SQS-22

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Weak supersymmetry and superconformal index

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WITTEN INDEX

(Witten, 1981, 1982)

Definition:

$$I_W \; = \; \tilde{Z}(\beta) \; \stackrel{\text{def}}{=} \sum_n (-1)^{F_n} e^{-\beta E_n} \equiv \langle\langle e^{-\beta H} \rangle\rangle \, .$$

$\beta = 1/T$ — inverse temperature. $\tilde{Z}$ — supersymmetric partition function, F_n is even for the bosonic states and odd for the fermionic states.

- Double degeneracy of excited levels $\rightarrow$ no β dependence and

$$I_W \; = \; n_B^{E=0} - n_F^{E=0} \, .$$

SUPERCONFORMAL INDEX FOR PEDESTRIANS

Take a $D = 4$ supersymmetric field theory.

• Compactification on $T^3 \times \mathbb{R}$ keeps the algebra.

• Compactification on $S^3 \times \mathbb{R}$ weakens supersymmetry. The best option:

$$
\begin{aligned}
[P_a, P_b] &= \frac{-2i}{\rho}\varepsilon_{abc}P_c, \\
[Q_\alpha, P_a] &= -\frac{1}{\rho}(\sigma_a Q)_\alpha, \qquad [\bar{Q}^\alpha, P_a] = \frac{1}{\rho}(Q\sigma_a)^\alpha, \\
\{Q_\alpha, \bar{Q}^\beta\} &= 2\left(H - \frac{R}{\rho}\right)\delta_\alpha^\beta + 2(\sigma_a)_\alpha{}^\beta P_a, \\
[Q_\alpha, R] &= -Q_\alpha, \qquad [\bar{Q}^\alpha, R] = \bar{Q}^\alpha, \\
[H, P_a] &= [H, R] = [R, P_a] = 0, \\
\{Q_\alpha, Q_\beta\} &= \{\bar{Q}^\alpha, \bar{Q}^\beta\} = [Q_\alpha, H] = [\bar{Q}^\alpha, H] = 0,
\end{aligned}
$$

where $\alpha = 1, 2$; $a, b, c = 1, 2, 3$; $H \equiv P_0$, ρ — radius of S^3; R — extra $U(1)$ generator.

(*D. Sen, 1987*)

• $\{Q_\alpha, Q_\beta^\dagger\}$ involves extra terms besides H.
• No exact pairing of *all* excited states.
• The index $\mathrm{Tr}\{(-1)^F e^{-\beta H}\}$ depends on β !
(*Römelsberger, 2006*)

• This index is often called superconformal, because it was used to prove the exact dualities of superconformal 4-dimensional theories. But there is nothing superconformal in its definition!

• The algebra $SU(2|1,1)$ coinsides with the algebra of

WEAK SUPERSYMMETRY

The simplest weak SQM model (*A.S., 2004*)

$$\begin{aligned} Q_\alpha &= (p - ix)\chi_\alpha, \quad \bar{Q}^\beta = (p+ix)\bar{\chi}^\beta, \\ H &= \frac{p^2 + x^2}{2} + F - 1, \end{aligned}$$

where $\alpha = 1, 2$, $p = -i\partial/\partial x$, $\bar{\chi}^\alpha = \partial/(\partial \chi_\alpha)$, $F = \chi_\alpha \bar{\chi}^\alpha$ (fermion charge) and

$$Z_\alpha^\beta = 2\chi_\alpha \bar{\chi}^\beta - \delta_\alpha^\beta F.$$

The algebra:

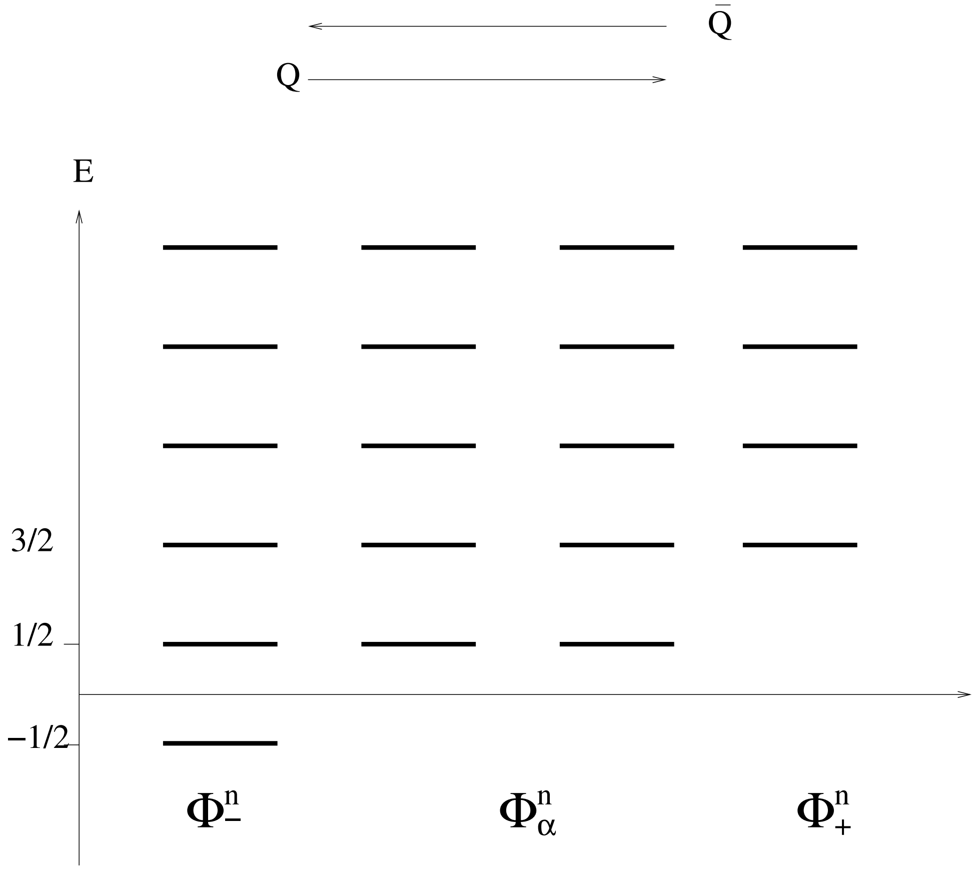
$$\begin{aligned} \{Q_\alpha, \bar{Q}^\beta\} &= (2H - Y)\delta_{\alpha\beta} + Z_\alpha^\beta\,, \\ [Q_\alpha, Z_\beta^\gamma] &= \delta_\beta^\gamma Q_\alpha - 2\,\delta_\alpha^\gamma Q_\beta, \\ [\bar{Q}^\alpha, Z_{\beta\gamma}] &= \delta_\beta^\alpha \bar{Q}^\gamma - \delta_\beta^\gamma \bar{Q}^\alpha, \\ [Q_\alpha, Y] = -Q_\alpha, &\qquad [\bar{Q}^\alpha, Y] = \bar{Q}^\alpha, \\ [Z_\alpha^\beta, Z_\gamma^\delta] &= 2\left(\delta_\gamma^\beta\, Z_\alpha^\delta - \delta_\alpha^\delta\, Z_\gamma^\beta\right)\,, \end{aligned} \tag{1}$$

where $Y = F - 1$.

It is equivalent to the SUSY algebra on $S^3 \times \mathbb{R}$ with

$$Z_{\alpha\beta} \equiv P_a, \qquad Y \equiv R\,.$$

SPECTRUM of H



- The states $\Phi^n_\pm$ are bosonic and the states Φ^n_α are fermionic.
- The ground state does not have zero energy.
- No pairing at the first excited level.
- 4-fold degeneracy starting from the 2-nd level.

The index:

$$I_W \;=\; e^{\beta/2} - e^{-\beta/2} \;=\; 2\sinh(\beta/2)\,.$$

INVARIANCE UNDER DEFORMATIONS

Theorem 1. *Let $Q_\alpha, \bar{Q}^\alpha$ and H but not P_a and R be functions of parameter γ such that the weak SUSY algebra stays intact. Then*

$$\frac{d}{d\gamma} \langle\langle e^{-\beta H} \rangle\rangle \; = \; 0. \tag{2}$$

Proof. By expanding $e^{-\beta H}$ into the series and using the cyclic property of the supertrace, we deduce

$$\frac{d}{d\gamma} \langle\langle e^{-\beta H} \rangle\rangle \; = \; -\beta \left\langle\left\langle \frac{dH}{d\gamma} e^{-\beta H} \right\rangle\right\rangle. \tag{3}$$

The first line in (1) reads

$$\{Q_\alpha, \bar{Q}^\beta\} \; = \; 2H\delta_\alpha{}^\beta \quad + \quad \text{central charges.}$$

Capitalizing on the assumed γ-independence of the central charges, we deduce

$$\frac{dH}{d\gamma} \; = \; \frac{1}{4}\left\{Q_\alpha, \frac{d\bar{Q}^\alpha}{d\gamma}\right\} + \frac{1}{4}\left\{\frac{dQ_\alpha}{d\gamma}, \bar{Q}^\alpha\right\}. \tag{4}$$

Lemma 1.

$$\langle\langle \{Q_\alpha, V\} e^{-\beta H} \rangle\rangle \; = \; 0 \tag{5}$$

for any (not too wild) V.

Proof. Take for definiteness $\alpha = 1$. By definition,

$$\langle\langle O \rangle\rangle \;=\; \sum_B \langle B|O|B\rangle - \sum_F \langle F|O|F\rangle,$$

where $|B\rangle$ and $|F\rangle$ are the bosonic and fermionic states.

The weak SUSY algebra (1) includes the ordinary $\mathcal{N} = 2$ SQM subalgebra $\mathcal{A}_1$ with the generators $Q_1, \bar{Q}_1$ and

$$H_1 = \frac{1}{2}\{Q_1, \bar{Q}^1\} \;=\; H + P_3 - \frac{R}{\rho}.$$

We choose the eigenstates of H_1 (which are also the eigenstates of H due to $[H_1, H] = 0$) as the basis in Hilbert space. From the viewpoint of $\mathcal{A}_1$, the spectrum includes:

i) The states annihilated by the action of both Q_1 and $\bar{Q}^1$. If the symmetry $\mathcal{A}_1$ is not broken spontaneously, these are the ground states of H_1,

ii) The doublets (B, F) satisfying

$$\begin{aligned} &Q_1|B\rangle = |F\rangle, \quad Q_1|F\rangle = 0, \\ &\langle B|Q_1 = 0, \quad \langle F|Q_1 = \langle B|. \end{aligned} \tag{6}$$

Using $[Q_1, H] = 0$, we may rewrite (5) as

$$\langle\langle\{Q_1, V\}e^{-\beta H}\rangle\rangle \;=\; \langle\langle Q_1 V e^{-\beta H}\rangle\rangle + \langle\langle V e^{-\beta H} Q_1\rangle\rangle$$
$$= \;\sum_B \langle B|Q_1 V e^{-\beta H} + V e^{-\beta H} Q_1|B\rangle -$$
$$\sum_F \langle F|Q_1 V e^{-\beta H} + V e^{-\beta H} Q_1|F\rangle.$$

It is immediately seen that the singlet states to not contribute. Next, using (6), we obtain

$$\langle\langle\{Q_1, V\}e^{-\beta H}\rangle\rangle \;=\;$$
$$\sum_{\text{doublets}} \langle B|V e^{-\beta H}|F\rangle - \sum_{\text{doublets}} \langle B|V e^{-\beta H}|F\rangle \;=\; 0.$$

By the same token we can prove

$$\langle\langle\{\bar{Q}^\alpha, V\}e^{-\beta H}\rangle\rangle \;=\; 0 \tag{7}$$

for any V. $\square$

By combining (3), (4), (5), (7), we arrive at (2). $\square$

Remarks.

1. The whole reasoning above also works for a *generalized* index

$$\tilde{I} \;=\; \langle\langle e^{-\beta H} e^{\mu M} \rangle\rangle$$

where M is an operator that commutes with the Hamiltonian and at least one pair of the supercharges. A rich enough dynamical system may involve many such extra integrals of motion M_j.

Then the index

$$I(\beta, \mu_j) \;=\; \langle\langle e^{-\beta H} e^{\mu_j M_j} \rangle\rangle$$

is invariant under the deformations described above.

2. This theorem represents a particular case of the so-called *equivariant index theorem* known to mathematicians. The notion of the equivariant index was first introduced back in 1950 by Cartan.

DEFORMATION

The algebra is kept intact if

$$Q_\alpha = [p - iV(x)]\chi_\alpha - ib(x)\,\chi_\alpha\chi_\gamma\bar\chi^\gamma,$$

$$\bar Q^\alpha = [p + iV(x)]\bar\chi^\alpha) + ib(x)\,\bar\chi^\alpha\chi_\gamma\bar\chi^\gamma,$$

$$H \;=\; \frac{p^2}{2} + \frac{V(x)^2}{2} + V'(x)\chi_\gamma\bar\chi^\gamma + b'(x)(\chi_\gamma\bar\chi^\gamma)^2\,,$$

where $V(x)$ and $b(x)$ satisfy the condition

$$V' - bV \;=\; 1\,.$$

• The ground state and the triplet of first excitations are not shifted.

• 4-fold degeneracy of higher excited states. The index is the same.

• Related to *quasi-exactly solvable* models (*Ushveridze, Turbiner*)

COMPLEX MODEL

• We start with the free model (*Ivanov + Sidorov, 2014*)

$$
\begin{aligned}
Q_\alpha &= \sqrt{2}\,\psi_\alpha(\pi - i\bar{\phi}),\\
\bar{Q}^\alpha &= \sqrt{2}\,\bar{\psi}^\alpha(\bar{\pi} + i\phi),
\end{aligned}
$$

$$
H = \bar{\pi}\pi + \bar{\phi}\phi + \frac{1}{2}(\psi_\alpha\bar{\psi}^\alpha - \bar{\psi}^\alpha\psi_\alpha)
$$

• Conserved fermion charge

$$
F = \psi_\alpha\bar{\psi}^\alpha
$$

and angular momentum

$$
L = i(\phi\pi - \bar{\phi}\bar{\pi})
$$

• $su(2|1,1)$ algebra

$$
\{Q_\alpha, \bar{Q}^\beta\} = 2\,(H + L)\,\delta_\alpha{}^\beta + 2Z_\alpha{}^\beta,
$$

$Z_\alpha{}^\beta = \psi_\alpha\bar{\psi}^\beta + \psi^\beta\bar{\psi}_\alpha.$

The spectrum: $E^F_{nl} = 2n + |l| + F$.

• One bosonic ground state and excited SUSY quartets. $I_W = 1$.

MODIFICATION

$$H_\lambda = H + \lambda(L+F)\,.$$

Algebra is modified,

$$\{Q_\alpha, \bar{Q}^\beta\} \;=\; 2\delta_\alpha{}^\beta[H_\lambda - \lambda F + L(1-\lambda)] + 2Z_\alpha{}^\beta,$$

• It is still $SU(2|1,1)$, but the central charge $R = L - \lambda(L+F)$ depends on λ. Spectrum pattern and the index are modified.

Deformation by including superpotential $\gamma W(\Phi)$

.

• The weak SUSY algebra can only be kept provided

(i) $W(\Phi) \propto \Phi^n$.

(ii) $\lambda = 1 - 2/n$.

The spectrum:

$n-1$ singlet bosonic states with energies $E_m = 2m/n,\ m = 0,\ldots,n-2$; The other states are in the degenerate quartets.

The index

$$I(n) \;=\; \sum_{m=0}^{n-2} e^{-2\beta m/n}.$$

When $n = 3$,

$$Q_\alpha = \sqrt{2}\left[(\pi - i\bar{\phi})\psi_\alpha + i\gamma\bar{\psi}_\alpha\bar{\phi}^2\right],$$

$$\bar{Q}^\alpha = \sqrt{2}\left[(\bar{\pi} + i\phi)\bar{\psi}^\beta + i\gamma\psi^\alpha\phi^2\right],$$

$$H = \bar{\pi}\pi + \bar{\phi}\phi + \psi_\alpha\bar{\psi}^\alpha - 1 + \frac{1}{3}[i(\phi\pi - \bar{\phi}\bar{\pi}) + \psi_\alpha\bar{\psi}^\alpha]$$
$$+2\gamma(\psi_1\psi_2\phi + \bar{\psi}^2\bar{\psi}^1\bar{\phi}) + \gamma^2(\bar{\phi}\phi)^2,$$

where γ is the deformation parameter.

• The index is the same for all γ.
• The energies of the singlet boson states with the energies $E_0 = 0$ and $E_1 = 2/3$ are not shifted.
• The energies of excited degenerate quartets may depend on γ.

• Wess-Zumino model with $W(\Phi) \propto \Phi^3$ on $\mathbb{R}^3$: two degenerate vacuum bosonic states.
• Wess-Zumino model on S^3 of radius ρ: these states are split, $\Delta E \propto 1/\rho^2$.

$su(N|1)$ oscillator and its deformation

It is the system

$$
\begin{aligned}
Q_\alpha &= (p - ix)\chi_\alpha, \quad \bar{Q}^\beta = (p + ix)\bar{\chi}^\beta, \\
H &= \frac{p^2 + x^2}{2} + F - N/2,
\end{aligned}
$$

with $\alpha = 1, \ldots, N$.

The algebra:

$$
\begin{aligned}
\{Q_\alpha, \bar{Q}^\beta\}_+ &= 2\left(H - \frac{N-1}{N}Y\right)\delta_\alpha{}^\beta + Z_\alpha{}^\beta\,, \\
[Q_\alpha, Z_\beta{}^\gamma] &= \frac{2}{N}\delta_\beta{}^\gamma Q_\alpha - 2\,\delta_\alpha{}^\gamma Q_\beta, \\
[\bar{Q}^\alpha, Z_\beta{}^\gamma] &= 2\,\delta_\beta{}^\alpha \bar{Q}^\gamma - \frac{2}{N}\delta_\beta{}^\gamma \bar{Q}^\alpha, \\
[Q_\alpha, Y] &= -Q_\alpha, \qquad [\bar{Q}^\alpha, Y] = \bar{Q}^\alpha, \\
[Z_\alpha{}^\beta, Z_\gamma{}^\delta] &= 2\left(\delta_\gamma{}^\beta\, Z_\alpha{}^\delta - \delta_\alpha{}^\delta\, Z_\gamma{}^\beta\right)\,.
\end{aligned}
$$

- Let $N = 3$. The spectrum involves:

1. The ground bosonic state Ψ_0 with zero fermion charge and the energy $E = -1$.

2. One bosonic state with $F = 0$ and 3 fermionic states with $F = 1$ at the level $E = 0$.

3. A state with $F = 0$, 3 states with $F = 1$ and 3 states with $F = 2$ at the level $E = 1$.

4. Complete supersymmetric octets with still higher energies.

The index:

$$I_W = e^{\beta} - 2 + e^{-\beta} = [2\sinh(\beta/2)]^2 \,.$$

Generically,

$$\begin{aligned} I_W &= \sum_{k=0}^{N-1} (-1)^k C_{N-1}^k \exp\left\{\beta\left(\frac{N-1}{2} - k\right)\right\} \\ &= [2\sinh(\beta/2)]^{N-1}. \end{aligned}$$

DEFORMATION

$N = 3$

$$Q_\alpha = [p - iV(x)]\chi_\alpha - ib(x)\,\chi_\alpha\chi_\gamma\bar{\chi}^\gamma - ic(x)\,\chi_\alpha(\chi_\gamma\bar{\chi}^\gamma)^2$$

with the conditions

$$\begin{aligned} V' - bV - \frac{bc}{2} &= 1\,, \\ b' - b^2 - 2cV - \frac{c^2}{2} &= 0\,. \end{aligned}$$

• First three levels unshifted: quasiexact solvability. The same index.

• For $N = 4$, we also have to add the term $-id(x)\,\chi_\alpha(\chi_\gamma\bar{\chi}^\gamma)^3$ in Q_α

CFIV index

Besides the Witten index, one can also consider another index,

$$I_{CFIV} \;=\; \langle\langle F e^{-\beta H} \rangle\rangle \,,$$

introduced by *Cecotti, Fendley, Intriligator* and *Vafa* back in 1992 in the context of $2D$ supersymmetric theories.

• It makes sense only for extended SQM systems, where it is invariant under deformations of the type discussed above.

Proof. Consider the operator $M_1 = \chi_1 \bar{\chi}^1$. It commutes with H, $Q_{\alpha>1}$ and $\bar{Q}^{\alpha>1}$. Hence, according to the theorem above, $\langle\langle M_1 e^{-\beta H}\rangle\rangle$ is invariant under deformations. And the same concerns $\langle\langle (\sum_{\alpha=1}^N M_\alpha)\, e^{-\beta H}\rangle\rangle$ □

• $N = 2$: only fermion states contribute, $I_{CFIV}^{N=2} \;=\; -2e^{-\beta/2}$.

Arbitrary N:

$$I_{CFIV} \;=\; -N e^{-\beta/2}\left(2\sinh\frac{\beta}{2}\right)^{N-2}$$

More complicated $su(N|1)$ systems:

(*Ivanov + Lechtenfeld + Sidorov, 2018*),
(*Bellucci + Krivonos + Nersessian, 2018*),
(*Khastyan + Krivonos + Nersessian, 2022*)

The analysis of their spectrum would be interesting.

!! THANKS TO THE ORGANIZERS !!