Cosmological solutions of integrable $F(R)$ gravity models

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based on

V.R. Ivanov and S.Yu. Vernov, *Eur. Phys. J.* C **81** (2021) 985, arXiv:2108.10276

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$F(R)$ gravity models

The $F(R)$ model, described by the action

$$S_R = \int d^4x\sqrt{-g}F(R), \qquad (1)$$

where $F(R)$ is a double differentiable function of the Ricci scalar R, has the following evolution equations:

$$F'R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}F - (\nabla_\mu\nabla_\nu - g_{\mu\nu}\Box)F' = 0. \qquad (2)$$

To get solutions of these equations one can use the connection with the General Relativity.

$F(R)$ gravity models and General Relativity

On the one hand, the General Relativity is particular case at $F(R) = R$.
On the another hand, $F(R)$ gravity models can be transform to the General Relativity models with scalar fields.
For following action with a scalar field σ without the kinetic term,

$$\tilde{S}_J = \int d^4x\sqrt{-g_J}\left[\frac{dF(\sigma)}{d\sigma}(R_J - \sigma) + F(\sigma)\right]. \qquad (3)$$

is equivalent to S_R, because $\sigma = R$ at $\frac{d^2F(\sigma)}{d\sigma^2}(\sigma) \neq 0$.
By the conformal transformation of the metric $g_{\mu\nu} = \frac{2f(\sigma)}{M_{Pl}^2}\tilde{g}_{\mu\nu}$,
where $f \equiv \frac{dF(\sigma)}{d\sigma}$, one gets the following action:

$$S_E = \int d^4x\sqrt{-g}\left[\frac{M_{Pl}^2}{2}R_E - \frac{h(\sigma)}{2}g^{\mu\nu}\partial_\mu\sigma\partial_\nu\sigma - V_E\right], \qquad (4)$$

where

$$h(\sigma) = \frac{3M_{Pl}^2}{2}\left(\frac{d\ln(f)}{d\sigma}\right)^2, \qquad V_E = M_{Pl}^4\frac{f\sigma - F}{4f^2}.$$

Introducing the scalar field

$$\phi = \sqrt{\frac{3}{2}} M_{Pl} \ln\left(\frac{2}{M_{Pl}^2} f(\sigma)\right), \qquad (5)$$

we obtain the action S_E as follows:

$$S_E = \int d^4x \sqrt{-g}\left[\frac{M_{Pl}^2}{2} R_E - \frac{1}{2}\partial_\mu\phi\partial_\mu\phi - V_E(\phi)\right]. \qquad (6)$$

So, we get the Einstein frame model with a standard scalar field.
The inverse transformation gives the function $F(R)$ in a parametric form:

$$R_J = \left[\frac{\sqrt{6}}{M_{Pl}}\frac{dV_E}{d\phi} + \frac{4V_E}{M_{Pl}^2}\right] \mathrm{e}^{\frac{\sqrt{6}\phi}{3M_{Pl}}}, \qquad (7)$$

$$F = \frac{M_{Pl}^2}{2}\left[\frac{\sqrt{6}}{M_{Pl}}\frac{dV_E}{d\phi} + \frac{2V_E}{M_{Pl}^2}\right] \mathrm{e}^{2\frac{\sqrt{6}\phi}{3M_{Pl}}}. \qquad (8)$$

$$\tilde{S}_{\rm J} = \int d^4x\sqrt{-g}\left[U(\sigma)R - \frac{\theta_\sigma}{2}g^{\mu\nu}\partial_\mu\sigma\partial_\nu\sigma - V_J(\sigma)\right], \qquad (9)$$

How to get integrable modified gravity cosmological model (9)?

1. Use the known integrable model in the Einstein frame and make a conformal transformation.
   In the Einstein frame, P. Fré, A. Sagnotti, A.S. Sorin, *Nucl.Phys.B* **877** (2013) 1028 arXiv:1307.1910.
   In the Jordan frame, A.Yu. Kamenshchik, E.O. Pozdeeva, A. Tronconi, G. Venturi, S.Yu. Vernov, *Class. Quant. Grav.* **33** (2016) 015004 [arXiv:1509.00590].
2. Use the integral of motion. For example, $R = const$.
   B. Boisseau, H. Giacomini, D. Polarski and A. A. Starobinsky, *JCAP* **07** (2015) 002 [arXiv:1504.07927].
3. Use the Painlevé analysis.
   A. Paliathanasis and P.G.L. Leach, *Phys. Lett. A* **380** (2016) 2815 [arXiv:1605.04204].
   G. Leon, A. Paliathanasis and P. G. L. Leach, *New analytic solutions in $f(R)$-Cosmology from Painlevé analysis*, arXiv:2205.01363.

Chiral Cosmological Models

Chiral Cosmological Models are actively use in cosmology both to describe inflation, and dark energy:

S.V. Chervon, Russ. Phys. J. **38** (1995) 539

S. Nojiri and S.D. Odintsov, Gen. Rel. Grav. **38** (2006) 1285,

R.R. Abbyazov, S.V. Chervon, V. Mueller, Mod. Phys. Lett. A **30** (2015) 1550114,

A. Paliathanasis, G. Leon and S. Pan, Gen. Rel. Grav. **51** (2019) 106,

S.V. Chervon, I.V. Fomin, E.O. Pozdeeva, M. Sami and S.Yu. Vernov, Phys. Rev. D **100** (2019) 063522

V. Zhuravlev and S. Chervon, Universe **6** (2020) 195

A. Paliathanasis, Dynamics of Chiral Cosmology, Class. Quant. Grav. **37** (2020) 195014

L. Anguelova, E.M. Babalic, C.I. Lazaroiu, JHEP **09** (2019) 007

M. Braglia, D. K. Hazra, F. Finelli, G. F. Smoot, L. Sriramkumar and A. A. Starobinsky, J. Cosmol. Astropart. Phys. **2008** (2020) 001 [arXiv:2005.02895].

A. Paliathanasis and G. Leon, Class. Quant. Grav. **38** (2021) 075013 [arXiv:2009.12874]; arXiv:2105.03261.

Chiral Cosmological Models

Our goal is to find integrable modified gravity cosmological models with an additional scalar field minimally coupled to gravity:

$$\tilde{S}_{\rm J} = \int d^4x\sqrt{-\tilde{g}}\left[U(\sigma)\tilde{R} - \frac{\theta_\sigma}{2}\tilde{g}^{\mu\nu}\partial_\mu\sigma\partial_\nu\sigma - \frac{\varepsilon_\psi}{2}\tilde{g}^{\mu\nu}\partial_\mu\psi\partial_\nu\psi - V_J(\sigma)\right], \tag{10}$$

where $U(\sigma) > 0$ and $V_J(\sigma)$ are double differentiable functions, the constant θ_σ equals either ±1 or 0, whereas $\varepsilon_\psi = \pm1$. The case of $\theta_\sigma = 0$ corresponds to $F(R)$ gravity models.
Models (10) are connected with chiral cosmological models:

$$S_{\rm E} = \int d^4x\sqrt{-g}\left[\frac{M_{\rm Pl}^2}{2}R - \frac{1}{2}g^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi - \frac{\varepsilon_\psi}{2}K(\phi)g^{\mu\nu}\nabla_\mu\psi\nabla_\nu\psi - V_E(\phi)\right], \tag{11}$$

where the $K(\phi) > 0$ and V_E are differentiable functions.

From

$$S_{\mathrm{E}} = \int d^4x\sqrt{-g}\left[\frac{M_{\mathrm{Pl}}^2}{2}R - \frac{1}{2}g^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi - \frac{\varepsilon_\psi}{2}K(\phi)g^{\mu\nu}\nabla_\mu\psi\nabla_\nu\psi - V_E(\phi)\right],$$

using the metric transformation:

$$g^{\mu\nu} = K(\phi)\tilde{g}^{\mu\nu}, \qquad \sqrt{-g} = \frac{\sqrt{-\tilde{g}}}{K^2},$$

we obtain

$$S_J=\int d^4x\sqrt{-\tilde{g}}\left[\frac{M_{\mathrm{Pl}}^2}{2K}\tilde{R} - \left(\frac{1}{2K} - \frac{3M_{\mathrm{Pl}}^2K_{,\phi}^2}{4K^3}\right)\tilde{g}^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - \frac{\varepsilon_\psi}{2}\tilde{g}^{\mu\nu}\partial_\mu\psi\partial_\nu\psi - \frac{V_E}{K^2}\right].$$

If

$$2K^2 > 3M_{\rm Pl}^2 K_{,\phi}^2,$$

then we can introduce a new scalar field σ:

$$\frac{d\sigma}{d\phi} = \sqrt{\frac{1}{K}\left(1 - \frac{3M_{\rm Pl}^2 K_{,\phi}^2}{2K^2}\right)} \qquad (12)$$

and get a positive kinetic term,

$$U(\sigma) = \frac{M_{\rm Pl}^2}{2K(\phi(\sigma))}, \qquad V(\sigma) = \frac{V_E(\phi(\sigma))}{K^2(\phi(\sigma))}\,. \qquad (13)$$

If

$$2K^2 < 3M_{\rm Pl}^2 K_{,\phi}^2,$$

then we get action with a phantom scalar field $z = i\sigma$, that corresponds to a standard scalar field ϕ.

The exponential function $K(\phi)$

Let

$$K(\phi) = K_0 \mathrm{e}^{\kappa\phi},$$

where K_0 and κ are constants[1].
If $\kappa^2 < 2/(3M_{\rm Pl}^2)$, then using Eq. (12) and

$$\frac{d\sigma}{d\phi} = \sqrt{\frac{2 - 3M_{\rm Pl}^2\kappa^2}{2K_0}}\,\mathrm{e}^{\kappa\phi/2},$$

we get:

$$\sigma = \sqrt{\frac{C_\kappa}{K_0}}\,\mathrm{e}^{\kappa\phi/2}, \qquad C_\kappa = \frac{2\left[2 - 3M_{\rm Pl}^2\kappa^2\right]}{\kappa^2}\,.$$

and an induce gravity model

$$\tilde{S} = \int d^4x\sqrt{-\tilde{g}}\left[\frac{M_{\rm Pl}^2}{2C_\kappa}\sigma^2\tilde{R} - \frac{\tilde{g}^{\mu\nu}}{2}\nabla_\mu\sigma\nabla_\nu\sigma - \frac{\varepsilon_\psi}{2}\tilde{g}^{\mu\nu}\nabla_\mu\psi\nabla_\nu\psi - C_\kappa^2 V_E(\phi(\sigma))\sigma^4\right].$$

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[1] M. Braglia, D.K. Hazra, F. Finelli, G.F. Smoot, L. Sriramkumar and A.A. Starobinsky, J. Cosmol. Astropart. Phys. **2008** (2020) 001 [arXiv:2005.02895].
A. Paliathanasis and G. Leon, Class. Quant. Grav. **38** (2021) 075013 [arXiv:2009.12874]; arXiv:2105.03261.

$F(R)$ gravity

$2K^2 = 3M_{\rm Pl}^2 K_{,\phi}^2$ for all ϕ if and only if

$$K(\phi) = K_0 \mathrm{e}^{\kappa_1 \phi},$$

where K_0 is a constant, $\kappa_1 = \pm \dfrac{\sqrt{2}}{\sqrt{3}M_{\rm Pl}}$.

We get the action

$$S_J = \int d^4x \sqrt{-\tilde{g}} \left[\frac{M_{\rm Pl}^2}{2K_0} \mathrm{e}^{-\kappa_1\phi} \tilde{R} - \frac{\varepsilon_\psi}{2} \tilde{g}^{\mu\nu} \nabla_\mu \psi \nabla_\nu \psi - \frac{V_E(\phi)}{K_0^2} \mathrm{e}^{-2\kappa_1\phi} \right].$$

It is a $F(R)$ gravity model. For example, at $V_E = \Lambda > 0$, we get

$$\mathrm{e}^{-\kappa_1\phi} = \frac{M_{\rm Pl}^2 K_0}{4\Lambda} \tilde{R} \qquad (14)$$

and

$$S_F = \int d^4x \sqrt{-\tilde{g}} \left[\frac{M_{\rm Pl}^4}{16\Lambda} \tilde{R}^2 - \frac{\varepsilon_\psi}{2} \tilde{g}^{\mu\nu} \nabla_\mu \psi \nabla_\nu \psi \right]. \qquad (15)$$

$F(R)$ is stable at $F' > 0$ and $F'' > 0$, so S_F is stable at $\tilde{R} > 0$.

The simplest integrable CCM

Let $V_E(\phi) = \Lambda$ and $K(\phi) = K_0 \mathrm{e}^{\kappa\phi}$.
In the spatially flat FLRW metric with

$$ds^2 = -dt^2 + a_E^2(t)\left(dx^2 + dy^2 + dz^2\right),$$

we obtain the following system of equations:

$$3M_{\mathrm{Pl}}^2 H_E^2 = \frac{1}{2}\dot{\phi}^2 + \frac{\varepsilon_\psi}{2} K\dot{\psi}^2 + \Lambda, \tag{16}$$

$$2M_{\mathrm{Pl}}^2 \dot{H}_E + 3M_{\mathrm{Pl}}^2 H_E^2 + \frac{1}{2}\dot{\phi}^2 + \frac{\varepsilon_\psi}{2} K\dot{\psi}^2 = \Lambda, \tag{17}$$

$$\ddot{\phi} = -3H_E\dot{\phi} + \frac{\varepsilon_\psi}{2} K'_{,\phi}\dot{\psi}^2, \tag{18}$$

$$\ddot{\psi} = -3H_E\dot{\psi} - \frac{K'_{,\phi}}{K}\dot{\phi}\dot{\psi}, \tag{19}$$

where $H_E = \dot{a}_E/a_E$,
dots and primes denote the derivatives with respect to the cosmic time t and to the scalar field ϕ respectively.
From Eqs. (16) and (17), we get

$$\dot{H} + 3H^2 = \frac{\Lambda}{M_{\mathrm{Pl}}^2} \equiv \lambda. \tag{20}$$

At $\Lambda > 0$ the general solution of Eq. (20) has the following form:

$$H(t) = \sqrt{\frac{\lambda}{3}}\,\frac{1 - C\mathrm{e}^{-2\sqrt{3\lambda}\,t}}{1 + C\mathrm{e}^{-2\sqrt{3\lambda}\,t}}, \qquad (21)$$

where C is an integration constant.
The form of the Hubble parameter depends on the sign of C:

- at $C > 0$,

$$H(t) = \sqrt{\frac{\lambda}{3}}\tanh\left(\sqrt{3\lambda}\,(t - t_0)\right),$$

- at $C = 0$,

$$H = \sqrt{\frac{\lambda}{3}},$$

- at $C < 0$,

$$H(t) = \sqrt{\frac{\lambda}{3}}\coth\left(\sqrt{3\lambda}\,(t - t_0)\right).$$

To get $\phi(t)$, we use Eq. (18) in the following form:

$$\ddot{\phi} = -3H\dot{\phi} + \frac{K'_{,\phi}}{K}\left(3M_{\rm Pl}^2H^2 - \frac{1}{2}\dot{\phi}^2 - \Lambda\right) = -3H\dot{\phi} + 3\kappa M_{\rm Pl}^2H^2 - \frac{\kappa}{2}\dot{\phi}^2 - \kappa M_{\rm Pl}^2\lambda.$$

and get a linear differential equation

$$\ddot{u} + 3H\dot{u} + \frac{\kappa^2 M_{\rm Pl}^2}{2}\left(\lambda - 3H^2\right)u = 0\,, \qquad (22)$$

for

$$u(t) = \sqrt{K(\phi)} = {\rm e}^{\kappa\phi/2}\,.$$

Its general solution can be presented in the following form:

$$u(t) = A\cos\left(\left|\frac{\kappa M_{\rm Pl}}{\sqrt{6}}\right| \arccos\left(\sqrt{3/\lambda}\,H(t)\right) + B\right),$$

where A and B are integration constants.

Therefore,

$$\phi(t) = \frac{2}{\kappa} \ln \left[A \cos \left(\left| \frac{\kappa M_{\rm Pl}}{\sqrt{6}} \right| \arccos \left(\frac{1 - C e^{-2\sqrt{3\lambda}\, t}}{1 + C e^{-2\sqrt{3\lambda}\, t}} \right) + B \right) \right]. \qquad (23)$$

Equation (19) gives:

$$\dot{\psi} = \frac{\tilde{C}_\psi e^{-\sqrt{3\lambda}\, t}}{\left(1 + C e^{-2\sqrt{3\lambda}\, t}\right) K(\phi)}, \qquad (24)$$

where $\tilde{C}_\psi$ is an integration constant.

Cosmic time in the Einstein frame is a parametric time in the Jordan frame:

$$ds^2 = -N_J^2(t)dt^2 + a_J^2(t)\left(dx_1^2 + dx_2^2 + dx_3^2\right), \qquad (25)$$

where,

$$N_J = \sqrt{K(\phi)}N_E, \qquad a_J = \sqrt{K(\phi)}a_E. \qquad (26)$$

So, we get the following general solution for R^2 model:

$$N_J(t) = \sqrt{K_0}\mathrm{e}^{\kappa\phi(t)/2}, \qquad (27)$$

$$a_J(t) = \sqrt{K_0}\mathrm{e}^{\kappa\phi(t)/2}a_E(t), \qquad (28)$$

$$\sigma(t) = \sqrt{\frac{C_\kappa}{K_0}}\,\mathrm{e}^{\kappa\phi(t)/2} \qquad (29)$$

and $\dot{\psi}(t)$ is given by (24).

Let us remind that the cosmic time in the Jordan frame is

$$\tilde{t} = \int \sqrt{K(\phi(t))}\,dt, \qquad (30)$$

and the Hubble parameter in the Jordan frame

$$H_J(\tilde{t}) = \frac{1}{\sqrt{K(\phi(\tilde{t}))}}\left[H_E(\tilde{t}) + \frac{1}{2}\frac{d\ln K}{dt}(\tilde{t})\right]. \qquad (31)$$

The behavior of the Hubble parameter in F_0R^2 gravity with a scalar field

We got solutions of modified gravity models in the parametric time $\tau = t$. To obtain solutions in the cosmic time $\tilde{t}$, we try to solve the evolution equations with the cosmic time $\tilde{t}$:

$$ds^2 = -\,d\tilde{t}^2 + \tilde{a}^2(\tilde{t})\left(dx^2 + dy^2 + dz^2\right),$$

$$18F_0\left(6H_J^2\dot{H}_J - \dot{H}_J^2 + 2H_J\ddot{H}_J\right) = \frac{\varepsilon_\psi}{4}\dot{\psi}^2, \quad (32)$$

$$6F_0\left(18H_J^2\dot{H}_J + 12H_J\ddot{H}_J + 9\dot{H}_J^2 + 2\dddot{H}_J\right) = -\,\frac{\varepsilon_\psi}{4}\dot{\psi}^2, \quad (33)$$

where

$$F_0 = \frac{M_{\mathrm{Pl}}^4}{16\Lambda}.$$

Excluding $\dot{\psi}$, we get the following third order differential equation in H_J:

$$\dddot{H}_J + 9H_J\ddot{H}_J + 18H_J^2\dot{H}_J + 3\dot{H}_J^2 = 0. \quad (34)$$

Multiplying Eq. (34) by $\dot{H}_J^2$ and factoring, we get

$$\left(\ddot{H}_J + 3H_J\dot{H}_J\right)\left(2H_J\dddot{H}_J + 6H_J^2\ddot{H}_J + 12H_J\dot{H}_J^2\right)$$
$$= \left(2H_J\ddot{H}_J + 6H_J^2\dot{H}_J - \dot{H}_J^2\right)\left(\dddot{H}_J + 3H_J\ddot{H}_J + 3\dot{H}_J^2\right),$$

so,

$$\left(\ddot{H}_J + 3H_J\dot{H}_J\right)\frac{d}{d\tilde{t}}\left[\dot{H}_J^2 - 2H_J\ddot{H}_J - 6H_J^2\dot{H}_J\right]$$
$$= \left(\dot{H}_J^2 - 2H_J\ddot{H}_J - 6H_J^2\dot{H}_J\right)\frac{d}{d\tilde{t}}\left[\ddot{H}_J + 3H_J\dot{H}_J\right]. \qquad (35)$$

Thus, we got two equations with independent solution.

The first equation

$$\ddot{H}_J + 3H_J\dot{H}_J = 0 \qquad (36)$$

has the following integral:

$$2\dot{H}_J + 3H_J^2 = 2\tilde{C},$$

where $\tilde{C}$ is an integration constant.
For this case, Eq. (32) takes the following form:

$$\varepsilon_\psi\dot{\psi}^2 = -72F_0\dot{H}_J^2. \qquad (37)$$

From this relation, it follows that $\varepsilon_\psi = -1$, and

$$\dot{\psi} = \pm 6\sqrt{2F_0}\dot{H}_J = \pm 3\sqrt{2F_0}\left(2\tilde{C} - 3H_J^2\right). \qquad (38)$$

The model has de Sitter solutions with $H_J = \sqrt{2\tilde{C}/3}$ that correspond to a constant ψ.

The type of solutions obtained depends on the sign of $\tilde{C}$, see Table 1.
The values of constants B and $\tilde{t}'$ are defined by the initial value H_{J0}.
The third line of Table 1 includes the solutions from lines 1 and 2 in a different form and de Sitter solutions at $B = 0$.
The solution from the fifth line is so that the scalar curvature $\tilde{R}$ changes sign at $\tilde{t} = \tilde{t}' \pm \frac{\pi}{3}\sqrt{-2/(3\tilde{C})}$.

Table: List of the R^2 gravity exact solutions.

$\tilde{C}$	$H_J(\tilde{t})$	$\dot{\psi}(\tilde{t})$
$\tilde{C} > 0$, $\dot{H}_{J0} > 0$	$\sqrt{\frac{2\tilde{C}}{3}}\tanh\left(\sqrt{\frac{3\tilde{C}}{2}}\left(\tilde{t}-\tilde{t}'\right)\right)$	$\frac{6\tilde{C}\sqrt{2F_0}}{\cosh^2\left(\sqrt{\frac{3C}{2}}\left(\tilde{t}-\tilde{t}'\right)\right)}$
$\tilde{C} > 0$, $\dot{H}_{J0} < 0$	$\sqrt{\frac{2\tilde{C}}{3}}\coth\left(\sqrt{\frac{3\tilde{C}}{2}}\left(\tilde{t}-\tilde{t}'\right)\right)$	$\frac{6\tilde{C}\sqrt{2F_0}}{\sinh^2\left(\sqrt{\frac{3C}{2}}\left(\tilde{t}-\tilde{t}'\right)\right)}$
$\tilde{C} > 0$	$\frac{\sqrt{6\tilde{C}}\left(1-B\mathrm{e}^{-\sqrt{6\tilde{C}}\tilde{t}}\right)}{3\left(1+B\mathrm{e}^{-\sqrt{6\tilde{C}}\tilde{t}}\right)}$	$\frac{24BC\sqrt{2F_0}\mathrm{e}^{-\sqrt{6\tilde{C}}\tilde{t}}}{\left(B\mathrm{e}^{-\sqrt{6\tilde{C}}\tilde{t}}+1\right)^2}$
$\tilde{C} = 0$	$\frac{2}{3\left(\tilde{t}-\tilde{t}'\right)}$	$\frac{4\sqrt{2F_0}}{\left(\tilde{t}-\tilde{t}'\right)^2}$
$\tilde{C} < 0$	$-\frac{\sqrt{-6\tilde{C}}}{3}\tan\left[\frac{\sqrt{-6\tilde{C}}}{2}\left(\tilde{t}-\tilde{t}'\right)\right]$	$\frac{6C\sqrt{2F_0}}{\cos^2\left(\sqrt{\frac{-3C}{2}}\left(\tilde{t}-\tilde{t}'\right)\right)}$

If $\ddot{H}_J + 3H_J\dot{H}_J \neq 0$, then one can integrate Eq. (35) and get the following equation:

$$\frac{\dot{H}_J^2}{\ddot{H}_J + 3H_J\dot{H}_J} - 2H_J = C_1, \tag{39}$$

where C_1 is a constant of integration.
Integrating the equation,

$$(C_1 + 2H_J)\ddot{H}_J + 3H_J(C_1 + 2H_J)\dot{H}_J - \dot{H}_J^2 = 0, \tag{40}$$

one gets:

$$\dot{H}_J = C_2\sqrt{|C_1 + 2H_J|} + (C_1 + 2H_J)(C_1 - H_J), \tag{41}$$

where C_2 is also a constant of integration.
Equation (41) with arbitrary constants C_1 and C_2 can be solved in quadratures.

Also, there are some particular solutions of Eq. (41) at $C_2 = 0$

1. At $C_1 = 0$,

$$H_J(t) = \frac{1}{2\left(\tilde{t} - \tilde{t}'\right)}, \tag{42}$$

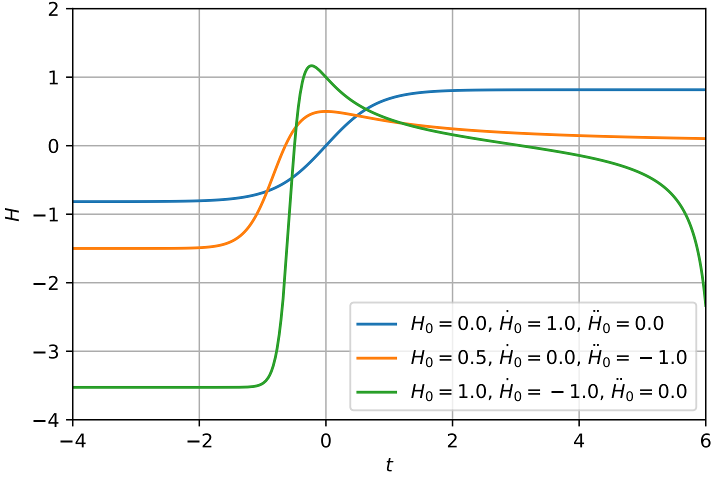
2. At $C_1 \neq 0$,

$$H_J(t) = C_1 \frac{\tilde{C} + \mathrm{e}^{-3C_1\tilde{t}}}{\tilde{C} - 2\,\mathrm{e}^{-3C_1\tilde{t}}}, \tag{43}$$

where $\tilde{C}$ is a constant of integration.
Combining Eqs. (32) and (40), we obtain

$$\dot{\psi}^2 = -72F_0C_1\varepsilon_\psi\left(\ddot{H}_J + 3H_J\dot{H}_J\right). \tag{44}$$

So, the case of $C_1 = 0$ corresponds to R^2 without additional scalar field.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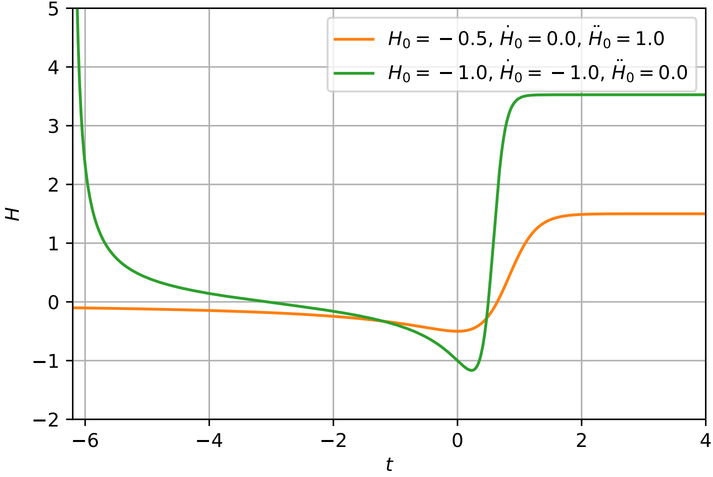
$H_0 = 1.0, \dot{H}_0 = -1.0, \ddot{H}_0 = 0.0$



A continues function $H_J(\tilde{t})$ change the sign only if ψ is a phantom field.

CONCLUSIONS

- We have found the general solution of CCM, described by

$$S_{\rm E} = \int d^4x\sqrt{-g}\left[\frac{M_{\rm Pl}^2}{2}R - \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - \frac{\varepsilon_\psi}{2}{\rm e}^{\kappa\phi}g^{\mu\nu}\partial_\mu\psi\partial_\nu\psi - \Lambda\right],$$

and the corresponding induced gravity models:

$$\tilde{S}_J = \int d^4x\sqrt{-\tilde{g}}\left[\frac{M_{\rm Pl}^2}{2C_\kappa}\sigma^2\tilde{R} - \frac{\tilde{g}^{\mu\nu}}{2}\nabla_\mu\sigma\nabla_\nu\sigma - \frac{\varepsilon_\psi}{2}\tilde{g}^{\mu\nu}\nabla_\mu\psi\nabla_\nu\psi - \Lambda C_\kappa^2\sigma^4\right],$$

and R^2 gravity:

$$S_F = \int d^4x\sqrt{-\tilde{g}}\left[\frac{M_{\rm Pl}^4}{8\Lambda}\tilde{R}^2 - \frac{\varepsilon_\psi}{2}\tilde{g}^{\mu\nu}\nabla_\mu\psi\nabla_\nu\psi\right].$$

- For R^2 gravity model we have integrated evolution equations with the cosmic time.
- There exist different solutions, including bounce solutions with non-monotonic behaviour of the Hubble parameter.

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$$S_{\rm E}=\int d^4x\sqrt{-g}\left[\frac{M_{\rm Pl}^2}{2}R-\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi-\frac{\varepsilon_\psi}{2}{\rm e}^{\kappa\phi}g^{\mu\nu}\partial_\mu\psi\partial_\nu\psi-\Lambda\right],$$

and the corresponding induced gravity models:

$$\tilde{S}_J=\int d^4x\sqrt{-\tilde{g}}\left[\frac{M_{\rm Pl}^2}{2C_\kappa}\sigma^2\tilde{R}-\frac{\tilde{g}^{\mu\nu}}{2}\nabla_\mu\sigma\nabla_\nu\sigma-\frac{\varepsilon_\psi}{2}\tilde{g}^{\mu\nu}\nabla_\mu\psi\nabla_\nu\psi-\Lambda C_\kappa^2\sigma^4\right],$$

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$$S_F=\int d^4x\sqrt{-\tilde{g}}\left[\frac{M_{\rm Pl}^4}{8\Lambda}\tilde{R}^2-\frac{\varepsilon_\psi}{2}\tilde{g}^{\mu\nu}\nabla_\mu\psi\nabla_\nu\psi\right].$$

• For R^2 gravity model we have integrated evolution equations with the cosmic time.

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Thank you