{\lambda_r i},\tilde{b}_{\lambda_r j}]=[\tilde{b}^+_{\lambda_r i},\tilde{b}^+_{\lambda_r j}]=0, \\
[\tilde{b}_{\lambda_r i},\tilde{b}^+_{\lambda_r j}]=\delta_{ij}\tilde{I},
\end{array}\tag{8}$$

for $r=1,2,\ldots,n$ and $i,j=\Lambda_r+1,\Lambda_r+2,\ldots,\Lambda_r+\lambda_{r+1}$, where $\Lambda_r:=\sum_{s=1}^{r}\lambda_s$.

In the case of a complete cliffonic cross-dressing, when $n = \Lambda$, $\Lambda_r = r$ $(r = 1, 2, \ldots, \Lambda)$, the formulas (6) take the form

$$\tilde{b}_j = c_j b_j, \quad \tilde{b}_j^+ = c_j b_j^+, \quad j = 1, 2, \ldots, \Lambda. \tag{9}$$

These cross-dressing generators satisfy the defining relations:

$$\{\tilde{b}_i, \tilde{b}_j\} = \{\tilde{b}_i^+, \tilde{b}_j^+\} = \{\tilde{b}_i, \tilde{b}_j^+\} = 0, \tag{10}$$

for $i \neq j$, $(i, j = 1, 2, \ldots, \Lambda)$, and

$$[\tilde{b}_i, \tilde{b}_i^+] = \tilde{I} \tag{11}$$

for $i = 1, 2, \ldots, \Lambda$.

In the case of a complete cliffonic cross-dressing, when $n=\Lambda$, $\Lambda_r=r$ $(r=1,2,\ldots,\Lambda)$, the formulas (6) take the form

$$\tilde{a}_j=c_j a_j,\quad \tilde{a}_j^+=c_j a_j^+,\quad j=1,2,\ldots,\Lambda. \tag{12}$$

The cross-dressing generators (15) satisfy the defining relations:

$$[\tilde{a}_i,\tilde{a}_j]=[\tilde{a}_i^+,\tilde{a}_j^+]=[\tilde{a}_i,\tilde{a}_j^+]=0 \tag{13}$$

for $i\neq j$, $(i,j=1,2,\ldots,\Lambda)$, and

$$\begin{aligned}&\{\tilde{a}_i,\tilde{a}_i\}=\{\tilde{a}_i^+,\tilde{a}_i^+\}=0,\\ &\{\tilde{a}_i,\tilde{a}_i^+\}=\tilde{I}\end{aligned} \tag{14}$$

for $i=1,2,\ldots,\Lambda$.

C. *Cliffons and fermions.*

Take once again the standard fermionic algebra F_n generated by the creation and annihilation operators a_i^+, a_i $(i=1,2,\ldots,n)$

$$\{a_i,a_j\}=\{a_i^+,a_j^+\}=0,\quad \{a_i,a_j^+\}=\delta_{ij}I,\quad i=1,2,\ldots,n. \qquad (15)$$

Let us introduce the new generators:

$$c_{2i-1}:=(a_i+a_i^+),\quad c_{2i}:=\imath(a_i-a_i^+),\quad i=1,2,\ldots,n. \qquad (16)$$

These new generators c_i $(i=1,2,\ldots,2n)$ satisfy the Clifford algebra defining relations

$$\{c_i,c_j\}:=c_ic_j+c_jc_i=\eta_{ij}e,\quad i,j=1,\ldots 2n, \qquad (17)$$

where e is identity element, $\eta=||\eta_{ij}||=\mathrm{diag}(1,1,\ldots,1)$ is the diagonal $(2n\times 2n)$-matrix.

The complex $\mathbb{Z}_2$-graded superalgebra

The complex $\mathbb{Z}_2$-graded Lie superalgebra (LSA) $\mathfrak{g}$, as a linear space over $\mathbb{C}$, is a direct sum of two graded components

$$\mathfrak{g} = \bigoplus_{a=0,1} \mathfrak{g}_a = \mathfrak{g}_0 \oplus \mathfrak{g}_1 \tag{1}$$

with a bilinear operation (the general Lie bracket), $[\![\cdot,\cdot]\!]$, satisfying the identities:

$$\deg([\![x_a, y_b]\!]) = \deg(x_a) + \deg(x_b) = a + b \pmod 2, \tag{2}$$

$$[\![x_a, y_b]\!] = -(-1)^{ab}[\![y_b, x_a]\!], \tag{3}$$

$$[\![x_a, [\![y_b, z]\!]]\!] = [\![[\![x_a, y_b]\!], z]\!] + (-1)^{ab}[\![y_b, [\![x_a, z]\!]]\!], \tag{4}$$

where the elements x_a and y_b are homogeneous, $x_a \in \mathfrak{g}_a$, $x_b \in \mathfrak{g}_b$, and the element $z \in \mathfrak{g}$ is not necessarily homogeneous. The grading function $\deg(\cdot)$ is defined for homogeneous elements of the subspaces $\mathfrak{g}_0$ and $\mathfrak{g}_1$ modulo 2, $\deg(\mathfrak{g}_0) = 0$, $\deg(\mathfrak{g}_1) = 1$. The first identity (2) is called the grading condition, the second identity (3) is called the symmetry property and the condition (4) is the Jacobi identity. From (2) and (3) it is follows that the general Lie bracket $[\![\cdot,\cdot]\!]$ for homogeneous elements posses two value: commutator $[\cdot,\cdot]$ and anticommutator $\{\cdot,\cdot\}$. From (2) it is follows that $\mathfrak{g}_0$ is a Lie subalgebra in $\mathfrak{g}$, and $\mathfrak{g}_1$ is a $\mathfrak{g}_0$-module:

$$[\mathfrak{g}_0, \mathfrak{g}_0] \subseteq \mathfrak{g}_0, \quad [\mathfrak{g}_0, \mathfrak{g}_1] \subseteq \mathfrak{g}_1, \quad (\{\mathfrak{g}_1, \mathfrak{g}_1\} \subseteq \mathfrak{g}_0). \tag{5}$$

The complex $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded superalgebra

The complex $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded LSA $\tilde{\mathfrak{g}}$, as a linear space, is a direct sum of four graded components

$$\tilde{\mathfrak{g}} = \bigoplus_{\mathbf{a}=(a_1,a_2)} \tilde{\mathfrak{g}}_{\mathbf{a}} = \tilde{\mathfrak{g}}_{(0,0)} \oplus \tilde{\mathfrak{g}}_{(1,1)} \oplus \tilde{\mathfrak{g}}_{(1,0)} \oplus \tilde{\mathfrak{g}}_{(0,1)} \tag{6}$$

with a bilinear operation $[\![\cdot,\cdot]\!]$ satisfying the identities (grading, symmetry, Jacobi):

$$\deg([\![x_{\mathbf{a}}, y_{\mathbf{b}}]\!]) = \deg(x_{\mathbf{a}}) + \deg(x_{\mathbf{b}}) = \mathbf{a}+\mathbf{b} = (a_1+b_1, a_2+b_2), \tag{7}$$

$$[\![x_{\mathbf{a}}, y_{\mathbf{b}}]\!] = -(-1)^{\mathbf{ab}}[\![y_{\mathbf{b}}, x_{\mathbf{a}}]\!], \tag{8}$$

$$[\![x_{\mathbf{a}}, [\![y_{\mathbf{b}}, z]\!]]\!] = [\![[\![x_{\mathbf{a}}, y_{\mathbf{b}}]\!], z]\!] + (-1)^{\mathbf{ab}}[\![y_{\mathbf{b}}, [\![x_{\mathbf{a}}, z]\!]]\!], \tag{9}$$

where the vector (a_1+b_1, a_2+b_2) is defined $\mod(2,2)$ and $\mathbf{ab} = a_1b_1 + a_2b_2$. Here in (6)-(8) $x_{\mathbf{a}} \in \tilde{\mathfrak{g}}_{\mathbf{a}}$, $x_{\mathbf{b}} \in \tilde{\mathfrak{g}}_{\mathbf{b}}$, and the element $z \in \tilde{\mathfrak{g}}$ is not necessarily homogeneous. From (6) and (7) it is follows that the general Lie bracket $[\![\cdot,\cdot]\!]$ for homogeneous elements posses two value: commutator $[\cdot,\cdot]$ and anticommutator $\{\cdot,\cdot\}$ as well as in the previous $\mathbb{Z}_2$-case. From (7) it is follows that $\tilde{\mathfrak{g}}_{(0,0)}$ is a Lie subalgebra in $\tilde{\mathfrak{g}}$, and the subspaces $\tilde{\mathfrak{g}}_{(1,1)}$, $\tilde{\mathfrak{g}}_{(1,0)}$ and $\tilde{\mathfrak{g}}_{(0,1)}$ are $\tilde{\mathfrak{g}}_{(0,0)}$-modules:

$$[\tilde{\mathfrak{g}}_{(0,0)}, \tilde{\mathfrak{g}}_{(0,0)}] \subseteq \tilde{\mathfrak{g}}_{(0,0)}, \quad [\tilde{\mathfrak{g}}_{(0,0)}, \tilde{\mathfrak{g}}_{(a,b)}] \subseteq \tilde{\mathfrak{g}}_{(a,b)}, \quad \{\mathfrak{g}_{(a,b)}, \mathfrak{g}_{(a,b)}\} \subseteq \mathfrak{g}_{(0,0)}. \tag{10}$$

It should be noted that $\tilde{\mathfrak{g}}_{(0,0)} \oplus \tilde{\mathfrak{g}}_{(1,1)}$ is a Lie subalgebra in $\tilde{\mathfrak{g}}$ and the subspace $\tilde{\mathfrak{g}}_{(1,0)} \oplus \tilde{\mathfrak{g}}_{(0,1)}$ is a $\tilde{\mathfrak{g}}_{(0,0)} \oplus \tilde{\mathfrak{g}}_{(1,1)}$-module, and moreover $\{\tilde{\mathfrak{g}}_{(1,1)}\tilde{\mathfrak{g}}_{(1,0)}\} \subset \tilde{\mathfrak{g}}_{(0,1)}$ and vice

Let us take the complex $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded Lie superalgebra (LSA) $\mathfrak{g}$, as a linear space over $\mathbb{C}$:

$$\tilde{\mathfrak{g}} = \bigoplus_{\mathbf{a}=(a_1,a_2)} \tilde{\mathfrak{g}}_{\mathbf{a}} = \tilde{\mathfrak{g}}_{(0,0)} \oplus \tilde{\mathfrak{g}}_{(1,1)} \oplus \tilde{\mathfrak{g}}_{(1,0)} \oplus \tilde{\mathfrak{g}}_{(0,1)} \tag{11}$$

and apply to it the following cross-dressing

$$\begin{aligned}\tilde{\mathfrak{g}}' &= \big(e\tilde{\mathfrak{g}}_{(0,0)} \oplus c_1 c_2 \tilde{\mathfrak{g}}_{(1,1)}\big) \oplus \big(c_1 \tilde{\mathfrak{g}}_{(1,0)} \oplus c_2 \tilde{\mathfrak{g}}_{(0,1)}\big) \\ &= \tilde{\mathfrak{g}}'_0 \oplus \tilde{\mathfrak{g}}'_1.\end{aligned} \tag{12}$$

Because of the Lie symmetries AdS_5 and dS_5 are real forms of the simple complex Lie algebra $\mathfrak{so}(5;\mathbb{C}) \simeq \mathfrak{sp}(4;\mathbb{C})$ it is natural to belive that AdS_5 and dS_5 $(N=1)$ supersymmeries are the real forms of the Lie superalgebra $\mathfrak{osp}(1|4;\mathbb{C})$ that is a minimal superextension of $\mathfrak{sp}(4;\mathbb{C})$.

Root systems $\Delta_+(\mathfrak{g})$, $\Delta_-(\mathfrak{g})$=-$\Delta_+(\mathfrak{g})$ and $\Delta(\mathfrak{g})=\Delta_+(\mathfrak{g})\cup\Delta_-(\mathfrak{g})$ for the complex $\mathbb{Z}_2$- and $\mathbb{Z}_2\times\mathbb{Z}_2$-graded Lie superalgebras $\mathfrak{g}=\mathfrak{osp}(1|4;\mathbb{C})$, $\tilde{\mathfrak{osp}}(1|2,2;\mathbb{C})$ are the same "geometrically", but they have different grading "algebraically". These root system can be expresed in terms of the orthogonal vectors e_1, e_2 and reduced to

$$\Delta_+(\mathfrak{g})=\{2e_1, 2e_2, e_1\pm e_2, e_1, e_2\}. \tag{1}$$

We see that the roots of $\Delta_+(\mathfrak{g})$ have different lengths l_α ($l_\alpha^2=(\alpha,\alpha)$, $\alpha\in\Delta_+(\mathfrak{g})$): long $l_\alpha=2$ $(\alpha\in\{2e_1, 2e_2\})$, medium $l_\alpha=\sqrt{2}$ $(\alpha\in\{e_1\pm e_2\})$ and short $l_\alpha=1$ $(\alpha\in\{e_1, e_2\})$ roots. The long vectors have $\deg(2e_1)=\deg(2e_2)=0$ for the $\mathbb{Z}_2$-case and $\deg(2e_1)=\deg(2e_2)=(0,0)$ for the $\mathbb{Z}_2\times\mathbb{Z}_2$-case. The mean vectors have $\deg(e_1\pm e_2)=(0)$ for the $\mathbb{Z}_2$-case and $\deg(e_1\pm e_2)=(1,1)$ for the $\mathbb{Z}_2\times\mathbb{Z}_2$-case. The short vectors have $\deg(e_1)=\deg(e_2)=(1)$ for the $\mathbb{Z}_2$-case and $\deg(e_1)=(1,0)$, $\deg(e_2)=(0,1)$ for the $\mathbb{Z}_2\times\mathbb{Z}_2$-case. We also recall that the grading function $\deg(\cdot)$ on the positive and negative roots of each countragredient Lie superalgebra $\mathfrak{g}$ coincides: $\deg(\alpha)=\deg(-\alpha)$ $(\forall\alpha\in\Delta_+(\mathfrak{g}))$.
For clarity and convenience, we will depict the root systems (1) for $\mathbb{Z}_2$- and $\mathbb{Z}_2\times\mathbb{Z}_2$-graded superalgebras $\mathfrak{osp}(1|4;\mathbb{C})$ and $\tilde{\mathfrak{osp}}(1|2,2;\mathbb{C})$ and their even Lie subalgebras in the form of so-called (color) root diagramms Figs.1-4.

In Fig.1 we present the root diagram of the Lie algebra $\mathfrak{sp}(4;\mathbb{C})$, which is the even part of the $\mathbb{Z}_2$-graded superalgebra $\mathfrak{osp}(1|4;\mathbb{C})$.

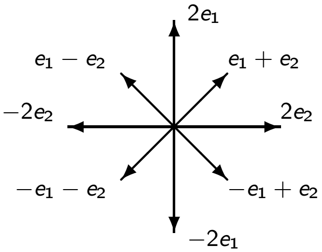

Fig.1. The root diagram of the simplectic Lie algebra $\mathfrak{sp}(4;\mathbb{C})$: all roots are even, $\deg(\Delta(\mathfrak{sp}(4;\mathbb{C}))) = 0$.

The root system for the $\mathbb{Z}_2$-graded orthosymplectic superalgebra $\mathfrak{osp}(1|4;\mathbb{C})$ is shown in Fig.2.

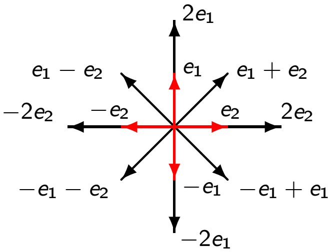

Fig.2. The root diagram of the Lie superalgebra $\mathfrak{osp}(1|4;\mathbb{C})$: the black roots are even, the red roots are odd.

Fig.3 shows the root diagram of the symplectic algebra $\tilde{\mathfrak{sp}}(2,2;\mathbb{C})\sim\mathfrak{sp}(4;\mathbb{C})$ with $\mathbb{Z}_2\times\mathbb{Z}_2$-grading, that is the even part of the superalgebra $\mathfrak{o\tilde{s}p}(1|2,2;\mathbb{C})$.

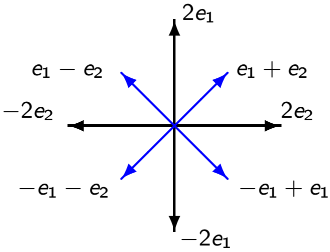

Fig.3. The root diagram of the symplectic algebra $\mathfrak{sp}(2,2;\mathbb{C})\sim\mathfrak{sp}(4;\mathbb{C})$: $\deg(\pm 2e_i)=(0,0)$, $\deg(\pm e_1\pm e_2)=(1,1)$.

Fig.4 depicts the root diagram of the $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded orthosymplectic superalgebra $\tilde{\mathfrak{osp}}(1|2,2;\mathbb{C})$.

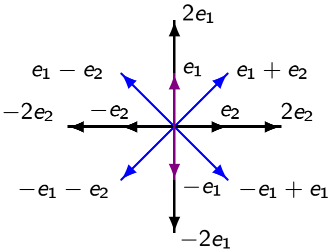

Fig.4. The root diagram of the superalgebra $\tilde{\mathfrak{osp}}(1|2,2;\mathbb{C})$:

$\deg(\pm 2e_i) = (0,0)$, $\deg(\pm e_1 \pm e_2) = (1,1)$,

$\deg(\pm e_1) = (1,0)$, $\deg(\pm e_2) = (0,1)$.

Let be the standard bosonic algebra Bo_2 generated by two oscillators b_i^+, b_i $(i=1,2)$:

$$[b_i,b_j]=[b_i^+,b_j^+]=0,\quad [b_i,b_j^+]=\delta_{ij}I. \tag{2}$$

The corresponding cross-dressing bosonic algebra $\tilde{Bo}_2$ is generated by the following elements $\tilde{b}_i^+$, $\tilde{b}_i$ $(i=1,2)$:

$$\{\tilde{b}_i,\tilde{b}_j\}=\{\tilde{b}_i^+,\tilde{b}_j^+\}=0,\quad [\tilde{b}_i,\tilde{b}_i^+]=I. \tag{3}$$

The elements

$$\begin{array}{llll}
E_{11}=(b_1^+)^2, & E_{-1-1}=(b_1)^2, & E_{22}=(b_2^+)^2, & E_{-2-2}=(b_2)^2,\\
H_{1-1}=b_1^+b_1, & H_{2-2}=b_2^+b_2, & & \\
E_{12}=b_1^+b_2^+, & E_{-2-1}=b_2b_1, & E_{1-2}=b_1^+b_2, & E_{2-1}=b_2^+b_1,\\
E_{01}=b_1^+, & E_{-10}=b_1, & E_{02}=b_2^+, & E_{-20}=b_2,
\end{array} \tag{4}$$

generate the $\mathbb{Z}_2$-graded superalgebra $osp(1|4;\mathbb{C})$. The elements

$$\begin{array}{llll}
\tilde{E}_{11}=(\tilde{b}_1^+)^2, & \tilde{E}_{-1-1}=(\tilde{b}_1)^2, & \tilde{E}_{22}=(b_2^+)^2, & \tilde{E}_{-2-2}=(b_2)^2,\\
\tilde{H}_{1-1}=\tilde{b}_1^+\tilde{b}_1, & \tilde{H}_{2-2}=\tilde{b}_2^+\tilde{b}_2, & & \\
\tilde{E}_{12}=\tilde{b}_1^+\tilde{b}_2^+, & \tilde{E}_{-2-1}=\tilde{b}_2\tilde{b}_1, & \tilde{E}_{1-2}=\tilde{b}_1^+\tilde{b}_2, & \tilde{E}_{2-1}=\tilde{b}_2^+\tilde{b}_1,\\
\tilde{E}_{01}=\tilde{b}_1^+, & \tilde{E}_{-10}=\tilde{b}_1, & \tilde{E}_{02}=\tilde{b}_2^+, & \tilde{E}_{-20}=\tilde{b}_2,
\end{array} \tag{5}$$

generate the $\mathbb{Z}_2\times\mathbb{Z}_2$-graded superalgebra $\tilde{\mathfrak{osp}}(1|2,2;\mathbb{C})$ arclength

Our orthosymplectic superalgebras $\mathfrak{osp}(1|4;\mathbb{C})$ and $\tilde{\mathfrak{osp}}(1|2,2;\mathbb{C})$ can be represented as a direct sum of spaces

$$\mathfrak{g} = \mathfrak{osp}_1(1|2;\mathbb{C}) \oplus Lin\{E_{\pm1\pm2}\} \oplus \mathfrak{osp}_2(1|2;\mathbb{C}) \qquad (1)$$

where:
(a) for $\mathfrak{osp}(1|4;\mathbb{C})$-case: $\mathfrak{osp}_1(1|2;\mathbb{C})$ and $\mathfrak{osp}_2(1|2;\mathbb{C})$ are the $\mathbb{Z}_2$-graded supersubalgebras, and $\deg(E_{\pm1\pm2}) = 0$.
(b) for $\tilde{\mathfrak{osp}}(1|2,2;\mathbb{C})$-case: $\mathfrak{sp}_1(1|2;\mathbb{C})$ is $(\mathbb{Z}_2, 0)$ and $\mathfrak{osp}_2(1|2;\mathbb{C})$ is $(0,\mathbb{Z}_2)$-graded supersubalgebras, and $\deg(E_{\pm1\pm2}) = (1,1)$.

Let $\{E_{\pm i\pm i}, H_{i-i}, E_{0\pm i}\}$ $(i=1,2)$ be a Cartan-Weyl (CW) basis of $\mathfrak{osp}_i(1|2;\mathbb{C})$:

$$[H_{i-i}, E_{\pm i\pm i}] = \pm E_{\pm i\pm i}, \quad [E_{ii}, E_{-i-i}] = 2H_{i-i}, \tag{2}$$

$$[H_{i-i}, E_{0\pm i}]] = \pm\frac{1}{2}E_{0\pm i}, \quad [E_{\mp i\mp i}, E_{0\pm i}] = E_{0\mp i} \tag{3}$$

$$[E_{\pm i\pm i}, E_{0\pm i}] = 0, \quad \{E_{0i}, E_{0-i}\} = -\frac{1}{2}H_{i-i}. \tag{4}$$

The CW basis H_{i-i}, E_{ii}, E_{0i} has the conjugation properties:
(a) compact case:

$$(H_{i-i})^{\dagger} = H_{i-i}, \quad (E_{\pm i\pm i})^{\dagger} = E_{\mp i\mp i}, \tag{5}$$

$$(E_{0i})^{\dagger} = iE_{0-i}, \quad (E_{0-i})^{\dagger} = -iE_{0i} \tag{6}$$

(b) noncompact case:

$$(H_{i-i})^{\dagger} = H_{i-i}, \quad (E_{\pm i\pm i})^{\dagger} = -E_{\mp i\mp i}, \tag{7}$$

$$(E_{0i})^{\dagger} = E_{0-i}, \quad (E_{0-i})^{\dagger} = E_{0i} \tag{8}$$

Let $\{E_{\pm i\pm i}, H_{i-i}, E_{0\pm i}\}$ $(i=1,2)$ be a Cartan-Weyl (CW) basis of $\mathfrak{osp}_i(1|2;\mathbb{C})$:

$$[H_{i-i}, E_{\pm i\pm i}] = \pm E_{\pm i\pm i}, \quad [E_{ii}, E_{-i-i}] = 2H_{i-i}, \tag{9}$$

$$[H_{i-i}, E_{0\pm i}]] = \pm\frac{1}{2}E_{0\pm i}, \quad [E_{\mp i\mp i}, E_{0\pm i}] = E_{0\mp i} \tag{10}$$

$$[E_{\pm i\pm i}, E_{0\pm i}] = 0, \quad \{E_{0i}, E_{0-i}\} = -\frac{1}{2}H_{i-i}. \tag{11}$$

The CW basis H_{i-i}, E_{ii}, E_{0i} has the conjugation properties:
(a) compact case:

$$(H_{i-i})^\dagger = H_{i-i}, \quad (E_{\pm i\pm i})^\dagger = E_{\mp i\mp i}, \tag{12}$$

$$(E_{0i})^\dagger = iE_{0-i}, \quad (E_{0-i})^\dagger = -iE_{0i} \tag{13}$$

(b) noncompact case:

$$(H_{i-i})^\dagger = H_{i-i}, \quad (E_{\pm i\pm i})^\dagger = -E_{\mp i\mp i}, \tag{14}$$

$$(E_{0i})^\dagger = E_{0-i}, \quad (E_{0-i})^\dagger = E_{0i} \tag{15}$$

Real forms

$$E^{nc}_{1-2},\quad E^{nc}_{12},\quad E^{c}_{11},\quad E^{c}_{22},\quad E^{c}_{01},\quad E^{c}_{01} \tag{16}$$

$$E^{nc}_{1-2},\quad E^{c}_{12},\quad E^{nc}_{11},\quad E^{nc}_{22},\quad E^{nc}_{01},\quad E^{nc}_{01} \tag{17}$$

$$E^{c}_{1-2},\quad E^{nc}_{12},\quad E^{nc}_{11},\quad E^{nc}_{22},\quad E^{nc}_{01},\quad E^{nc}_{01} \tag{18}$$

Analyzing

$$E^{\dagger}_{1\pm 2} = \{E_{01}, E_{0\pm 2}\}^{\dagger},\quad forarclength\ \mathbb{Z}_2 - case \tag{19}$$

$$E^{\dagger}_{1\pm 2} = [E_{01}, E_{0\pm 2}]^{\dagger},\quad for\ \mathbb{Z}_2\times\mathbb{Z}_2 - case. \tag{20}$$

Application A. $\mathbb{Z}_2$- and $\mathbb{Z}_2\times\mathbb{Z}_2$-graded Poincarè superalgebras

Using the standard contraction procedure: $L_{\mu 4}=R\,P_{\mu}\ (\mu=0,1,2,3)$, $Q_k\to\sqrt{R}\,Q_k$ and $\bar{Q}_k\to\sqrt{R}\,\bar{Q}_k\ (k=1,2)$ for $R\to\infty$ we obtain the super-Poincarè algebra (standard and alternative) which is generated by $L_{\mu\nu}$, P_{μ}, Q_{α}, $Q_{\dot{\alpha}}$ where $\mu,\nu=0,1,2,3$; $\alpha,\dot{\alpha}=1,2$ with the relations (we write down only those which are changed in the standard and alternative Poincarè SUSY.

(I) *for the standard Poincarè SUSY:*

$$[P_{\mu},Q_{\alpha}]\ =\ [P_{\mu},\bar{Q}_{\dot{\alpha}}]=0\ ,\quad \{Q_{\alpha},\bar{Q}_{\dot{\alpha}}\}\ =\ 2\sigma^{\mu}_{\alpha\beta}P_{\mu}, \tag{1}$$

(II) *for the altrenative Poincarè SUSY:*

$$\{P_{\mu},Q_{\alpha}\}\ =\ \{P_{\mu},\bar{Q}_{\dot{\alpha}}\}=0\ ,\quad [Q_{\alpha},\bar{Q}_{\dot{\alpha}}]\ =\ 2\sigma^{\mu}_{\alpha\beta}P_{\mu}, \tag{2}$$

Application B. $\mathbb{Z}_2$- and $\mathbb{Z}_2\times\mathbb{Z}_2$-graded superspaces and superfields

Let us consider the supergroups associated to the $\mathbb{Z}_2$- and $\mathbb{Z}_2\times\mathbb{Z}_2$-graded Poincarè superalgebras. A group element g is given by the exponential of the super-Poincarè generators, namely

$$g(x^\mu,\omega^{\mu\nu},\theta^\alpha,\bar{\theta}^{\dot{\alpha}}) \;=\; \exp(x^\mu P_\mu+\omega^{\mu\nu}M_{\mu\nu}+\theta^\alpha Q_\alpha+\bar{Q}_{\dot{\alpha}}\bar{\theta}^{\dot{\alpha}}). \tag{1}$$

Because the grading of the exponent is zero (0 or (00)) we have the following.
1). $\mathbb{Z}_2$-case: $\deg P=\deg x=0$, $\deg Q=\deg\bar{Q}=\deg\theta=\deg\bar{\theta}=1$. This means that

$$[x_\mu,\theta_\alpha] \;=\; [x_\mu,\bar{\theta}_{\dot{\alpha}}] \;=\; \{\theta_\alpha,\bar{\theta}_{\dot{\beta}}\} \;=\; \{\theta_\alpha,\theta_\beta\} \;=\; \{\bar{\theta}_{\dot{\alpha}},\bar{\theta}_{\dot{\beta}}\} \;=\; 0. \tag{2}$$

2). $\mathbb{Z}_2\times\mathbb{Z}_2$-case: $\deg P=\deg x=(11)$, $\deg Q=\deg\theta=(10)$, $\deg\bar{Q}=\deg\bar{\theta}=(01)$. This means that

$$\{x_\mu,\theta_\alpha\} \;=\; \{x_\mu,\bar{\theta}_{\dot{\alpha}}\} \;=\; [\theta_\alpha,\bar{\theta}_{\dot{\beta}}] \;=\; \{\theta_\alpha,\theta_\beta\} \;=\; \{\bar{\theta}_{\dot{\alpha}},\bar{\theta}_{\dot{\beta}}\} \;=\; 0. \tag{3}$$

One defines the superspaces as the coset spaces of the standard alternative super-Poincarè groups by the Lorentz subgroup, parameterized the coordinates x^μ, θ^α, $\bar{\theta}^{\dot{\alpha}}$, subject to the condition $\bar{\theta}^{\dot{\alpha}}=(\theta^\alpha)^*$.
We can define a superfield $\mathcal{F}$ as a function of superspace.

Application C. Once more about SUGRA

In the theory of supergravity (SUGRA), there has been the following unsolved problem for about 30 years. All physical reasonable solutions of SURGA models with cosmological constants Λ have been constructed for the case $\Lambda < 0$, i.e. for the anti-de Sitter metric

$$g_{ab} \;=\; \operatorname{diag}(1,-1,-1,-1,1), \quad (a,b\;=\;0,1,2,3,4) \tag{4}$$

with the space-time symmetry $\mathfrak{o}(3,2)$. In the case $\Lambda > 0$, i.e. for the Sitter metric

$$g_{ab} \;=\; \operatorname{diag}(1,-1,-1,-1,-1), \quad (a,b\;=\;0,1,2,3,4) \tag{5}$$

with the space-time symmetry $\mathfrak{o}(4,1)$ no reasonable solutions have been find. For example, in SUGRA it was obtained the following relation

$$\Lambda = -3m^2, \tag{6}$$

where m is the massive parameter of gravitinos.

In my opinion these problems for the case $\Lambda > 0$ are connected with superextensions of anti-de Sitter $\mathfrak{o}(3,2)$ and de Sitter $\mathfrak{o}(4,1)$ symmetries. The symmetry $\mathfrak{o}(3,2)$ has the superextension - the superalgebra $\mathfrak{osp}(3,2|1)$. This is the usual $\mathbb{Z}_2$-graded superalgebra. In the case of $\mathfrak{o}(4,1)$ such superextension does not exist. However the Lie algebra $\mathfrak{o}(4,1)$ has another *alternative* superextension that is based on the $\mathbb{Z}_2 \times \mathbb{Z}_2$-grading and a preliminary analysis shows that we can construct the reasonable SUGRA models for the case $\Lambda > 0$.

Let us return to SUGRA formula which connects the cosmological constant Λ with the mass of gravitinos m

$$0 > \Lambda = -3m^2. \tag{7}$$

We can rewrite the formula in terms of the time-component of the four-momenta, P_0:

$$0 > \Lambda = -3P_0^2, \tag{8}$$

This formula is valid for $\mathbb{Z}_2$-graded case and we belive tbat it is valid for the $\mathbb{Z}_2\times\mathbb{Z}_2$-graded case also, that is

$$0 < \Lambda = -3\tilde{P}_0^2. \tag{9}$$

where $\tilde{P}_0$ is the time-component of the four-momenta for the $\mathbb{Z}_2\times\mathbb{Z}_2$-graded case. Because

$$\tilde{P}_0 = c_1c_2P_0 = c_1c_2m, \tag{10}$$

where c_1, c_2 are the real cliffons: $c_1^* = c_1$, $c_2^* = c_2$, $(c_1c_2)^* = c_1c_2$, $(c_1c_2)^2 = -1$, therefore

$$0 < \Lambda = -3\tilde{P}_0^2 = 3m^2. \tag{11}$$

THANK YOU FOR YOUR ATTENTION